Continuum Mechanics And Transport Phenomena
Prof. T. Renganathan
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture - 18
Reynolds Transport Theorem: Simplified Form

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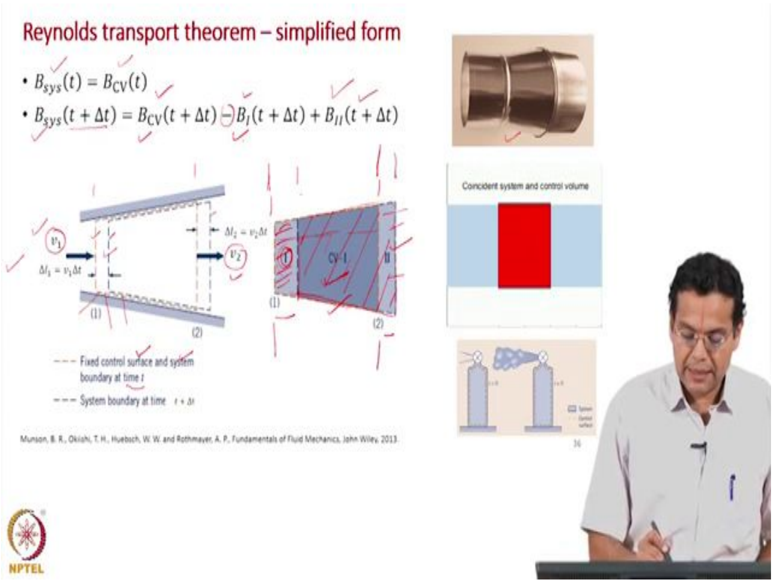

So, now we are ready to start deriving the Reynolds Transport Theorem, we have gathered all the required knowledge to start deriving the Reynolds transport theorem the simplified form of the Reynolds transport theorem. Now, I told you the reason why we are deriving simplified form first so that we understand that clearly and then extend to a more general form.

Now, we are going to consider the flow in an expanding section shown physically (in the above slide picture) in terms of a 3-dimensional view, shown schematically as well. So, we are going to consider flow in a diverging section and now, the dashed line represents the control surface at time t and always why control volume does not move. So, control volumes are always fixed at any time t the control volume remains fixed.

Now, what is my system as we have seen in the previous case? At time t, my system boundaries also the same as the control volume boundary, the control surface, surface of the system both coincide that is why this is a fixed control surface and system boundary at time t.

So, this line represents both my control volume and system at time t. Now, I have inflow through the left surface and outflow through the right surface.

Now, as the fluid is continuously flowing, we have identified whatever fluid in this region as my system next time $t + \Delta t$ sometime later system would have moved out of the control volume or trying to move out of the control volume. What is shown here is a coincident system and control volume (red colored picture) these are at time t the fluid is continuously flowing through the control volume. So, our system should always focus on the same fluid particles they also keep moving.

So, to begin with, we identified or a time t we identified system coincide with the control volume, but sometime later the system has moved. So, there is some inflow through this control surface and there is some outflow through this control surface and that is what is shown here in this diagram schematically. Now, this blue line represents the boundary of the system at some time later, to begin with, they were coinciding. Now, at some time later because the system has moved, this boundary and what is shown in this video which is now frozen are analogous (refer to lecture video for more clarification).

Now, the control surface always, of course, remains the same system has more. Now, what do we do? We will give some nomenclature for the different regions. Now, this is my control volume, shown as the dotted boundary. Now, after some time and of course, at the system to begin with at time t after some time let us say $t + \Delta t$ the system is moved. The boundary of the system, to begin with, was here (black dotted line) and now it has moved through the control volume and this is my boundary of the system at some time later (blue dotted line).

Now, given some nomenclature here region 1 corresponds to the volume of fluid which has entered the control volume over this period of time and region 2 is whatever volume of fluid which flows out of the control volume. Just to quickly summarize the nomenclature here we are considering a diverging section with inflow at one velocity v_1 and outflow at velocity v_2.

To begin with the control surface and the system are coinciding some time later the system has moved and in terms of nomenclature, we have region 1 which corresponds to the volume of fluid which has flown into the control volume. This corresponds to the volume of fluid which has left the control volume through the control surface and so, the new volume is control volume minus 1 ($CV - 1$).

