Continuum Mechanics And Transport Phenomena
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Lecture - 25
Differential Total Mass Balance Part 1

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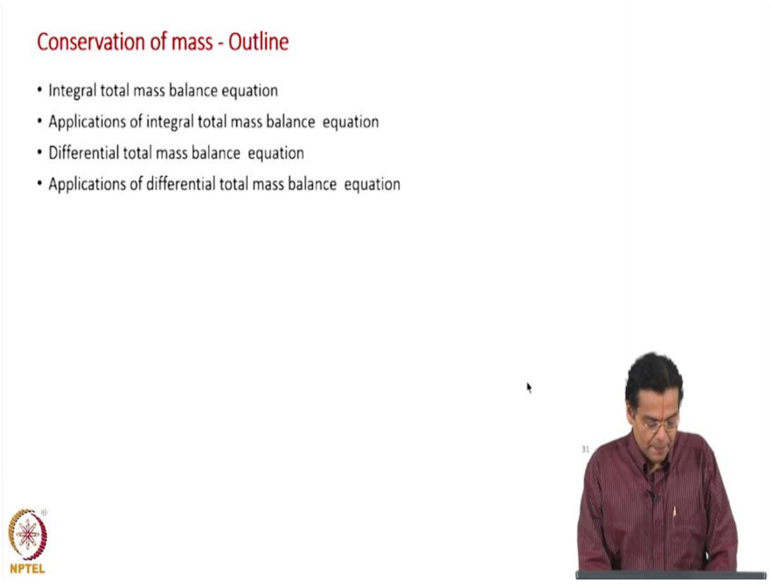

We will get started. We are discussing the conservation of mass and that we derived the integral total mass balance equation starting the law of physics using the Reynolds transport theorem. And then we looked at applications of that integral total mass balance equation under steady state and unsteady state conditions etcetera. Now we are going to derive the differential form of the total mass balance equation and also look at applications of differential total mass balance equation.

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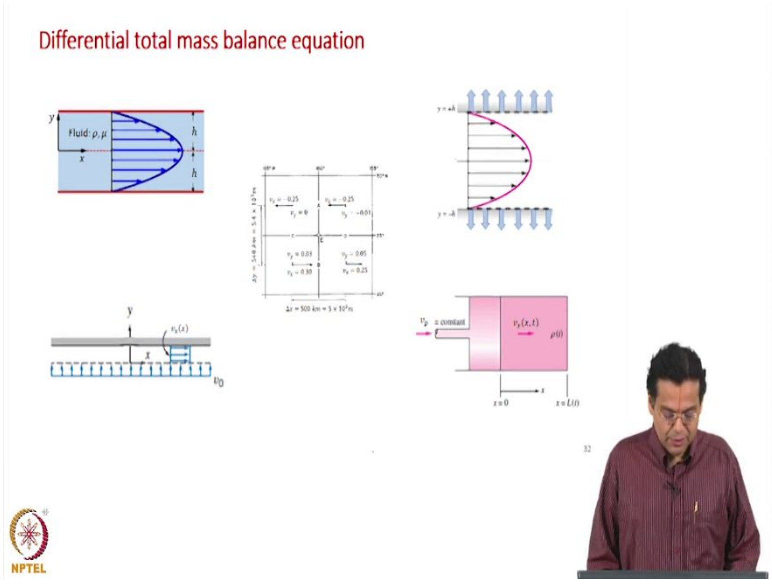

Now, what are the applications we will be looking at after sometime, we will be applying the differential form total balance equation for these kinds of configurations.

- One is our usual flow between the two parallel plates,
- Another configuration is same parallel plates but with porous walls and then
- The flow between the narrow gap and
- Another interesting example linking measurements in an ocean and then
- Of course, the last example would be once again on the transient.

This differential form of total mass balance equation puts a restriction on the components of velocity. So, all these examples will revolve around finding out one velocity component, details of course you will know as we go along.

