Continuum Mechanics And Transport Phenomena
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Lecture - 24
Integral Total Mass Balance: Examples

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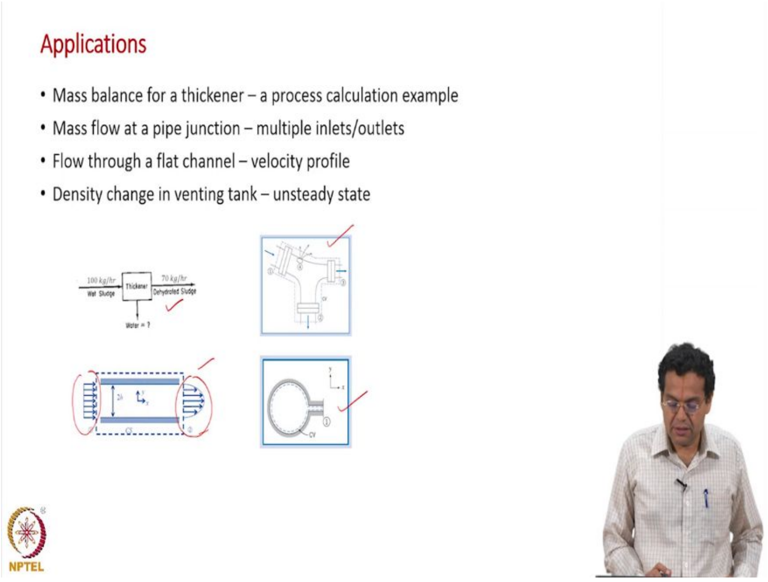


Let us discuss few Applications of the Integral Mass Balance Equation. In each application, we will highlight one aspect.

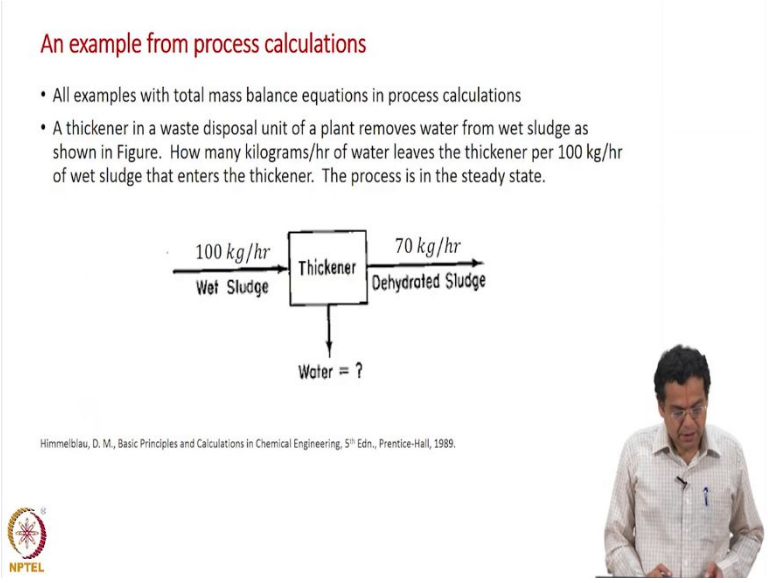
- The first application is a mass balance for a thickener it is a typical process calculation example. The whole idea is to show that what you have done earlier is use the equation which I derived now so, that you do not think they are two different things.

- The second example is a mass flow at a pipe junction case where you have multiple inlets, multiple outlets.

- The third example is flow through a flat channel, our well known example. Here the aspect to which we pay attention is that the velocity is uniform at the inlet, but the velocity is not uniform at the outlet. So, how do we take that into account and all these three cases, are for steady state operating condition.

- The fourth example will be considering the transient as well, we have a tank that went out how do the density varies as a function of time.

