Continuum Mechanics And Transport Phenomena
Prof. T. Renganathan
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture - 19
Reynolds Transport Theorem:
General Form - Part 1

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We are discussing the Reynolds Transport Theorem and we discussed and defined the system and control volume. The system is made up of specific fluid particles, control volume, different fluid particles pass through and we looked at the need for the Reynolds transport theorem, to extend the laws of physics for the conservation equations.

We derived the simplified form of the Reynolds transport theorem with few assumptions and now we are going to derive a more general form of the Reynolds transport theorem.

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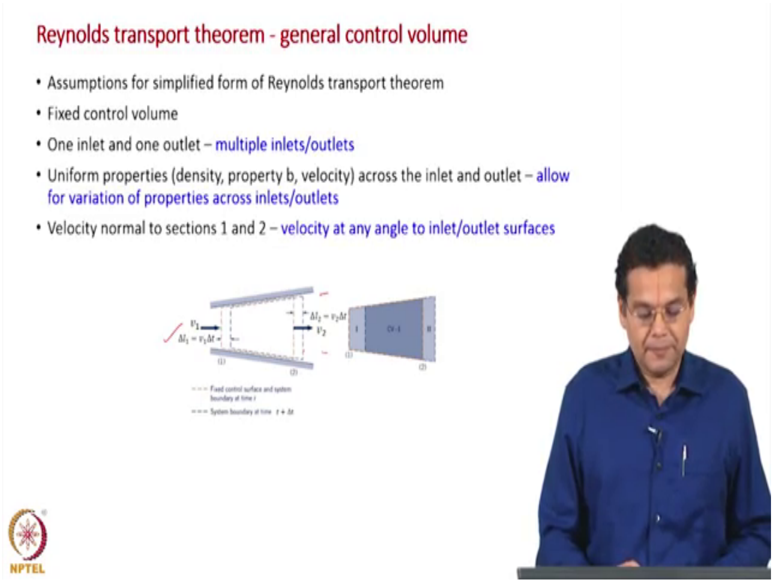

To understand in what way it is general, first let us look at the assumptions, which were used to derive the simplified form of the Reynolds transport theorem.

- Fixed control volume, we are going to continue with that and then
- one inlet and one outlet; we are going to extend to multiple inlets multiple outlets not alone that, we are going to extend to, the case where any part of the surface can be inlet; any part of the surface can be an outlet.
- The properties to be uniform across the cross section, of the inlet and outlet. Now, the properties being density, the property v itself the velocity. Now we are going to allow for variation of properties across inlets and outlets.
- Then we took the velocity, normal to the inlet and outlet phases, and now we are going to consider velocity at any angle to inlet and outlet surfaces.

So, we are going to extend to multiple inlet outlets, allow of a variation of properties across the inlet outlets and the velocity can be at any angle to inlet outlet surfaces.

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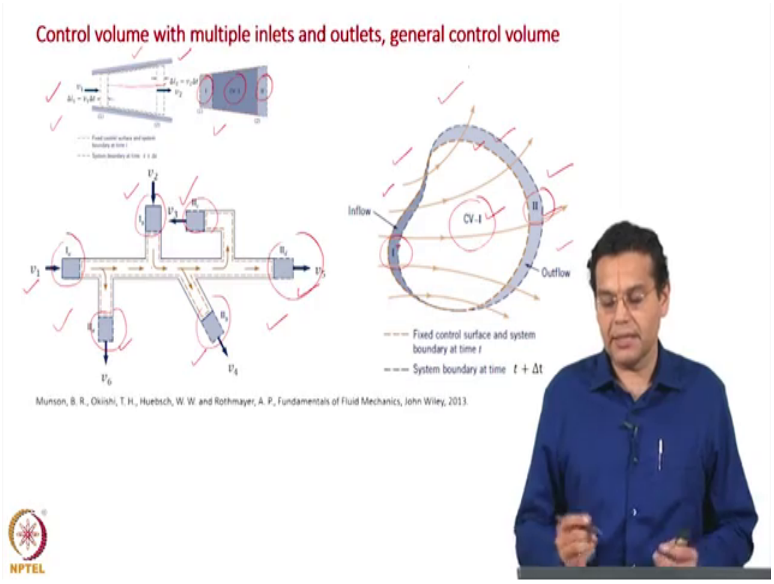