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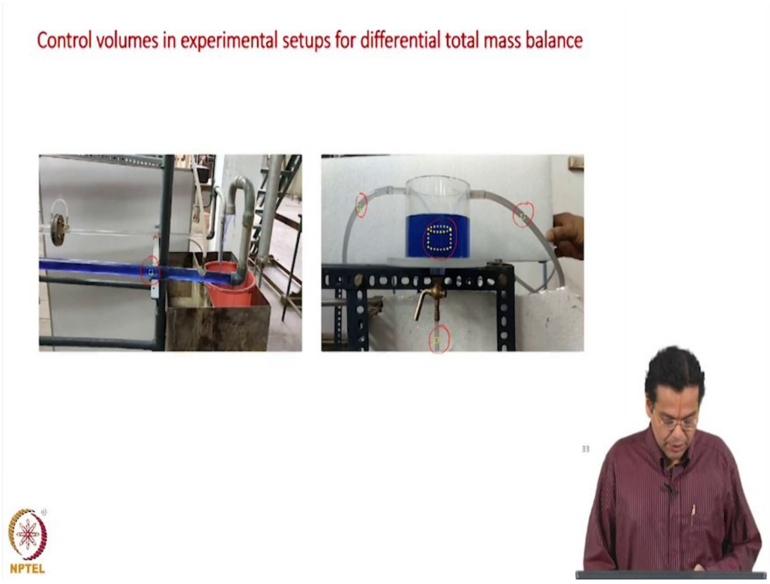

So, control volumes and experimental setups I think we have come across this setup a few times. So, I have shown only the control volume which we will be using for differential total mass balance, I have not shown that for the integral balance though. So, the yellow and dash lines (above slide images) represent the control volume which is now inside the equipment that is what is to be paid attention to either inside the tank or in the inlet pipe or in the outlet pipe. So, the control volume when we say for differential balance has to be imagined a very small volume we will see as we go along.

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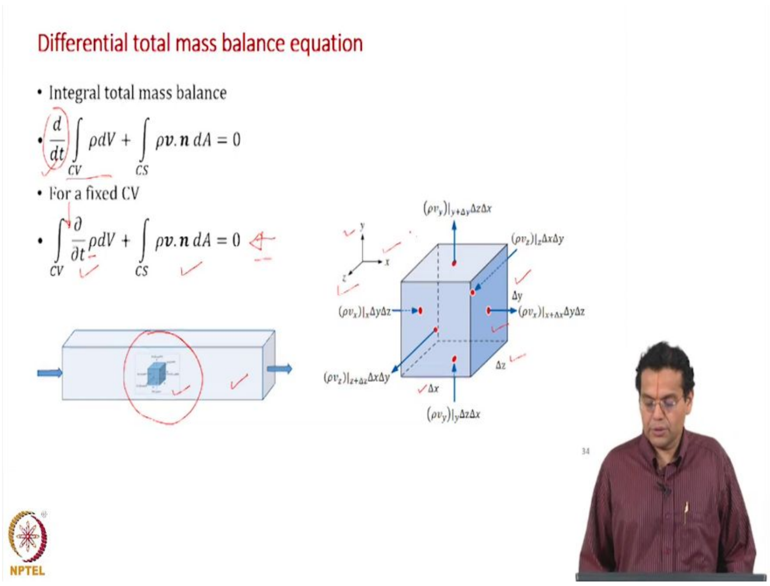

What is the starting point for this differential total mass balance equation, we start the integral total mass balance equation. So, this was obtained from the law of physics. So, which means that indirectly, we are obtaining the differential balance also from the law of physics only that we keep in mind always. Let us write the integral total mass balance equation which we derived

$$\frac{d}{dt}\int_{CV} \rho \, dV + \int_{CS} \rho \, v.n \, dA = 0$$

Now, I am going to apply this for a small control volume inside the equipment as shown here (left side in the slide image), a rectangular domain is shown. Usually of course, pipes are cylindrical but that will lead us to a cylindrical coordinate system and within the scope of this course mostly we are going to restrict to the Cartesian coordinate system and hence pipe of the rectangular cross section is shown here and a small control volume as we have seen in the experimental set up is also shown here. So, it is a small fixed control volume.

So, let us write this integral total mass balance equation for a fixed control volume, we have done this when we discussed the Reynolds transport theorem.

$$\int_{CV} \frac{\partial}{\partial t} \rho \, dV + \int_{CS} \rho \, v.n \, dA = 0$$

We bring in the differential sign inside the integral and when we do that we express that in terms of partial derivative because now the density can be a function of spatial coordinates and time. In this case, because we are integrating after integration only time remains as the only independent variable and hence it was $\frac{d}{dt}$. Now it is inside the integral so density can be a function of space and time hence partial derivative of course, the second term remains as it such.

Now, what is shown on right hand side is the enlarged view of this control volume. Let us see how do we discuss that control volume. First attention is on the coordinate axis the horizontal axis is x, the vertical axis is y, and z-axis points away from the board or slide or a paper. Now, so accordingly we have taken this control volume and we have taken a cuboidal control volume, as I discussed we are restricted into the Cartesian coordinate system hence we are taking a cuboidal control volume. Now the length along the x-axis is Δx and along the y-axis is Δy and along the z-axis is Δz.