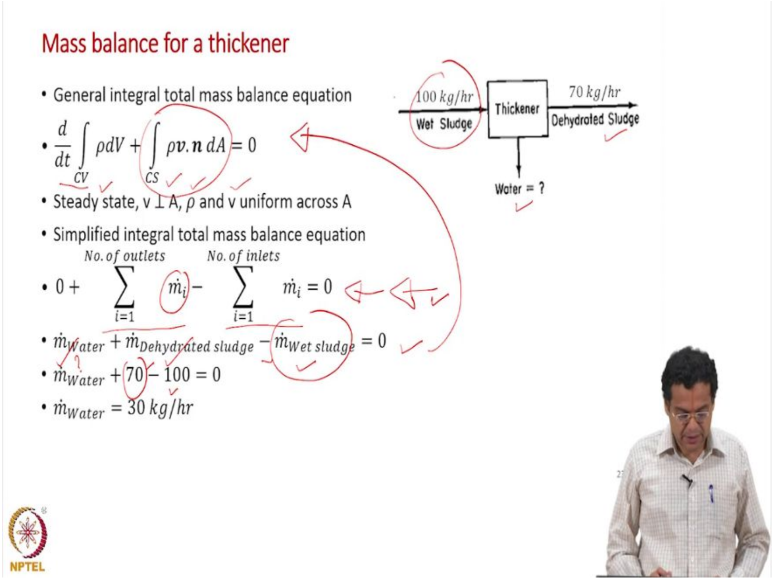
Example: (Refer Slide Time: 01:38)



So, first, take an example from process calculation, once again light emphasis that all the examples in your process calculation course where you would have done total mass balance fall under this category. You would have come across this term integral balance or integral mass balance in a process calculation course, now you would really understand what was really meant by the term integral balance in the process calculation course. Probably that time you would have not paid attention, but now you know what is really the meaning of that, what is a full pledged equation after many simplifications how does it get to the equations of process calculation book etcetera. We took a very specific example here,

A thickener in a waste disposal unit of a plant removes water from wet sludge as shown in figure (above slide image). So, in this figure you see 100 kg/hr of wet sludge entering, some amount of water is removed per hour, the dehydrated sludge flow rate is given which is 70 kg/hr. So, how many kilograms per hour of water leaves a thickener per 100 kg/hr of wet sludge that enters the thickener, the process is this steady state. As we discussed sometime back, all the values are given in terms of mass flow rates directly, not in terms of velocity densities etcetera.

Solution: (Refer Slide Time: 03:07)



So, let us write the general integral total mass balance equation which you have derived, to begin with,

$$\frac{d}{dt}\int_{CV} \rho\, dV + \int_{CS} \rho\, v.n\, dA = 0$$

The rate of change of mass in the control volume plus the net rate of mass leaving the control volume through the control surface equal to 0. And then we simplified it, we made a lot of assumptions relevant to the problem at hand.

- First is steady state and then
- We assumed that the velocity is perpendicular to area and then
- Density and velocity uniform across area.

Then we could write in terms of the mass flow rates,

$$\sum_{i=1}^{No.\ of\ outlets} \dot{m}_i - \sum_{i=1}^{No.\ of\ inlets} \dot{m}_i = 0$$

So, let us apply this equation for this example, we have two outlets streams, one through which water leaves others through which the dehydrated sludge leaves. So,

