Continuum Mechanics And Transport Phenomena

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Lecture - 26
Differential Total Mass Balance Part 2

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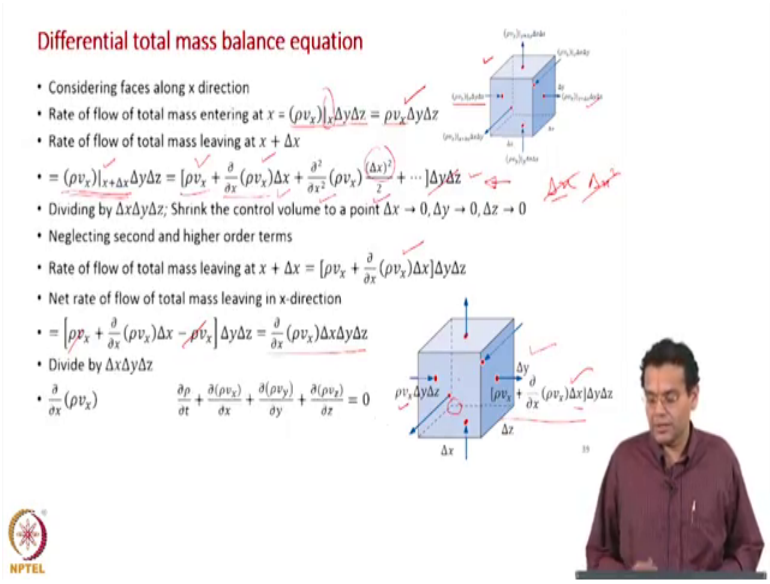

Now, the way in which we have derived is usually followed in chemical engineering books, transfer phenomena books. If you look at fluid mechanics books the same derivation is done, of course, inter-cell cannot change in and slightly different way. So, for you to enable to understand from fluid mechanics book, we will derive in that particular way also, some steps are different end result is the same. Let us do that I listed only for the x-direction so, that you can understand for other directions. Now, we said

The rate of flow of total mass entering at x = $\left(\rho v_x\right)\Big|_x \Delta y \Delta z = \rho v_x \Delta y \Delta z$

This diagram (above slide image) is our earlier representation, the control volume shown is the new representation. Now, this is how we express the total mass entering and that is what is shown here. The way in which it is done is this vertical bar and x or naught usually shown, it is just written as ρv_x the implication is that it is elevated at x. So, the vertical bar x is not shown and that is the representation shown here, physical significance is the same, representation is slightly different.

Now,

The rate of flow of total mass leaving at x + Δx = $\left(\rho v_x\right)\big|_{x+\Delta x}\Delta y\Delta z$

In fluid mechanics books, this $\left(\rho v_x\right)\big|_{x+\Delta x}$ is represented in terms of the Taylor series,

$$\left(\rho v_x\right)\big|_{x+\Delta x}\Delta y\Delta z = \left[\rho v_x + \frac{\partial}{\partial x}\left(\rho v_x\right)\Delta x + \frac{\partial^2}{\partial x^2}\left(\rho v_x\right)\frac{(\Delta x)^2}{2} + \ldots\right]\Delta y\Delta z$$

So, instead of writing this way expansion Taylor series, let us see further how do we proceed. Now, in the derivation, we have seen we are dividing by $\Delta x\Delta y\Delta z$ and then we string the control volume meaning, that $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$. So, if you divide this expression by $\Delta x\Delta y\Delta z$, $\Delta y\Delta z$ will cancel out then you will have Δx. Now, terms that have Δx^2 and higher, if you divide by Δx for the Δx^2 term Δx will remain, then Δx^2 will remain and then when you take $\Delta x \to 0$ all those terms will not contribute. So, which means that only the first two terms will contribute. So, the way in which is return is neglecting second and higher order terms.

You divide by $\Delta x\Delta y\Delta z$, in all other higher order terms second higher order terms Δx will remind and we are going to string and then say $\Delta x \to 0$; so, all those terms will not contribute. So, we neglect second and higher order terms and write the rate of mass leaving at x +Δx as only the first two terms, of course, multiplied by area.

The rate of flow of total mass leaving at x + Δx = $\left[\rho v_x + \frac{\partial}{\partial x}\left(\rho v_x\right)\Delta x\right]\Delta y\Delta z$

Now,

The net rate of flow of total mass in x-direction = $\left[\rho v_x + \frac{\partial}{\partial x}\left(\rho v_x\right)\Delta x\right]\Delta y\Delta z - \rho v_x\Delta y\Delta z$

So, ρv_x cancels out and after simplification we get,

The net rate of flow of total mass in x-direction = $\frac{\partial}{\partial x}\left(\rho v_x\right)\Delta x\Delta y\Delta z$

Now, divide by $\Delta x\Delta y\Delta z$ and then result in our same expression as we have got earlier.