So, those are the dimensions of the cuboidal control volume which we are considering. So, we are going to apply this form of the integral mass balance equation, if I say mass it means total mass as we go along ok because species mass has no confusion now until the very end of the course. So, let us use for simplicity mass. So, we are going to apply this form of the integral mass balance equation for the control volume shown here.

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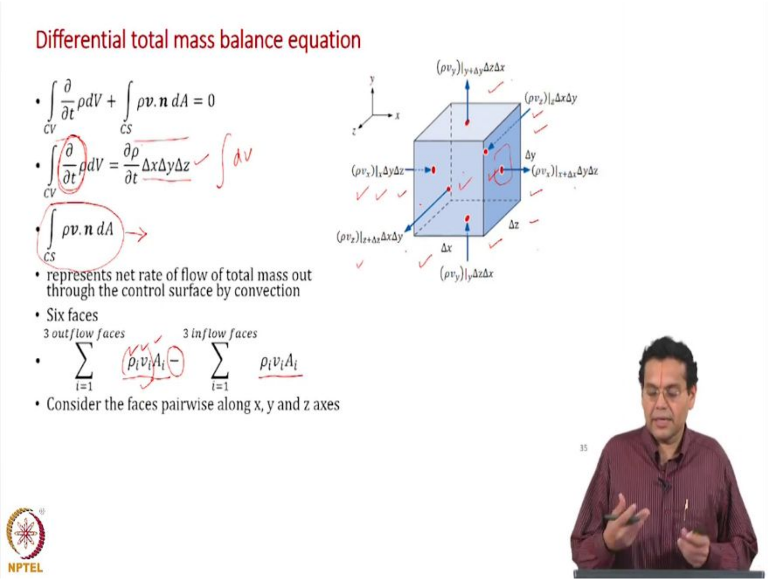

Now, express the two terms in an integral mass balance equation for this control volume.

$$\int_{CV} \frac{\partial}{\partial t} \rho \, dV + \int_{CS} \rho \, v.n \, dA = 0$$

Let us take the first term, now

$$\int_{CV} \frac{\partial}{\partial t} \rho \, dV = \frac{\partial \rho}{\partial t} \Delta x \Delta y \Delta z$$

The density has some average value ρ and when I say average value it is averaged over the entire volume and so I can take out $\frac{\partial \rho}{\partial t}$ outside the integral. Then what have left out is the integral of dV which is nothing but the volume of this small control volume which is $\Delta x \Delta y \Delta z$. So, what is the average value? You will understand as we go along. Right now within the small volume, I take there is a density which is an average value in the small volume which means that average means it is constant over the entire region.

So, this is the expression for the first term. Now let us consider the second term

$$\int_{CS} \rho\, v.n\, dA$$

we have seen the significance of this, let us repeat it represents the net rate of flow of total mass out through the control surface by convection. This is a meaning which you have discussed when we derived the integral mass balance equation. Now if you look at the control surface, we have six faces here. What are the faces? If you take in terms of along axis you have the right side face and the left side face and then you have the top face and then the bottom face and then the front face and the rear face in terms of a pairwise if you take.

Now, I will express this integral for this control volume. So, if we consider the six faces there are three outlet faces and then three inlet faces.

$$\sum_{i=1}^{3\ out\,flow\,faces} \rho_i v_i A_i - \sum_{i=1}^{3\ in\,flow\,faces} \rho_i v_i A_i$$

The outlet faces are the right side face, the top face, and the front face. So, these are the outlet faces or the inflow faces the left and then the bottom and then the rear so all those are the inflow faces. Obviously, here we should be expressed in terms of the densities and velocities and areas rather than mass flow rate that is not significance but m dot has no importance and relevance here. Our expression that differential equation is going to be expressed in terms of the measurable properties namely density, velocity. So, we express the mass in terms of density velocity that is why expression what you see is $\rho_i v_i A_i$ for the outflow faces. Similarly for the inflow faces and then we have taken care of the negative sign ourselves instead of leaving it to $v.n$.

Now, the second point is that we know that this expression is valid when the velocity is perpendicular to the face. So, for all the six faces shown here the velocity is perpendicular to the respective faces, so that is why we are writing this expression.

Thirdly remember we can write this expression when the density and velocity are uniforms across the area, just like we said this density is some average value in the volume. Similarly what we will say is this $\rho_i v_i$ has some average value over this face over any of the faces. So, that is why we write in terms of $\rho_i v_i A_i$.