$$\dot{m}_{water} + \dot{m}_{Dehydrated\ sludge} - \dot{m}_{wet\ sludge} = 0$$

Now, let us substitute the known values

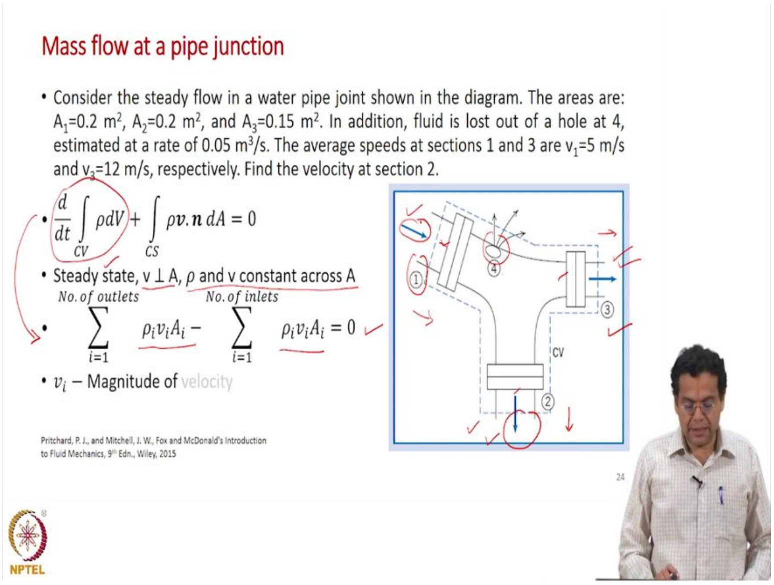
$$\dot{m}_{water} + 70 - 100 = 0$$

So,

$$\dot{m}_{water} = 30\ kg/hr$$

This problem is very simple, the idea of showing this example is once again I repeat to relate the equation which you would have started off in the process calculation course. This would have been your starting point in your process calculation course or straight away you would have written this equation. We are just showing that this equation is a very simplified version of the general integral total mass balance equation, of course, 30 kg per hour of water leaves the thickener.

Example: (Refer Slide Time: 05:32)



Let us move on to the next example which is mass flow at a pipe junction. We consider the steady flow in a water pipe joint shown in the diagram (above slide image), you have a pipe junction with one inlet and two outlets and the cross sectional areas are given, A_1 = 0.2 m^2, A_2 = 0.2 m^2 and A_3 = 0.15 m^2 and then you have a let us say a leak or a small hole through which water is lost.

In addition, fluid is lost out of a hole at location 4 (shown in the image), estimated at a rate of 0.05 m^3/s because it is a leak it is a small hole, it may not be regular also. So, usually, you know the volumetric flow rate through the hole that is why the volumetric flow rate is given, you are not given velocity through that hole or area of that hole. If it is a well defined inlet or outlet etcetera then you know the area different types of velocity. It is a small hole and the small leak over it, then you do not specify in terms of area and velocity just in terms of whatever is leaking out. In terms of let us say volumetric flow rate then you are given the average speeds at sections 1 and 3. So, the section 1, the velocity is $V_1 = 5$ m/s and section 3, a high velocity of $V_3 = 12$ m/s. It says average speed because we are not considering variation across the cross sectional area. So, we just give an average value that is why it says the average speed at sections 1 and 3 and we are asked to find out what is the velocity at section 2.

Solution:

To begin with, we say that water enters at section 1 and leaves at section 2 and section 3. We will have to verify based on the result what we get let us see what happens. Now let us start with the general form of the integral mass balance equation, we will make set of assumptions steady state and velocity is perpendicular to the area.

$$\frac{d}{dt}\int_{CV} \rho\, dV + \int_{CS} \rho\, v.n\, dA = 0$$

So, the assumptions are

- Across all the phases let us say a velocity and an area perpendicular to each other, similarly at the other inlets and outlets as well, and then
- The density and velocity are constant across the cross sectional area which we have done several times.

So, if you make these assumptions, these integral balance equation get simplified as

$$\sum_{i=1}^{No.\ of\ outlets} \rho_i v_i A_i - \sum_{i=1}^{No.\ of\ inlets} \rho_i A_i = 0$$

Now, unlike the previous examples, where this equation was written in terms of mass flow rates, the same equation now as may be written in terms of densities and velocities and areas. As I told you that is more a practical way of representing the conservation equation because those are the measurable values measured values and this is a good example where the specifications are as close to as possible to the reality meaning you measure velocity measure area etcetera.