Let see how do we extend to a control volume with multiple inlets and outlets and this is the old geometry which I have seen (upper right hand side image). Now, what is shown here (down right hand side image) is the network of pipes to represent a control volume with multiple inlets multiple outlets. You have inflow and then there is an outflow, and then there is another inflow and this combines with this fluid, and then it leaves through the different outlets. So, you have a network of pipes with 2 inlets and then 4 outlets. So, in general, you can have multiple inlets and then multiple outlets.

Why do we extend multiple inlets and outlets? Because of the usual control volume, usual equipment can have multiple inlets, multiple outlets. What is shown here is a physical model with multiple inlets and multiple outlets. Let us say you have flow entering here, flow entering through these 3 inlets, and then flow goes out through these outlets (object demonstrated in the lecture video). So, as a physical model, so that, we can easily understand what you mean by control volume with multiple inlets and then multiple outlets.

Now, we are going to make it a little more general by considering any arbitrary shape with some portion to be an inlet and then some portion to be outflow (left hand side image). How do we imagine? We combine all the surfaces through which there is inflow into an inflow surface, and then all the surfaces through which there is outflow into an outflow surface. So, our control volume is made equivalent to control volume where part of the surface is inflow, part of the surface is an outflow. So, that it is very general it not be a particular geometric shape, it could be any surface where part of the surface inlet, part of the surface is an outlet.

The control volume is drawn to an arbitrary shape. So, that the equation which we are going to derive is applicable for any arbitrary shape. The brown dashed line represents the fixed control surface. And then as we have seen earlier, it also represents the system and both of them are coinciding at time t.

So, just, in this case, the control volume and system had a simple, easily understandable geometry. In this present case, there is some arbitrary shape, but conceptually they are the same meaning that you have a fixed control surface, you have a system and both coincide at time t. Now just like we had the system moved, because of the fluid flow through the control volume. Here again, this blue dashed line represents the surface of the system which has moved through the control volume. And just like here, we have region 1, through which there was flow into the control volume; and then region 2 which represents the volume of fluid which as cross the control surface and left the control volume and as we have seen earlier we have control volume minus 1 ($CV-1$).

So, we have analogous descriptions between region 1, region 2, and the $CV-1$ for all three geometries. Conceptually they are the same, in terms of physical meaning they are the same, only in terms of geometry they are different. Now, well defined geometry in this case some arbitrary shape, an arbitrary configuration in this case. To better visualize taken a duster purposefully, which is easily available in a class; and which has some kind of arbitrary shape here and then of course, these are well defined shapes here.

So, part of the surface is an inlet, part of the surface is an outlet and that is what we have taken; and as the system moves through the control volume you have region 1, region 2, and then control volume minus 1 as the remaining region. I discuss the analogy between these two geometries. So, that most of the steps are common to both the derivations, except a few terms and hence I discussed the analogy between these two control volumes.

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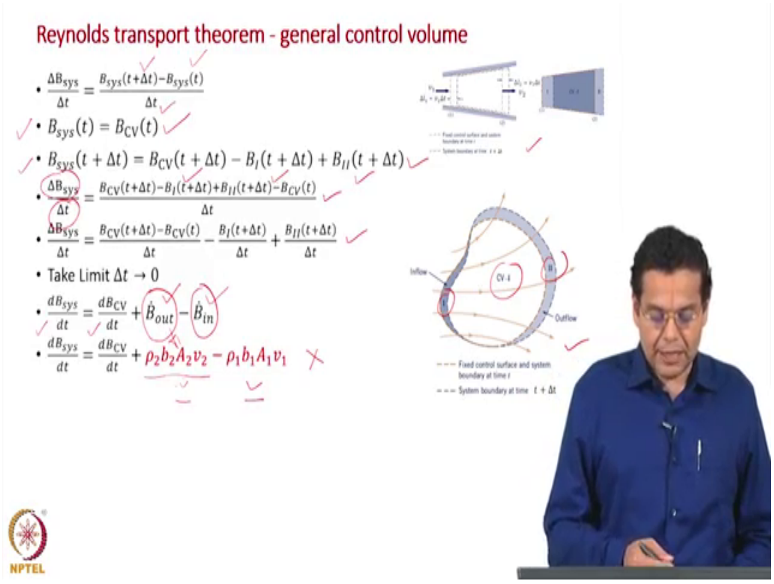