I will just repeat the last two points, the way we are writing this expression implies that the velocity is perpendicular to the surface area and that is what we have expressed as. We have shown all the velocities perpendicular to the respective faces and the second point is that this expression assumes that the density and velocity uniform across a surface area and so we take some average value over the entire surface area be it total etcetera. Now what we will do is these are in terms of rho and velocities let express in terms of further express this ok. Now let us consider the faces pairwise, we will first take the inlet and outlet faces along the x-axis then consider the y-axis, z-axis.

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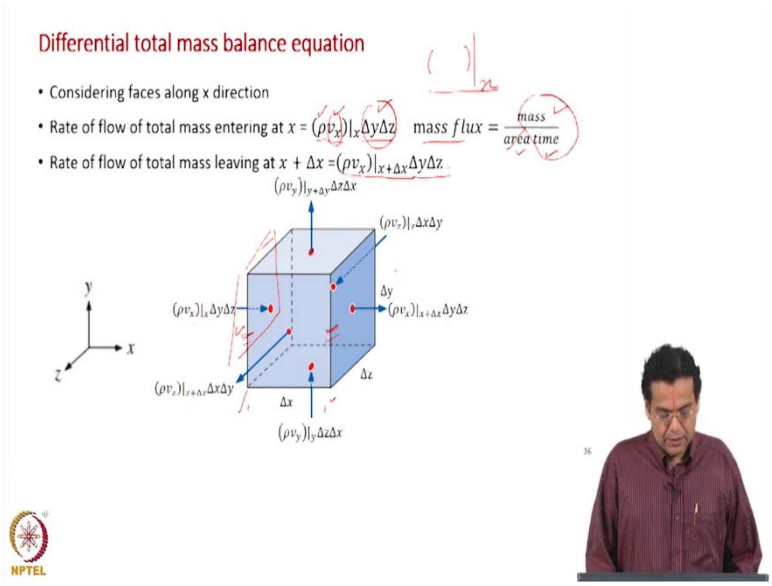

Let us do that now so considering faces along the x-direction. Now before proceeding that let us look at the coordinates, we are considering faces along the x-axis, so the left side face is an inlet face and the right side face is an outlet face. Now this face the inlet face is at an x location of x the outlet face because the length of the cuboid along the x as Δx the right hand side face location is $x + \Delta x$. So, entering the face at x the leaving face is at $x + \Delta x$. Now,

The rate of flow of total mass entering at x = $(\rho v_x)|_x \Delta y \Delta z$

There are two ways of interpreting this expression. Number one vx is a velocity in x-direction multiply by the area what is area this cross sectional area through which it enters and which is this $\Delta y \Delta z$ that is the area through which flows. So, the area of the faces $\Delta y \Delta z$, so velocity in the x-direction multiply by area that gives you volumetric flow rate multiply by density gives you mass flow rate. So, we are writing expression for rate at which mass enters the control surface. Now, what is the other way of interpreting? We have seen earlier that velocity represents volumetric flux. When you derived the integral balance we said velocity represents volumetric flux. So, when you multiply by density, you have mass flux of course, we know mass flux is mass per area per time. So, when you multiply by area you are left with mass per time which is the mass flow rate.

So, two ways of interpreting velocity into area volumetric flow rate multiply by density which is a mass flow rate or the volumetric flux multiply by density mass flux multiply by area gives mass flow rate.

Now coming to the nomenclature here what I have shown here is enclosed $(\rho v_x)_x$ and subscript x. What does it mean? Now, this density and velocity are changing with the spatial location. We are evaluating mass entering at the left face for which x coordinate is x. So, this tells you that ρv_x is evaluated at x or the value of rho and vx at the position x. So now,

The rate of flow of total mass leaving at x + Δ x = $(\rho v_x)|_{x+\Delta x}\,\Delta y \Delta z$

So, we have written expression for whatever mass entering the control volume through the left face and leaving through the right face.