Now, let us express all these in terms of equations.

$$B_{sys}(t) = B_{CV}(t)$$

So, the first one says that B system at time t is equal to B control volume at time t. This corresponds to the case where the control volume and system coincident at time t. Now, let us write this expression at time t + Δ t

$$B_{sys}(t + \Delta t) = B_{CV}(t + \Delta t) - B_1(t + \Delta t) + B_2(t + \Delta t)$$

And writing this statement only we introduced nomenclature 1, 2. Let us see how do we do. What is that I want to know the property of the system at some time t + Δ t.

Now, what does the region corresponding to the system t + Δ t, this region (CV-1). This region is the region corresponding to the time t + Δ t. How do I express in terms of known values. I take this region which is the control volume. So, B of control volume at t + Δ t and then I subtract the region – 1, that means subtracting the properties corresponding to that region, B_1(t + Δ t) and then add region - 2, B_2(t + Δ t).

So, in terms of expression B system at time t + Δ t becomes equal to B for the control volume a t + Δ t and then B in the region 1 at t + Δ t and then B in the region 2 plus t + Δ t. Why there a minus sign because this is a whatever has been brought in is minus and then whatever has been taken out is the plus sign.

Now, which came up from the geometry of this figure I took the control volume and then subtracted the value corresponds to region 1 and then added whatever corresponds to region 2.

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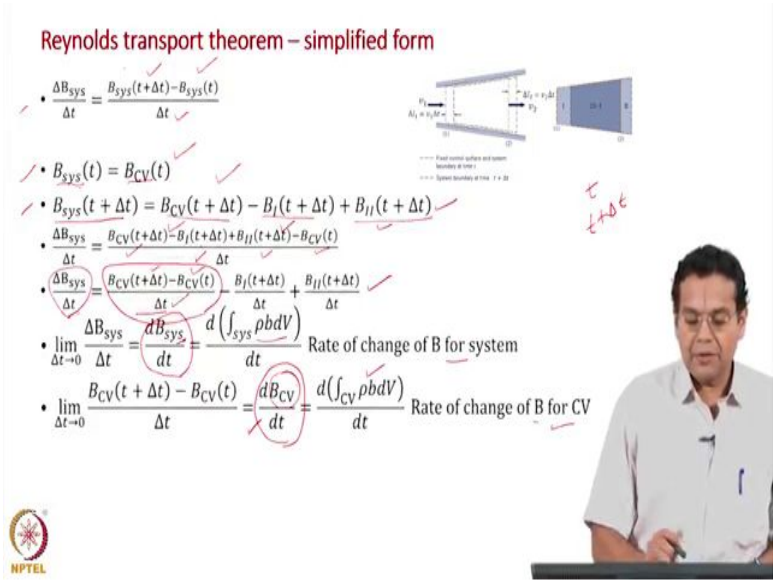

We said we want to relate the rate of change of property for the system to the rate of change in the property for the control volume. Now, the rate of change of property to the system is

$$\frac{\Delta B_{sys}}{\Delta t} = \frac{B_{sys}(t+\Delta t) - B_{sys}(t)}{\Delta t}$$

Of course, we will take limit delta t $\rightarrow$ 0 as we go along, this is something known to us. Now, what do we do we have seen the last light that

$$B_{sys}(t) = B_{CV}(t)$$

And,

$$B_{sys}(t + \Delta t) = B_{CV}(t + \Delta t) - B_1(t + \Delta t) + B_2(t + \Delta t)$$

The physical meaning of this is, If you take let us say mass of the system at time t and mass of the system at $t + \Delta t$, for the case of mass they are same, but if you take some other properties they need not be same. Now, $B_{CV}(t)$ is mass in the control volume at time t and $B_{CV}(t + \Delta t)$ is the mass in the control volume at $t + \Delta t$. $B_1(t + \Delta t)$ *and* $B_2(t + \Delta t)$ are the mass corresponding to region 1 and region 2 respectively.

So, mass of the system has been expressed in terms of mass in control volume and mass in regions 1 and mass in region 2 written for any property capital B. Now, let substitute these two expressions in the equation and see what happens.

$$\frac{\Delta B_{sys}}{\Delta t} = \frac{B_{CV}(t + \Delta t) - B_1(t + \Delta t) + B_2(t + \Delta t) - B_{CV}(t)}{\Delta t}$$

So, the left-hand side we have change in property for the system divided by small-time interval and for B system as substituted from the expression B_{CV}, B_1, B_2 etcetera and for B system at time t it is the same as B control volume and t. So, this states that just mass of systems equal to the mass in the control volume at the time it both are same because it coinciding. So, I am replacing B system at t by B control volume at time t.