The net rate of flow of total mass in x-direction = $\frac{\partial}{\partial x}\left(\rho v_x\right)$

Of course, you can extend to other terms as well and obtain

The net rate of flow of total mass in y-direction = $\frac{\partial}{\partial y}\left(\rho v_y\right)$

The net rate of flow of total mass in z-direction = $\frac{\partial}{\partial z}\left(\rho v_z\right)$

This same differential form of the total mass balance equation. So,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\, v_x)}{\partial x} + \frac{\partial(\rho\, v_y)}{\partial y} + \frac{\partial(\rho\, v_z)}{\partial z} = 0$$

This is what you will find in fluid mechanics books, other small variant is that we took x, y, z to one corner, they take in the center of the control volume. So, here you will have $\frac{\Delta x}{2}$, here you will have the same expression with $-\frac{\Delta x}{2}$ etcetera. So, those are other variants, but I have shown one variant here so, that you can follow other variants as well.

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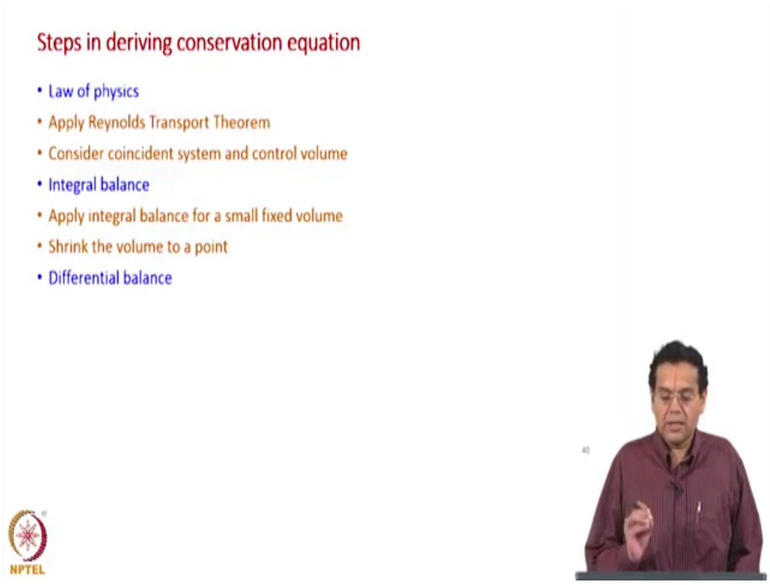

So, let us proceed further. So, what are the steps in deriving the conservation equation? I would say this is a very important slide, in the sense that we have arrived at the recipe of deriving conservation equations. And, because it is mass balance, is easier for us to follow, we keep this in mind we can extend it to other conservation equations as well. So, what are the steps?

- The first step is we started with the law of physics and then
- We apply the Reynolds transport theorem and, then
- We took consider coincidence system and control volume and then
- We arrived at the integral balance equation.

I have not used here mass momentum anything, because this is applicable for all the conservation equation. And, then

- We applied the integral balance for a small fixed control volume and then
- We shrink the volume to a point resulting in a differential balance equation.

So, these steps are universal across the derivation of all the conservation equation.

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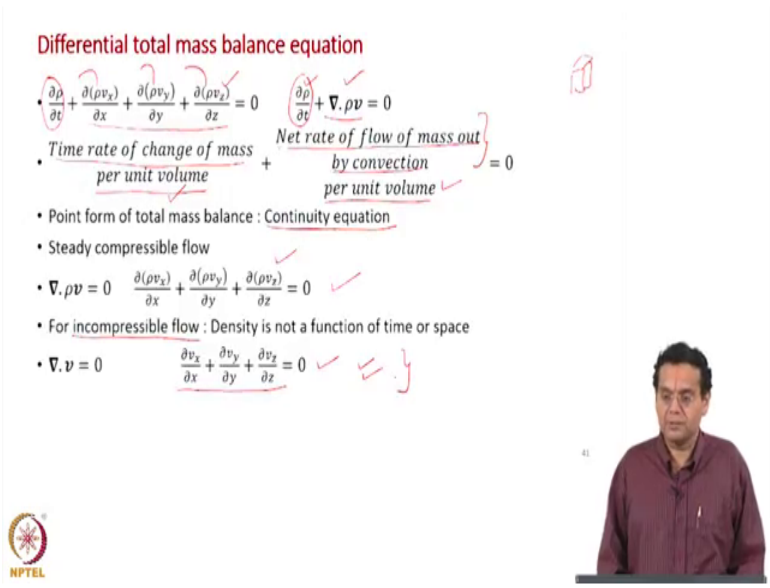

So, let us look at the physical significance of the terms in the differential total mass balance equation, by this time it should be clear the way in which wrote down the different terms.