Now, what are the steps which are common to both the control volumes? The step which tells that, the rate of change of property of a system in terms of

$$\frac{\Delta B_{sys}}{\Delta t} = \frac{B_{sys}(t+\Delta t)-B_{sys}(t)}{\Delta t}$$

this step is common to both the control volumes.

Now, here again, we have taken, the system and control volume to be coinciding. So,

$$B_{sys}(t) = B_{CV}(t)$$

This step is also valid. Now as in the earlier case, here also the system moved partly out of the control volume and we have corresponding regions 1, 2 and then control volume minus 1. So,

$$B_{sys}(t + \Delta t) = B_{CV}(t + \Delta t) - B_1(t + \Delta t) + B_2(t + \Delta t)$$

This expression which relates B system at $t + \Delta t$ to B control volume at $t + \Delta t$, B region 1 $t + \Delta t$ and B region 2 $t + \Delta t$, is also valid.

$$\frac{\Delta B_{sys}}{\Delta t} = \frac{B_{CV}(t + \Delta t) - B_1(t + \Delta t) + B_2(t + \Delta t) - B_{CV}(t)}{\Delta t}$$

Now, the difference in the property of the system is divided by Δt, where we substitute for B system $t + \Delta t$ expression, and for B system at t this expression is also valid.

$$\frac{\Delta B_{sys}}{\Delta t} = \frac{B_{CV}(t+\Delta t) - B_{CV}(t)}{\Delta t} - \frac{B_1(t+\Delta t)}{\Delta t} + \frac{B_2(t+\Delta t)}{\Delta t}$$

The next step is the rearrangement, of course, it's going to be valid and later we took a limit of this equation delta $t \to 0$ that is also valid,

$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \dot{B}_{out} - \dot{B}_{in}$$

The left hand side becomes the rate of change of property for the system, and right hand side becomes the rate of change of property for the control volume.

What is the significance of $\dot{B}_{out}$ is the rate at which the property leaves the control volume and $\dot{B}_{in}$ tells the rate at which property enters the control volume. Till this everything is the same as the derivation which I have done for the simple control volume, but now we have extended for arbitrary control volume with the corresponding meaning of 1, 2 and then control volume and system.

$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \rho_2 b_2 A_2 v_2 - \rho_1 b_1 A_1 v_1$$

Now, this last step, where we found an expression for $\dot{B}_{out}$ in terms of these properties namely density, the property per unit mass, the area, the velocity both for outflow and inflow is restricted to the geometry which I have considered or to the control volume which I have considered. So, only this expression, even in this expression only these two terms $\dot{B}_{out}$ *and* $\dot{B}_{in}$ are not applicable for any general control volume. What is it going to do now? Express $\dot{B}_{out}$ *and* $\dot{B}_{in}$ for any general control volume and that is what I am going to do now. Now we will extend the expressions here, so that is applicable for any general control volume.