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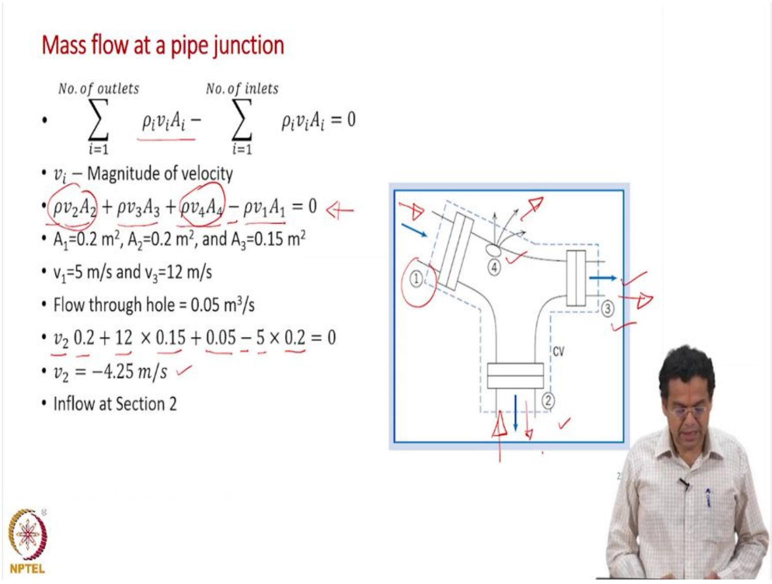


So, let us write down the simplified equation for this pipe junction in terms of outlets, we have considered section 2 to be an outlet and section 3 to be an outlet and 4 is also an outlet and there is only one inlet which is section 1. So, let us write down that because it is water I kept the density to be constant.

$$\rho\, v_2 A_2 + \rho\, v_3 A_3 + \rho\, v_4 A_4 - \rho\, v_1\, A_1 = 0$$

So, all terms represent the mass flow rate either leaving or entering respectively. Now, we will substitute the values, we are given the values of A_1 = 0.2 m^2, A_2 =0.2 m^2, A_3 =0.15 m^2, also, we are given the velocities v_1 =5 m/s, v_3 =12 m/s but v_2 has to be found out and then we are not given v_4, A_4 etcetera. We are given the product v_4 A_4 = 0.05 m^3/s as we have mentioned a few minutes back, we are given the volumetric flow rate, not mass flow rate.

$$v_2 A_2 + v_3 A_3 + v_4 A_4 - v_1\, A_1 = 0$$

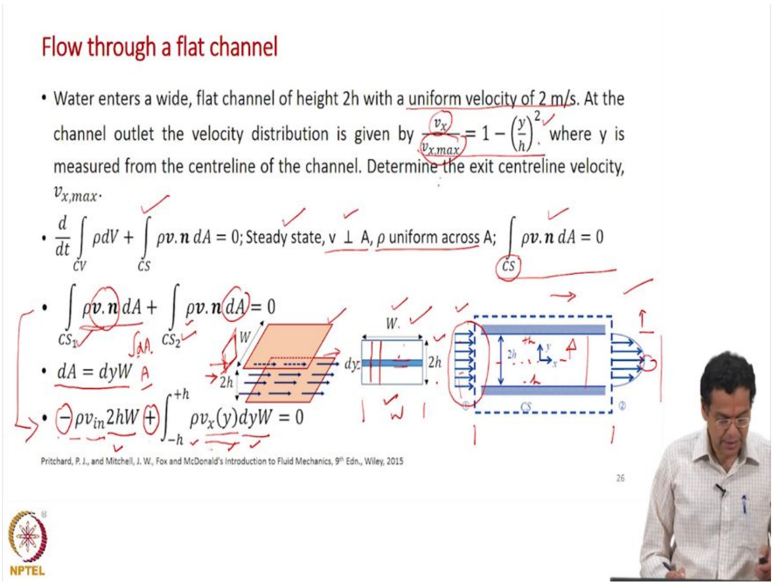
$$v_2\ x\ 0.2 + 12\ x\ 0.15 + 0.05 - 5\ x\ 0.2 = 0$$

$$v_2 =- 4.25\ m/s$$

As I told you to begin with we assumed that water is leaving through this phase and so, we included that in the outlet as a flow rate leaving the control volume, but now we found out that v_2 is negative which means that water is entering through the inlet 1, water is also entering through this surface 2 and leaving through this outlet surface 3 and of course, leaving through outlet hole 4, that is why the velocity v_3 is much higher.