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho\, v_x)}{\partial x}+\frac{\partial(\rho\, v_y)}{\partial y}+\frac{\partial(\rho\, v_z)}{\partial z}=0$$

This is the expanded form of differential total mass balance equation and

$$\frac{\partial \rho}{\partial t}+\nabla.\rho v=0$$

This is in terms of a vectorial notation, slowly we can get use the vectorial form as well; very nice representation. Now, we have two terms: one is the variation of the time term and the other is the net mass flow rate leaving term. So, let us look at the significance, the first term represents

$\frac{\partial \rho}{\partial t}$ = time rate of change of mass per unit volume, and

$\nabla . \rho v$ = the net rate of flow of mass out by convection per unit volume.

If you have observed the significance of the terms are the same as that of integral balance; obviously, it cannot change we have started from there so, significance should be same. But in what way it is different? It there it was just the time rate of change of mass, but it is here it is per unit volume, remember we divided by $\Delta x \Delta y \Delta z$. So, of course, it is density mass per volume. So, significant at the first term is the time rate of change of mass per unit volume. In next second term, the net rate of flow mass out by conversion is the same as integral balance; once again here it is per unit volume. So, the significance is same as that of integral balance, but now they are on a per unit volume basis. So, how do you imagine? Remember, we have the control volume and then we said we are shrinking to a point and then small cuboidal volume.

So, per unit volume of that region what is the rate of change of mass, per unit volume of that region what is the net mass flow leaving by convection that is the significance. So, per unit volume to imagine should imagine a very small control volume cuboidal control volume, per unit volume what is the time rate of change of mass and then of course, and then the net rate of flow of mass out by convection. And, remember all the terms in the differential mass balance equation are per unit volume basis, this also universal. Whenever we are going to write differential balance equations they are per unit volume basis. Now, what are the other names for this differential total mass balance equation?

Because we consider a control volume and then shrink to a point, it is also called as point form of mass balance equation. This you may not come across very frequently, but it is very obvious the way in which we have done it's also called as point form of mass balance equation of course, we have been repeating it saying the differential form of mass balance equation.

The most popular name is this continuity equation, the continuity equation is the most popular name. So, all these mean the same convey the same meaning, they all tell about mass balance in a small region under steady state and unsteady state condition. Three different names

- Differential total mass balance,
- Point form of total mass balance and
- The continuity equation.

You will come across the word continuity equation mostly always I would say. Now, let us simplify this consider two special cases: one is steady compressible flow, moment it is steady there is no transient term. So,

$$\frac{\partial \rho}{\partial t} = 0$$

So,

$$\nabla . \rho v = 0$$

Or,

$$\frac{\partial(\rho\, v_x)}{\partial x} + \frac{\partial(\rho\, v_y)}{\partial y} + \frac{\partial(\rho\, v_z)}{\partial z} = 0$$

What is an example of this? Can consider, for example, the flow of air in a duct for our case rectangular duct at relatively high velocity, not very low velocity; applicable for low velocity also. But, if the air is flowing at high velocity, you will have to consider this form of the mass balance equation under steady state condition. Now, for incompressible flow what are the examples we can take?

First, let us simplify this and then I will explain incompressible flow the density is not a function of time or space. So,

$$\frac{\partial \rho}{\partial t} = 0$$

The first term vanishes, not because it is steady state, just because density is a constant value; let us say water is flowing then the density is let us say 1000 kg/m^3 because it is not a function of time this term vanishes. Then because density is a constant I can take out the density of all the derivative terms. And so, there will also that term rho also will not appear in the equation and the differential equation becomes much simpler,

$$\nabla . v = 0$$

Or,

$$\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z} = 0$$

I will repeat the second part, we are considering incompressible flow; now I can give examples. The first example as I told you is the flow of water constant density; rho is not a function of time as a function of space also. So, the first time does not contribute ρ can be taken out to the second term, another example where this equation is valid is the flow of air at low velocities. Water is almost incompressible certainly you can use this air is of course, compressible it is a gas it is compressible, but if the air is flowing at low velocities then this equation is valid and that is why I use the word here incompressible flow.

So, when you say incompressible flow two categories are considered: one is the flow of incompressible fluid itself which is water for example. Another is the flow of a compressible fluid like air but at low velocities. So, if you want to summarize the application, suppose if it is the flow of water you use the second simplified equation. The flow of air at low velocities once again use used the simplified equation, but if the air is flowing at high velocities then you have to use the first equation which is a more general form. Also like to mention this simplified form of incompressible flow is applicable both for steady state and unsteady state because, $\frac{\partial \rho}{\partial t}$ vanished because density is constant.

Later on, in the course we will see another meaning of incomparable flow; until then we will stick on to this particular meaning. We will have a very nice physical representation of what we mean by incompressible flow which will result in of course same equation, but let us stick to this right now.s