Now, let us make a small rearrangement on the right-hand side I will take terms together for the control volume. So, I will take term 1 and term 2 and written them to begin with the first expression

$$\frac{\Delta B_{sys}}{\Delta t} = \frac{B_{CV}(t+\Delta t) - B_{CV}(t)}{\Delta t} - \frac{B_1(t+\Delta t)}{\Delta t} + \frac{B_2(t+\Delta t)}{\Delta t}$$

Why do I do this way because I am going to take limit delta t $\rightarrow$ 0,

$$\frac{\Delta B_{sys}}{\Delta t} = \frac{dB_{sys}}{dt} = \frac{d}{dt}\left(\int_{sys} \rho b \, dV\right)$$

This is the rate of change of B for the system. Now, we are going to take limit $\Delta t \rightarrow 0$ on the right-hand side, what we are discussing here is only the first two terms put together that is this term only we are going to see here.

$$\lim_{\Delta t \rightarrow 0} \frac{B_{CV}(t+\Delta t) - B_{CV}(t)}{\Delta t} = \frac{dB_{CV}}{dt} = \frac{d}{dt}\left(\int_{CV} \rho b \, dV\right)$$

So, if you take limit $\Delta t \rightarrow 0$ it becomes the derivative of the property for the control volume. We know how to express this in terms of the integral and then this represents the rate of change of B for the control volume.

As I told you we can tell this as the rate of change of B of the system and then the rate of change of B in the control volume or for the contents of the control volume. Now, why are we taking limits why should we take a limit that is question can answer. The equations we are going to derive are to be valid at every incident of time. So, I want an instantaneous equation because conservation equations that when derive are to be applicable even for transient systems.

So, the equation should be valid at every instant of time in the derivation we took t and then $t + \Delta t$. So, when I take $\Delta t \rightarrow 0$ I get a derivative which is instantaneous. These derivatives are instantaneous that is why I take limit $\Delta t \to 0$ so get so that I get a form of a Reynolds transport theorem which is valid at every instant of time.

We related B system and B control volume at time t and $t + \Delta t$; we wrote the expression for the rate of change of property for the system, substituted that and found out the left hand side is the rate of change the property of the system, on the right hand side you have rate of change the property of control volume both are not equal you have two additional terms. Next, we will see how to express those two terms in the limit of $\Delta t \to 0$.

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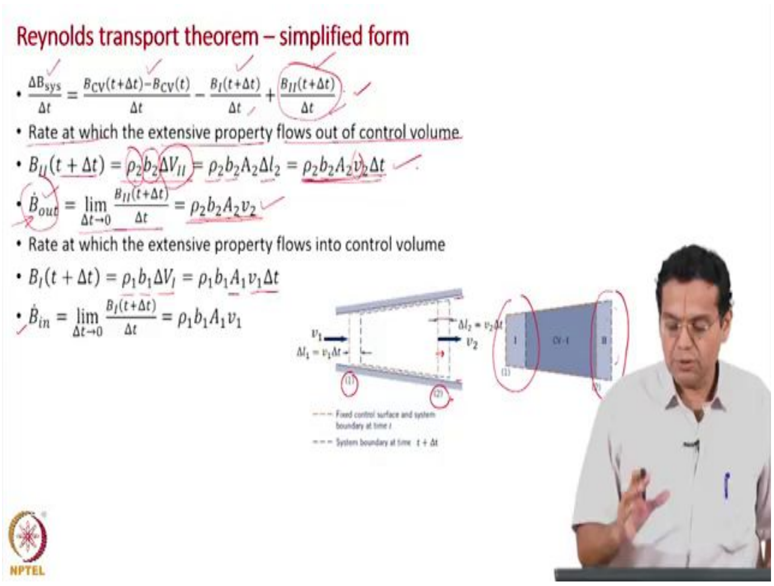

Now, the expression which we have seen and taken the limit of the left hand side term and the first term on the right hand side. Now, we are going to discuss the limit for the remaining two terms.