So, entering at through 1 inlet at 5 m/s through the other inlet at 4.25 m/s leaving at 12 m/s and their cross section area is also less so, resulting in a higher velocity. So, this is a good example where if you have we work in terms of velocities areas and then apply the integral mass balance equation.

Example: (Refer Slide Time: 12:58)



This example is to illustrate the use of velocity profile, we have so far assumed that the velocity is uniform across the cross sectional area of the inlet or the outlet. Suppose, if the velocity varies across the inlet or outlet how do we take that into account, remember the equation that we have derived can take this into consideration let us read the example, water enters a wide flat channel of height 2h with a uniform velocity of 2 m/s.

This geometry is very well known to us in which is shown here (above slide image) and the inlet velocity is 2 m/s remember understand the word uniform velocity. What does uniform velocity mean? The velocity does not vary over the height of the channel at the inlet. At the channel outlet, the velocity distribution meaning velocity varies along the y-direction, along with the height of the channel it varies. And, we are given the velocity profile it says v_x because velocity in the x-direction, and that is the function of y.

$$\frac{v_x}{v_{x,max}} = 1 - \left(\frac{y}{h}\right)^2$$

Velocity in the x-direction varies in the y-direction and $v_{x,}$ is the maximum velocity which is supposed to find out where y is measured from the centerline of the channel. So, y is measured so, for example, if 2h is the height of the channel, y is + h and then − h that is why we are given the position of the coordinate axis, y starts from the centerline axis. Now, we asked to find out the, determine the exit centerline velocity which is maximum at the centerline. So, we are asked to find out what is the exit centerline velocity $v_{x,}$ why is it centerline, when you substitute y = 0 in this equation, you get $v_x = v_{x,}$ that is why centerline velocity.

$$\frac{d}{dt}\int_{CV} \rho\, dV + \int_{CS} \rho\, v.n\, dA = 0$$

Now, let us start with the most general form of the integral balance equation, we will make some assumption, but not all of them

- Steady state
- Velocity is perpendicular to the area through the velocity either the inlet or the outlet. In the inlet, the velocity is uniform across the area, but still perpendicular to the area, in the outlet it varies across the channel height, but still, it is perpendicular. So, at the inlet, it is uniform throughout it is perpendicular at the outlet it varies, but still, it is perpendicular to the area.
- So, this assumption is valid and now in the earlier examples, we took the density and velocity to be uniform across the area. In this example, we are taking only the density to be uniform across the area, we are not taking as that the velocity is also uniform which is not valid at the exit.

Now, when you simplify the equation becomes

$$\int_{CS_1} \rho\, v.n\, dA + \int_{CS_2} \rho\, v.n\, dA = 0$$

So, now, let us split this term for the control surfaces 1 and 2, the inlet is denoted as control surface 1, outlet as control surface 2. So, what I have done here, I split this control surface as control surface 1 and control surface 2, and of course, right hand side is 0 wherever there is a wall velocity is 0 and does not contribute.

$$dA = dyW$$

$$-\rho v_{in} 2hW + \int_{-h}^{+h} \rho v_x(y)\ dyW = 0$$

Now, let us express this for the inlet, density is a constant, now $v.n$ remember for inlet it is negative. So, we have a negative sign in the first term and minus the magnitude of velocity which is v_{in} and then the area (2hW). Now, let us take the second term, the second term we have ρ which is written as such. Now, $v.n$ is positive because the flow is leaving the control volume and what is the velocity now, it is $v_x(y)$ earlier v_{in} was just a constant, but now v_x to be more explicit, we also mention $v_x\ (y)$. Now, what is the area we should take, this is the outflow and the width is still W, but now because velocity varies along the y-direction along with the height I cannot take the entire to which I take a small strip of height dy. So, the area of this strip disc dy W. So, that area is denoted as dA which is dy W. So, the area dA = d y W because the velocity varies along with the height of the channel along the y-direction. So, I cannot take the entire area W 2h, I only take a strip which is of height dy so that then I integrate to get cover the entire area.