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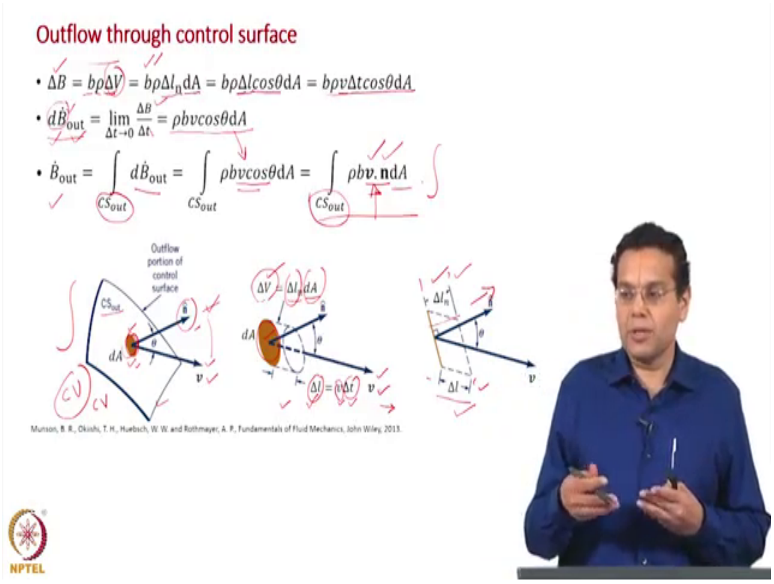

Now, let us consider the outflow for the control surface which easier to discuss first. What is shown here is the outflow control surface denoted by CS_{out}; what do you mean by that? The control volume is to the left of this. So, you have the control volume and this is the control surface and flow takes place out of the control volume through this control surface. That is important because we identify the direction of an area with a normal and that normal is always outward drawn normal.

So, the control volume, LHS is inside and RHS is outside. So, what I have taken here (upper right side image in slide) is a small area; small area dA and then the direction of the area is indicated by the normal to the area and the normal is drawn outward or in short, we say outward normal, why is it outward normal? The control volume is on this side and this is the outside to the control volume. Now, what is shown here (middle image in slide) is also a velocity vector; which represents the velocity of fluid flow through this small area. And now unlike the earlier case; earlier case the velocity vector was along the normal to the surface.

Now, we have allowed for variation in the angle between the velocity vector and the normal to the face. We said, we are going to extend to the case where the velocity vector can make any angle with the normal to the plane and that is why the normal is in one direction. They have a plane here and then the normal and that normal and the velocity through the face are making angle θ between each other.

Now, why did we consider a small area dA? We said we want to extend where the properties change across the surface area. So, that is why, I take a small area, so that, I can integrate over the control surface so that I allow for variation across the surface. That is why, we take a small area dA and we will find out what is the rate of flow of property through the small area, then integrate for the entire control surface. So, we allow for variation across the control surface.

Let us do that quantitatively in terms of expressions.

$$\Delta B = b\rho\Delta V = b\rho\Delta l_n dA = b\rho\Delta l\, cos\theta\, dA = b\rho v\Delta t\, cos\theta\, dA$$

ΔB represents the amount of property which has flown out of the control volume through the control surface. Now, just like in the earlier case, we will express that in terms of ΔV, where ΔV is the volume of liquid which has flown out of the control volume, you multiply by density, multiply by the property value per unit mass. So, these three terms $b\rho\Delta V$ put together gives you the amount of property associated with the volume which has left the control volume.

Now, how do we express this ΔV, the volume of liquid which has left the, which is leaving the control volume. And that is what is shown in this figure, the area dA is shown here and then this volume ΔV is equal to the length $\Delta l_n dA$. The Δl_n is the length along the normal to the surface; we have a surface a small area, we have a normal to that, the volume is that area multiply by the length perpendicular to that area. And that is what gives us $\Delta V = \Delta l_n dA$; what has to be kept in mind is that the Δl_n is perpendicular to the area.

Now, what is that we know, what is the length we know? We know the velocity of the fluid. So, what we know is this distance Δl_n which represents, the distance traveled by velocity over a time interval Δt. To repeat we are given the velocity vector v and what is the length we know, this length Δl; and that length represents, the distance travel by the fluid along the direction of velocity. So, we know this Δl which is given by the velocity into the time interval Δt.

What we want is the length along the normal; what we know is the length along the velocity vector. Which means that to find out this length, I take a projection of this Δl along the direction of the normal; which is nothing, but this length multiplies by the cos of the angle between them, which is $cos(\theta)$. And that is what is shown here, we have b as such rho as such ΔV has been represented in terms of dA the small area multiply by the length perpendicular to the area. That length perpendicular to the area has been represented in terms of the length Δl, which represents the distance travel by the fluid in a time Δt multiplied by the cosine of the angle between the two directions. So, I take the projection.