Now, what does B_2 represent? As I told you B_2 represents the amount of property it is not rated, only when you divide by Δ t it becomes rate. So, B is amount mass, the momentum it is all amount and which has crossed this control surface and has which has flown out of the control volume through this control surface. So, rate when you divide by Δ t it becomes rate.

So, $\frac{B_2(t + \Delta t)}{\Delta t}$ represents the rate at which the extensive property that is why I said amount, extensive property flows out of the control volume. Why do we say that remember this so much amount of volume has left the control volume or so much volume has come out of the control volume that is why and when we divide the by Δt we get the rate at which the extensive property flows out of the control volume.

Now, nice to define like this, but we always want to express in terms of measurable variables what are they or in and in terms of variables which are known to us some geometric variables we have found out the significance of this term, but we need to express in terms of let us say velocity, density, area etcetera. Let us see how do we do that. Until you do that there is no use for this term we know the significance, but we need to express it in terms of the usual properties. Let us do that now.

$$B_2(t + \Delta t) = \rho_2 b_2 \Delta V_2 = \rho_2 b_2 A_2 \Delta l_2 = \rho_2 b_2 A_2 v_2 \Delta t$$

B_2 is the amount of property corresponding to the volume. How do I express? I take that volume ΔV_2 that is a volume of the region multiply by density so that I get mass, I used 2 because rho is the density of the fluid in section 2; rho 1 is the density of the fluid in section 1 density can vary from section 1 to section 2. So, I multiply this volume with this density and then now I get mass of this volume, mass corresponds to this volume and then multiply by the intensive property so that I get the value of the extensive property.

So, I have taken the volume, multiply by density, get the mass, multiplied by property per unit mass intensive property. So, when you multiply all these you get the amount of property, extensive value associated with this volume. Now, let us express further. This ΔV_2 is volume can be expressed in terms of this area which is A_2 into the distance Δl_2. So, The volume has been expressed in term expressed in terms of area which is A_2, and then multiply by this distance Δl_2.

The significance of the distance Δl_2 is the distance moved by the fluid over the period Δt. So, express this Δl_2 which is the length, the significance is the distance traveled by the fluid over the time Δt which is nothing, but the velocity into Δt.

So, this expression gives you the amount of the property let us say mass which corresponds to region 2 and that mass has crossed the surface and has come out of the control volume. We want the rate. So, we divide by Δt taking limit $\Delta t \rightarrow 0$, which is just dividing by Δt. So, now, what is the significance of this? The rate at which extensive property flows out of control volume $\dot{B}_{out}$ is given by this expression.

$$\dot{B}_{out} = \lim_{\Delta t \to 0} \frac{B_2(t + \Delta t)}{\Delta t} = \rho_2 b_2 A_2 v_2$$

When I take $\Delta t \to 0$, all these become instantaneous values, that is a distinction between these two steps. When I take $\Delta t \to 0$ until then these values all average over a small Δt. When I take $\Delta t \to 0$ they become point values, point in terms of the time they become instantaneous values.

So, it may look like dividing by Δt, but the physical significance is that you take $\Delta t \to 0$, these are an average value over the period Δt, but now they become instantaneous values at time t. As I have been telling we want expressions which are instantaneous at every time t. So, this represents the rate at which extensive properties flows out of control volume. So, we will quickly go through whatever rate at which property flows into the control volume just analogously.

So, the rate at which extensive property flows into control volume B_1

$$B_1\,(t + \Delta t) = \rho_1 b_1 \Delta V_1 = \rho_1 b_1 A_1 \Delta l_1 = \rho_1 b_1 A_1 v_1 \Delta t$$

$$\dot{B}_{in} = \lim_{\Delta t \to 0} \frac{B_1(t + \Delta t)}{\Delta t} = \rho_1 b_1 A_1 v_1$$

Once again this is an instantaneous value. So, $\dot{B}_{out}$ is the rate at every instant of time t; similarly, $\dot{B}_{in}$ also.

So, what is that we have done now? We have taken limits of these two terms. In this slide, we have seen the limit for these two terms more importantly we have also seen the significance of those two terms. Like to repeat, $\dot{B}_{out}$ represents the rate at which extensive property flows out of control volume and $\dot{B}_{in}$ represents the rate at which the extensive property flows into the control volume through the control surface.