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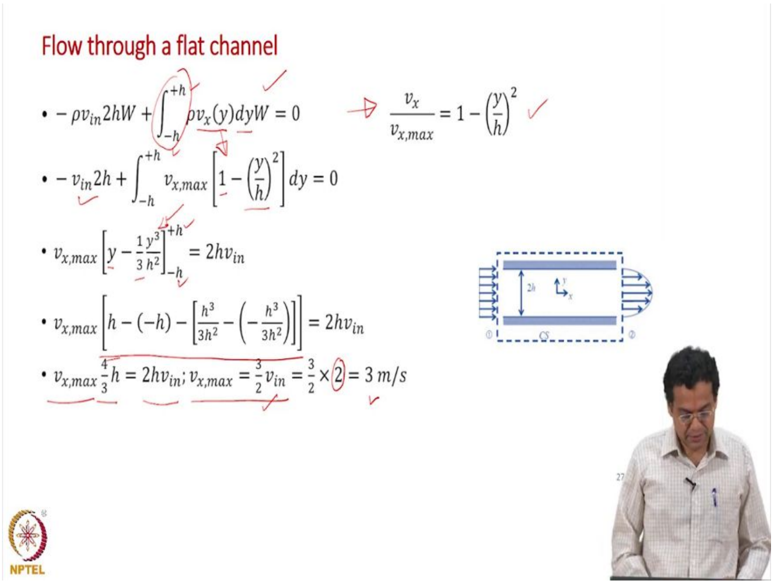


Rewriting the same equation here so,

$$-\rho v_{in} 2hW + \int_{-h}^{+h} \rho v_x(y)\ dyW = 0$$

Now, we are given the velocity as a function of y, W cancels out in the in this equation, and let us substitute for vx as a function of y from the given velocity distribution. So,

$$\frac{v_x}{v_{x,max}} = 1 - \left(\frac{y}{h}\right)^2$$

$$-v_{in} 2h + \int_{-h}^{+h} v_{x,max} \left[1 - \left(\frac{y}{h}\right)^2\right] dy = 0$$

This equation has been integrated and we get

$$v_{x,max} \left[y - \frac{1}{3}\left(\frac{y^3}{h^2}\right) \right]_{-h}^{+h} = 2hV_{in}$$

Now, coming to the limits first we set the limits was in terms of control surface 2, now the independent variable is y. So, the limits are written in terms of y and the y value varies from − h to +h because y = 0 represents a centerline and we substitute those limits here.

Now, substitute the limits and then

$$v_{x,max} \left[h - (-h) - \left[\frac{h^3}{3h^2} - \left(-\frac{h^3}{3h^2}\right)\right]\right] = 2hv_{in}$$

$$v_{x,max} \left(\frac{4}{3}\ h\right) = 2hv_{in}$$

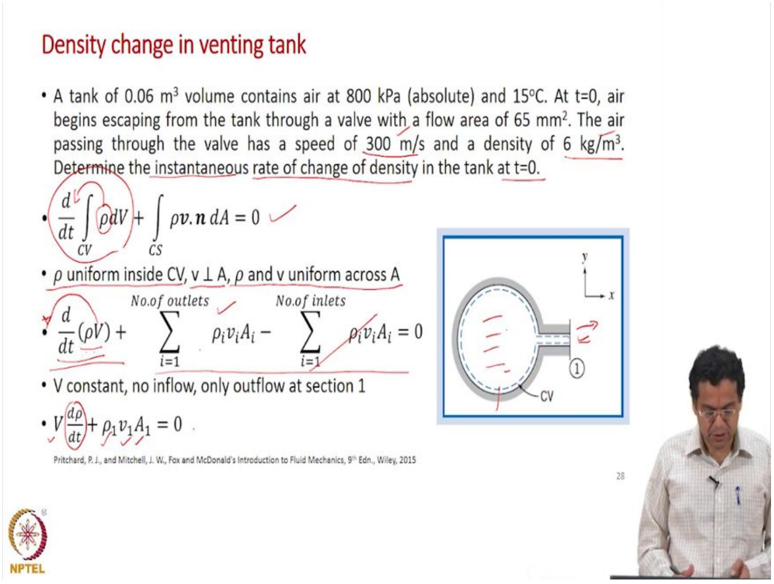
$$v_{x,max} = \frac{3}{2} v_{in}$$

The maximum velocity is 1.5 times to the inlet velocity. So, inlet velocity is given as $v_{in} = 2$ m/s, you can find out what is the maximum velocity at the exit.