Now, this Δl we said is the distance travel with the fluid in time Δt. So, multiply the velocity into Δt. So, finally, the expression $b\rho v\Delta t\, cos\theta\, dA$ represents the amount of property that has left the control volume through the outflow control surface. Now, what do we do, as we have done earlier we will take the limit $\Delta t \rightarrow 0$,

$$d\dot{B}_{out} = \frac{\Delta B}{\Delta t} = \rho b v\, cos\theta\, dA$$

This represents the rate at which the property leaves the control volume through the outflow control surface. But what does this rate represent, whatever rate living through this small area that is why it is denoted as d$\dot{B}_{out}$; dB represents that this rate is for a small area and the dot represents it is the rate and out represents that is it is leaving the control volume.

We want this rate for the entire control surface. So, we integrate this d$\dot{B}_{out}$ over the control surface through which there is an outflow.

$$\dot{B}_{out} = \int_{CS_{out}} d\dot{B}_{out} = \int_{CS_{out}} \rho b v\, cos\theta\, dA = \int_{CS_{out}} \rho b\, v.n\, dA$$

So, d$\dot{B}_{out}$ represents rated which property leaves through the outflow control surface. We know $v\, cos\theta$ can be represented in terms of the dot product between the v vector and the n vector. So, v is the velocity vector, n represents the direction of the normal. So, $v.n$ gives you $v\, cos\theta$, n is the unit normal vector. So, $v.n$ becomes $v\, cos\theta$.

So, this integral expression represents the rate at which the property leaves the control volume through the outflow control surface. Now we have taken two things into account by this integral expression; one, the angle between the velocity vector and the normal vector. Secondly, by taking a small area and then integrating it, we allowed for various in the properties namely density, b, and velocity within the area those were objectives. And by considering a very general shape of the control volume, we are allowed for multiple inlets multiple outlets that is where we started off; and not even multiple inlets multiple outlets, any part of the surface can be inlet or outlet. Now, let us repeat this analogously for the inflow control surface.

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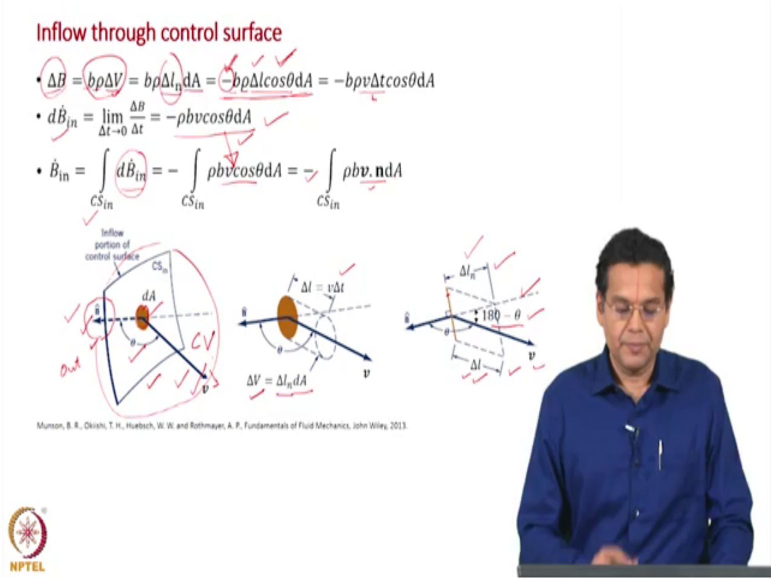

Now, what is shown here is the inflow control surface, what does it mean, the control volume is on an outward region or outside the region for this control volume. And that is why as we have discussed earlier, the normal is always drawn as an outward normal which means it should point away from the control volume. The control volume is in the right and that is why this is the inflow surface through which there is fluid inflow entering the control volume. And n always should be drawn, so convention that, you always draw n as an outward normal and that is why it is pointing outwards.