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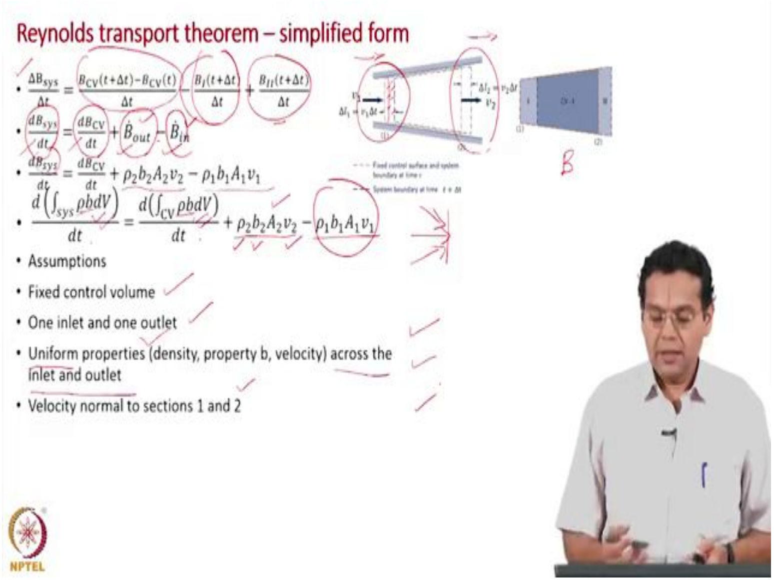

Let us put them all together.

$$\frac{\Delta B_{sys}}{\Delta t} = \frac{B_{CV}(t+\Delta t) - B_{CV}(t)}{\Delta t} - \frac{B_1(t+\Delta t)}{\Delta t} + \frac{B_2(t+\Delta t)}{\Delta t}$$

$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \dot{B}_{out} - \dot{B}_{in}$$

So, what does this tell you? Rate of change of property of the system, rate of change of property in the control volume is equal to the rate at which the property leaves the control volume through the control surface. And, then rate at which of course, minus rate at which the property enters the control volume through the control surface. If it is mass then the rate of change of mass of the system, rate of change of mass in the control volume or the contents of the control volume, the rate at which mass leaves the control volume through the control surface, the rate at which mass enters the control volume through the control surface.

$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \rho_2 b_2 A_2 v_2 - \rho_1 b_1 A_1 v_1$$

So, we will also write in terms of the integral expressions. I have expressed this B system in terms of integral B control volume in terms of integral.

$$\frac{d}{dt}\left(\int_{sys} \rho\, b\, dV\right) = \frac{d}{dt}\left(\int_{CV} \rho\, b\, dV\right) + \rho_2 b_2 A_2 v_2 - \rho_1 b_1 A_1 v_1$$

Why do I do that? this is a Reynolds transport theorem for this simplified case and the variable that should appear here should be ρ, b, geometric values, velocities etcetera that is why I replaced B system in terms of this integral which involves ρ, b etcetera. So, this is for the system, this was a control volume. That is what we have done though we started with B finally, my expression is in terms of the intensive properties of course, b and the density. Remember these are at the control surfaces entering control surface through which there is outflow here and these are properties at the inflow control surface. So, this is for the system inside the control volume and control surface through which there is an outflow control surface through which there is inflow.

Now, why did I call this as a simplified form? We have made a lot of assumptions though I did not very explicitly state that let us look at what are the assumptions we have made so that we called this a simplified form.

- The first assumption is fixed control volume; we have taken a control volume which is fixed that is very obvious our control volume has been fixed. We did not move the control volume; the control volume did not move.
- Second, we are considered the case where there is only one inlet, there is only one outlet, and then we took uniform properties and I say properties the density, the property b, and the velocity. All these were taken uniform across the inlet and outlet. So, there could be variation the property b could vary, the velocity could vary, density could vary, but we have not taken that into account. We said all these properties are uniform across the inner cross section, outer cross section. So, uniform properties across the inlet and outlet.
- The third assumption, what we have done is that we took the area, and then the velocity was normal to this area. Velocity could make some angle with this area, we have not considered that. So, we have the way in which even we have shown the velocity is just normal to the area. If you have phase here the velocity is normal to this.