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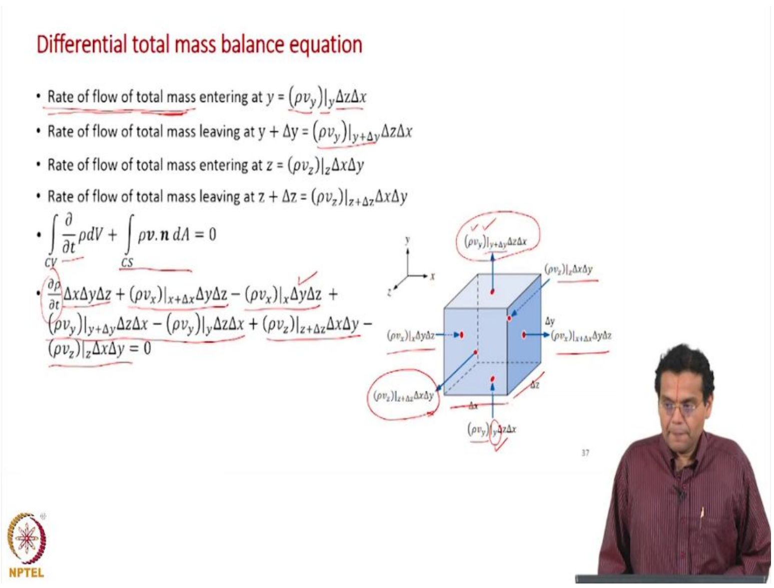

Now, write similar expressions for mass flow rate entering and leaving through the other faces.

The rate of flow of total mass entering at y = $(\rho v_y)|_y \Delta x \Delta z$

The rate of flow of total mass leaving at y + Δy = $(\rho v_y)|_{y+\Delta y} \Delta x \Delta z$

So, let me just show for the z-direction

The rate of flow of total mass entering at z = $(\rho v_z)|_z \Delta x \Delta y$

The rate of flow of total mass leaving at z + Δ z = $(\rho v_z)|_{z+\Delta z} \Delta x \Delta y$

Just to like to mention, the description used here says the rate of flow of total mass, I could have just told written as a mass flow rate this particular nomenclature is used. So, that later on when you see similar statements for momentum balance, energy balance, species balance, I would say rate of flow of momentum, energy, species mass.

So, the same words are used so that the analogy also becomes clear as we go along. Instead of writing mass flow rate the words used the rate of flow of total mass, species mass, momentum energy etcetera. Now let us put it all together in the integral balance equation

$$\int_{CV} \frac{\partial}{\partial t} \rho \, dV + \int_{CS} \rho \, v.n \, dA = 0$$

Now, we have expressed this as

$$\frac{\partial \rho}{\partial t} \Delta x \Delta y \Delta z + (\rho v_x)|_{x+\Delta x} \Delta y \Delta z - (\rho v_x)|_x \Delta y \Delta z + (\rho v_y)|_{y+\Delta y} \Delta x \Delta z - (\rho v_y)|_y \Delta x \Delta z + (\rho v_z)|_{z+\Delta z} \Delta x \Delta y - (\rho v_z)|_z \Delta x \Delta$$

So, I like to mention that for the shake of formality and to be in line with usual books derivations are done taking all the three directions into account in terms of understanding. If you understand for one direction most of the time you can just extend the using words like analogously similarly and easily extend. But for the shake of completeness in all the lectures, we will see terms accounting for all the three directions. So, when you look at it may look little complex, but if you focus on one particular direction; you can easily extend to other directions as well. So, let us proceed.

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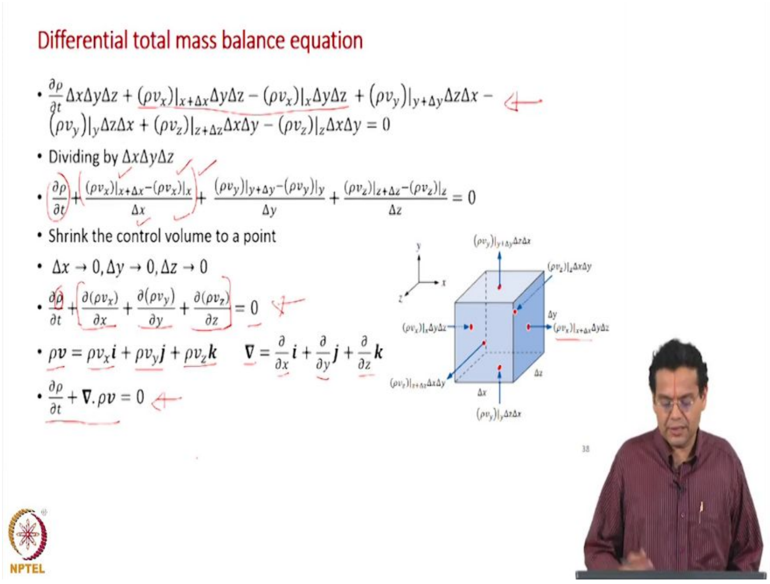