Now, as we have done in the earlier case, we have considered a small area dA and the direction of that normal to the phase is n vector, the direction of the velocity through this small area is v vector and θ as an angle and the earlier case between the normal to the phase and the v vector; normal to the area dA and to the v vector.

Let us see how do we proceed in this particular case, it is almost analogous which a small difference.

$$\Delta B = b\rho\Delta V = b\rho\Delta l_n dA = -b\rho\Delta l\, cos\theta\, dA = -b\rho v\Delta t\, cos\theta\, dA$$

So now, ΔB represents the amount of property which is entering the control volume. We represent as, that into the ΔV which is the volume of liquid which enters the control volume through the inflow control surface multiply by density gives the mass of fluid multiply with a property per unit mass. So, this will give you the amount of property which associated with the volume which has entered the control volume through the inflow control surface.

Now, as we have done earlier, we will represent this volume in terms of the length normal to the area Δl_n into dA and that length is shown here is Δl_n. Remember that length Δl_n is perpendicular to the area. So, we represent the volume in terms of the length perpendicular to the area and then multiply by the small area dA and that volume is what is given.

Now, what is known to us, as in the earlier case, is known is this Δl which is the length along the velocity vector and that represents the distance traveled by the fluid in a time interval Δt. So, we know the Δl along the velocity vector, what we want is the Δl_n normal to the area. So, we take a projection of Δl along this direction and these are related by the cos of the angle between them. But now the difference between the earlier case and the present case is that; in the earlier case, the angle was θnow the angle between them is $180 - \theta$.

So, if you take a cosine of that you get minus cos(θ) and that is what is shown here I have $b\rho$ as in the earlier expression this Δl_n is represented as $\Delta l\, cos\theta$ with the negative sign. Because the angle between these two lengths is $180 - \theta$ and not θ as in the earlier case, that results in a minus sign. But keep in mind, in this case, because n vector and velocity vector are in a different direction, the angle between them, θ still represents the angle between the velocity and the normal vector; that angle is going to be more than 90. So, this will be negative, the expression final value will be positive; because we are taken this as an amount that cannot be negative. So, the final value will be a positive value, because this negative sign and cosθ is negative they make it as a positive value.

We will discuss more on that in the next to the next slide and let us go ahead. Now the Δl is the distance travel by fluid along the direction of velocity, we express that in terms of the velocity into Δt. So,

$$d\dot{B}_{in} = \frac{\Delta B}{\Delta t} = - \rho b v\, cos\theta\, dA$$

As in the earlier case we divide this expression by Δt and then take limit $\Delta t \rightarrow 0$; and what we get is the rate at which the property enters the control volume over the small area dA as were in the previous case.

$$\dot{B}_{in} = \int_{CS_{in}} d\dot{B}_{in} = -\int_{CS_{in}} \rho b v \cos\theta \, dA = -\int_{CS_{in}} \rho b \, v.n \, dA$$

Now, as we have done for the outflow surface, we will integrate this expression over the entire inflow surface; So, we will integrate over the entire area of the inflow control surface. What I have done is, substituted for $d\dot{B}_{in}$ from earlier expression and $v\cos\theta$ is expressed as earlier in terms of $v.n$ remember. Even in this case are always θ is the angle between the v vector and the n vector. So, remember we have still a minus sign and $v.n$ will be negative eventually $\dot{B}_{in}$ is positive.