So, just to repeat the assumptions: fixed control volume, one inlet - one outlet; uniform properties across the inlet and outlet, and then velocity normal to sections 1 and 2. The general form will take out all these assumptions. So, we will derive a general form without making any of these assumptions. Of course, we will restrict back to fixed control volume I said the whole scope the causes restrict to fixed control volume, we will keep that. But, we will derive a general expression where there could be any number of inlet and outlets in fact, any part of this surface can be inlet-outlet, where the properties change across the inlet or outlets and velocity can make any angle with the area.

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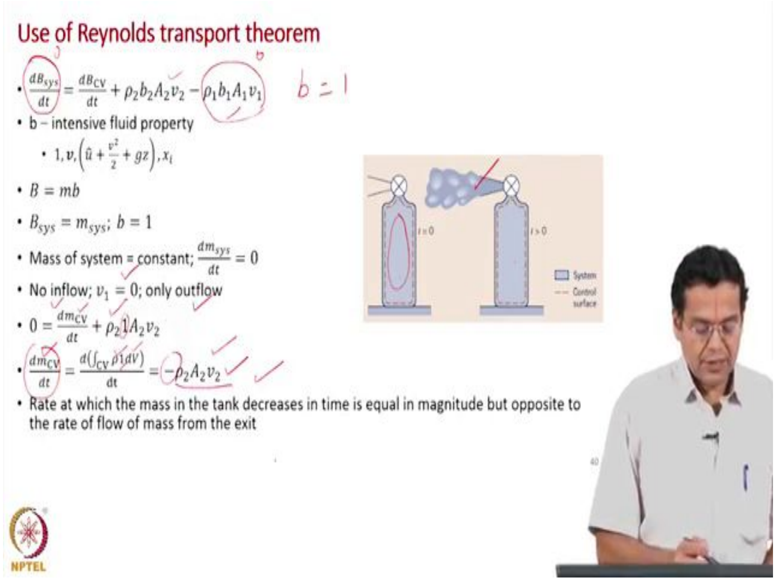

Just to understand that understand the application of Reynolds transport theorem. The actual application is in deriving all the conservation equations just to get a feel for what it implies will apply this Reynolds transport theorem the simplified form for this fire extinguisher case.

So, let us write the Reynolds transport theorem which we derived in the previous few slides and b is an intensive fluid property.

$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \rho_2 b_2 A_2 v_2 - \rho_1 b_1 A_1 v_1$$

We are going to write for the case of mass, we said b could take on different values. So, because we are going to consider mass in our case, b = 1. So, in this expression everywhere b = 1; $b_1 = b_2 = 1$.

Now, we already discussed several times the mass of the system is constant. So, the left-hand side derivative is 0,

$$\frac{dm_{sys}}{dt} = 0$$

Now, in this particular case to began with we had the gas filling up the entire control volume. Sometime later we opened this valve and hence there was only outflow there was no inflow at all. So, there is no inflow. So, $v_1 = 0$ and v_2 has some value. So, there is the only outflow.

$$0 = \frac{dm_{CV}}{dt} + \rho_2 (1) A_2 v_2$$

So, we should know what is the area, what is the velocity, rho 2 etcetera. So, if you rearrange in a more physically meaningful way

$$\frac{dm_{CV}}{dt} = \frac{d}{dt}\left(\int_{CV} \rho \, dV\right) = -\rho_2 A_2 v_2$$

This tells the rate of change of mass in the control volume. Once again I have written that in terms of the integral expression.

So, this is something you would have written in your process calculation course may not be used it. What it does says is the rate of change of mass in the control volume is equal to the rate of mass flow leaving the control volume; of course, the negative sign this tells that; obviously, all these are positive values. So, this tells you that the mass decreases with time in the control volume that is why I have a negative sign.

If you want to put in more formula is a statement rate at which the mass in the tank decreases is equal in magnitude in terms of magnitude is, but opposite to the rate of flow of mass from the exit. So, you would write the law of conservation of mass the conservation equation has the rate of accumulation is equal to the rate in minus rate out. We do not have any rate in the term, we have only rated out term.

So, this is a more practical form of the Reynolds transport theorem as applied for mass conservation in this example of fire extinguisher. Which relates the rate of change of mass to the rate whichever is flowing. I want to emphasize that this is what is happening in the for the contents of the control volume this what is happening across the control surface that takes place across the control surface, flow takes place across the control surface.