$$v_{x,max} = \frac{3}{2} x2 = 3\ m/s$$

So, a good example where at one inlet it was uniform velocity, and the outlet the velocity varied across the cross sectional area. By using this integral sign we have been so far telling that the integral sign accounts for the variation of velocity, density. So, a good example to demonstrate that as well.

Example: (Refer Slide Time: 23:14)



The last example where our attention is to consider a case where there is transient where is a unsteady state that is the focus of this example. Let us read the example, a tank of 0.06 m^3 volume contains air at 800 kilo Pascal absolute and 15 ^{0}C. So, given volume of the tank, you have given the pressure and temperature and at time t equal to 0, air begins escaping the tank through a valve, and with a flow area you have given small area 65 mm^2.

The air passing through this valve at a speed you are given the speed so, varying at speed of 300 m/s because high pressure and it is a small area as well and the density of air leaving is 6 kg/m^3. Determine the instantaneous rate of change of density in the tank. The rate of change of density tells you that the rate keeps changing because as air escapes the pressure inside falls and the velocity can change the density can change. So, whatever you are finding out is the instantaneous rate of change of density at time t = 0, the moment you open the valve how to do the density change with time that is why it says the instantaneous rate of change of density in the tank moment say instantaneous you mention a time and that time is t = 0, moment you open the valve what happens.

Solution:

$$\frac{d}{dt}\int_{CV} \rho \, dV + \int_{CS} \rho \, v.n \, dA = 0$$

Now, let us start once again with the most general form of the integral mass balance equation, for all the earlier examples we have assumed steady state but we cannot do that here because it is a transient case so that assumption is not listed, all other assumptions are listed.

- Density is uniform inside CV
- Velocity is perpendicular to the area and,
- Velocity is uniform across cross section area

Now, because we are going to consider the accumulation term that a transient term, we need to make another assumption that the density is uniform inside the control volume no longer we so far we need not mention that because that term was anywhere 0, but now that term is going to contribute. So, now we should say whether density is varying within the control volume or it is a constant within the control volume that will enable us to take out whether should we keep it inside or can I take it outside. As usual, to simplify the analysis, we will say density is uniform inside the control volume.

So, these assumptions are made depending on what aspect we were trying to focus on a particular problem. Now,

$$\frac{d}{dt}(\rho V) + \sum_{i=1}^{No.\ of\ outlets} \rho_i v_i A_i - \sum_{i=1}^{No.\ of\ inlets} \rho_i A_i = 0$$

$$V\frac{d\rho}{dt} + \rho_1 v_1 A_1 = 0$$

So, I can take the volume to be constant so that I can take this out of the derivative, and then there is no inflow at all; obviously, there is only outflow. So, the last term will not contribute there is no inflow, only outflow will contribute and there is only one outflow.

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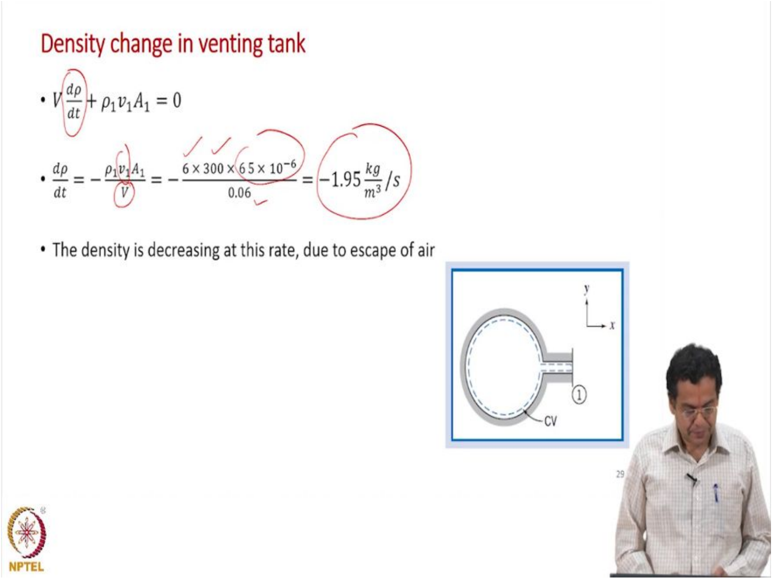