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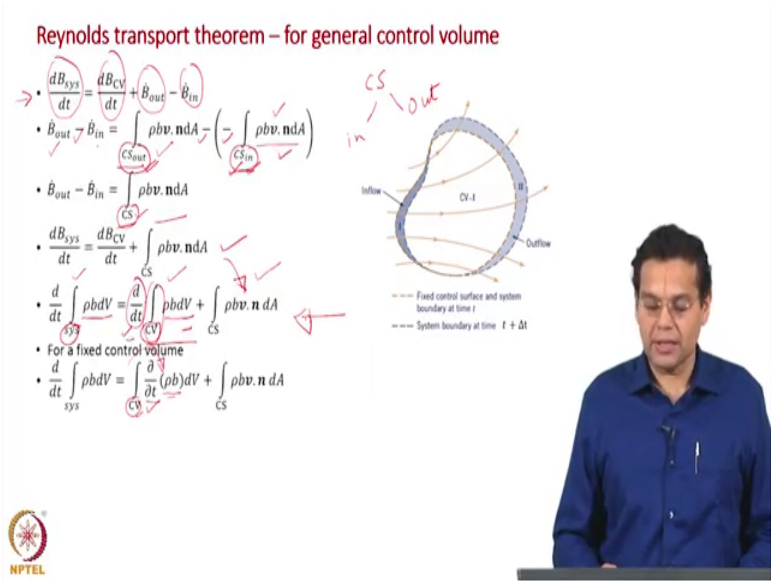

Now, let us put them all together,

$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \dot{B}_{out} - \dot{B}_{in}$$

We have written this expression for the Reynolds transport theorem which relates the rate of change of property for system, rate of change of property for control volume, and what we discussed was this $\dot{B}_{out}$ and $\dot{B}_{in}$ cannot be represented in terms of simplified expressions.

$$\dot{B}_{out} - \dot{B}_{in} = \int_{CS_{out}} \rho b\, v.n\, dA - \left(- \int_{CS_{in}} \rho b\, v.n\, dA \right)$$

Now we have got more general expressions for them and let us see how do, we simplify that.

Now, two minuses becomes plus and then the entire control surface is split into partly inflow and then partly outflow.

$$\dot{B}_{out} - \dot{B}_{in} = \int_{CS} \rho b\, v.n\, dA$$

So, what do I do, I add this control surface out and control surface in into the entire control surface. Now the question arises, there can be part of the control surface where there is no inflow or outflow that would not contribute because the velocity will be 0. We will discuss this also in detail in the next slide.

But for the moment, we have summed up the control surface out, control surface in as the entire control surface. So, this is the control volume, all these put together the control surface. Let us say there are 2, 3 inflow control surfaces, and 3 outflow control surfaces, and some surfaces where there is no flow, at all the surfaces there is this integral will become 0. So, a summed up and represented as CS_{out} and CS_{in}, as the CS.

Let us substitute that in the earlier expression and then we get

$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \int_{CS} \rho b\, v.n\, dA$$

Let us substitute the integral form of B system and B control volume which you have seen earlier.

$$\frac{d}{dt}\left(\int_{sys} \rho\, b\, dV \right) = \frac{d}{dt}\left(\int_{CV} \rho\, b\, dV \right) + \int_{CS} \rho b\, v.n\, dA$$

So, this represents rate of change of property for system is equal to the rate of change of property for control volume and then last term represents the net rate of outflow of property through the control surface. We will see why is that, shortly. Now, for a fixed control volume, I can change the order of this differentiation and integration. Let me do that, but with a small change, when I integrate and then differentiate once you integrated volume all the spatial changes are taken care. So, after integrate, only time is the only independent variable.

$$\frac{d}{dt}\left(\int_{sys} \rho\, b\, dV\right) = \int_{CV} \frac{\partial}{\partial t}(\rho\, b)\, dV + \int_{CS} \rho b\, v.n\, dA$$

So, I will repeat again, this is a volume integral, once you are integrated over volume all the spatial variants are taken into account. So, after integrate it is a function of time only; and that is why we are used $\frac{d}{dt}$. Now how am I able to bring the $\frac{d}{dt}$ inside? Because the boundary of the control volume the control surface is not changing, so the volume you can bring in the $\frac{d}{dt}$ inside the volume integral. But when you do that now ρ b, both ρ and b are ρ b put together can be a function of both space and time; and hence this becomes a partial derivative.

You can bring in, because the boundary is fixed, but when you bring in, these can be functions of space; and hence the total derivative becomes a partial derivative, because their function of both space and time anyway. And this is the general form of the Reynolds transport theorem. We look at the significant shortly, but before that what we will see now is the, this particular term $\rho b\, v.n\, dA$ what does it represent, how does it take care of automatically inflow and outflow.