So let us rewrite that equation here

$$\frac{\partial \rho}{\partial t}\Delta x\Delta y\Delta z + (\rho v_x)|_{x+\Delta x}\Delta y\Delta z - (\rho v_x)|_x\Delta y\Delta z + (\rho v_y)|_{y+\Delta y}\Delta x\Delta z - (\rho v_y)|_y\Delta x\Delta z + (\rho v_z)|_{z+\Delta z}\Delta x\Delta y - (\rho v_z)|_z\Delta x\Delta$$

This the expression which you have seen in the last slide. Now what we will do is divide by $\Delta x\Delta y\Delta z$ and we get

$$\frac{\partial \rho}{\partial t} + \frac{(\rho v_x)|_{x+\Delta x}-(\rho v_x)|_x}{\Delta x} + \frac{(\rho v_y)|_{y+\Delta y}-(\rho v_y)|_y}{\Delta y} + \frac{(\rho v_z)|_{z+\Delta z}-(\rho v_z)|_z}{\Delta z} = 0$$

If you look at the second term alone that term represents the net rate of mass flow leaving the control volume in x-direction that is the significance of that. I tell you this because you will see how this term get simplified. So, that when you look at this term that is what should come to your mind. It tells you that the net rate of mass flow leaving the control volume at the direction and then that is per unit volume. It is not just the mass flow rate, it represents net mass flow rate leaving the control volume in the x-direction.

Because we have divided by the $\Delta x\Delta y\Delta z$ at this stage after dividing it represents per unit volume, please keep this in mind so that when you are interpreting the physical significance of the terms the differential equation it will be helpful. So, once again to emphasize at this stage of the equation they represent mass flow rate or net mass flow rate in x-direction etcetera. At this stage after dividing, they represent net mass flow rate, of course leaving per unit volume.

Now, what is that we are going to do now remember the differential equation the form the total mass balance equation which we are approaching to should be validate every point in the equipment, at every instant of time we are writing this equation at any instant of time and for to get an equation which is valued at every point inside the equipment we shrink this control volume. When I say shrink that shrink should be proportionally done. For example, we say that we maintained the aspect ratio, maintaining the aspect ratio, so the cuboid remain as a cuboid and then you shrink to the small point.

So, the way you should imagine is you have a small point, but that small point as a cuboidal shape with the same delta $\Delta x \Delta y \Delta z \rightarrow 0$. So, that is what we write formally as $\Delta x \rightarrow 0$ $\Delta y \rightarrow 0$, $\Delta z \rightarrow 0$. So, the entire control volume is made to a point, but the point also the shape of a cuboid that is the point to be remembered.

Now, sometime back we said I consider an average density in the control volume, now that average has now become a point value because the small control volume has become now a point. So, no longer average has to be told just becomes a point value. In fact, if you are more formal more precise we should use a different symbol earlier and some other symbol now.

Now it was ρ some average value in the control volume now because we have shrunk to a point it just becomes a point value and that is what we are interested in. Similarly these $\rho\, v_x$ values we said that average values across the face and those faces become now a point a small face so they also become a point values.

So, every variable in the equation has become a point value now, that is why I said you will understand this average and point value little later. To begin with they were all average values either over the volume or over the face. Now just because they became a point the entire control volume has become an infinitesimal point.

So, every variable also becomes a point value. So now, let us see what happens

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho\, v_x)}{\partial x} + \frac{\partial (\rho\, v_y)}{\partial y} + \frac{\partial (\rho\, v_z)}{\partial z} = 0$$

Now, what happens to the second term, of course very obvious limit $\Delta x \rightarrow 0$ some function at $x + \Delta x$ minus at x use the derivative this case because we are considering the only variation along x-direction keeping other variables constant. We result in a partial derivative similarly in the y-direction, similarly in z-direction of course right hand side is 0. So, this is the Differential form of the total mass balance equation.

We will put little more in a nice representation; if we use vectorial representation, we know that

$$\rho\, v = \rho\, v_x\, i + \rho\, v_y\, j + \rho\, v_z\, k$$

Then the gradient term vector can be expressed as

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

So, the simplified form of the differential mass balance equation is

$$\frac{\partial \rho}{\partial t} + \nabla . \rho v = 0$$

So, this is a more short representation; a simple representation of the same differential total mass balance equation. Of course, here the variables are explicitly seen, but if you want to represent them in a short way, then their vectorial representation helps us. It has some other utility representing in vectorial representation which we will see towards the very end of the course.