So, let us simplify this equation, what is required is an instantaneous rate of change of density.

$$V\frac{d\rho}{dt} + \rho_1 v_1 A_1 = 0$$

$$\frac{d\rho}{dt} = -\frac{\rho_1 v_1 A_1}{V}$$

Let us substitute all the values $\rho_1 = 6$ kg/m^3, $v_1 = 300$ m/s and $A_1 = 65$ mm^2 represented in terms of SI units and the volume = 0.06 m^3.

$$\frac{d\rho}{dt} = -\frac{6 \, x \, 300 \, x \, 65 \, x \, 10^{-6}}{0.06} = -1.95\frac{kg}{m^3}/s$$

So, this gives you the rate at which density keeps changing of course, obliviously it should decrease because air escapes and how does it decrease roughly at the rate of 2 $\frac{kg}{m^3}/s$. So, the density is decreasing at this rate due to the escape of air. Once again a very simple example to demonstrate the unsteady state term all others are something known to us from previous examples.

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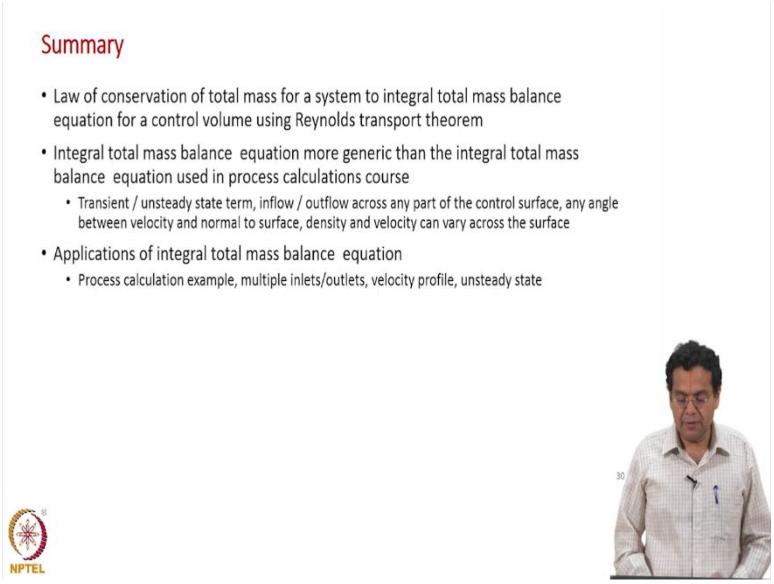

To summarize the conservation of mass part, the total mass balance part the integral mass balance alone. So, the law of conservation of total mass for a system is started from that and express that in terms of the integral total mass balance equation for a control volume using Reynolds transport theorem with that objective is derived the Reynolds transport theorem.

We have seen the first application of Reynolds transport theorem, conservation of mass for a system the law of physics has been expressed in terms of a governing equation as a conservation equation, but in integral form for a control volume, we use Reynolds transport theorem for that. We will see several applications of that for other properties as well as momentum, energy, and then spacious mass.

Now, the integral total mass balance equation that you have derived is more generic than the integral total mass balance equation used in the process calculation course. Transient, unsteady state term, inflow, outflow across any part of the control surface, then any angle between velocity and normal to the surface and of course, density, velocity can vary across the surface. And, looked at the application of this integral total mass balance equation, process calculation example, an example with multiple inlets and outlets, example considering velocity profile and example where we include the unsteady state term.