Continuum Mechanics And Transport Phenomena
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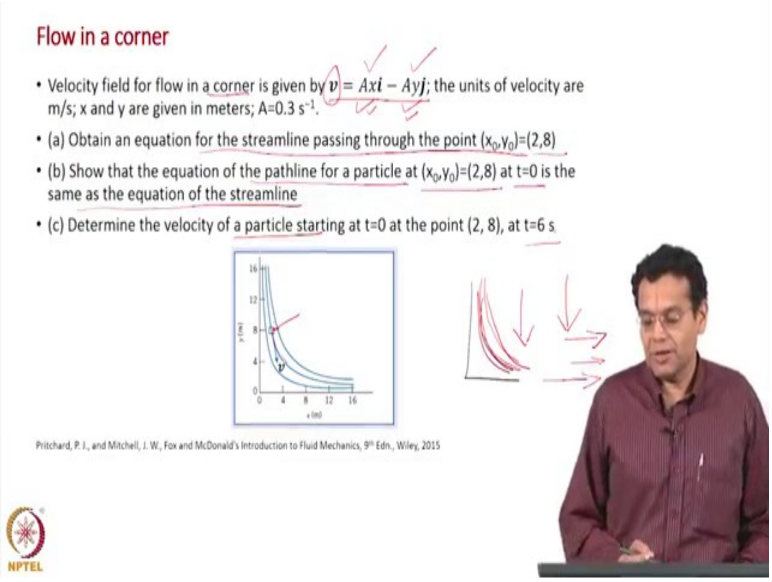
Lecture – 14
Streamline, Pathline: Steady Flow Example

Example: (Refer Slide Time: 00:14)



We have defined what are streamlines, what are path lines, what are streak lines and then you also seen physically what do they mean, represent terms of figures, had some animation and simulations. We also saw the mathematical expressions for them. What we will do now is take two examples and illustrate how do you really calculate them.

First example is steady state velocity field, second is a unsteady state velocity field. This example is from Fox and McDonald. What you have here is flow in a corner. So, velocity field for flow in a corner is given by the equation,

$$v = Axi - Ayj$$

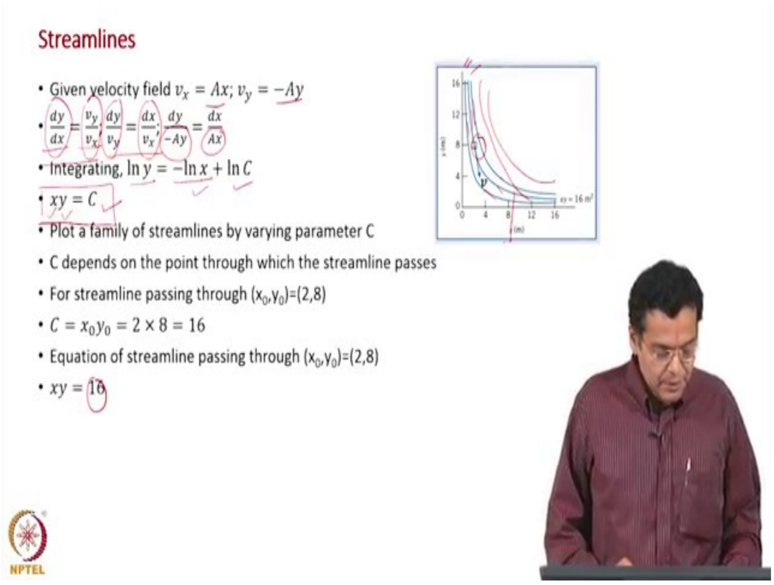
So, velocity is a vector. The first time we are coming across a velocity field where both the x component and the y component are present. The earlier example we are just only one component. The x component is the function of x only and the y component as a function of y only. How do you visualize this? This x velocity increases with x. So, if you move along the x direction the velocity keeps increasing.

Now, what about this y component? This y component in terms of first of, all it is in the negative y direction, it is $-Ay$ and its magnitude increases if you go further with increase with y, though the magnitude of this component increases as Ay increases. So, x component is directed in this direction, y component is directed in this direction, that also can be easily visualized because we have flow in this way. Now, the units of velocity are meters per second, and A the constant is given as 0.3 second inverse.

Obtain equation for the stream line passing through the point $(x_0, y_0)=(2,8)$ and that point is shown in the given figure. We are asked to get an equation for a streamline passing through the point $(2,8)$. Then because it is a steady state velocity field, it has to show that equation of the path line for a particle, as we are discussed not any stream line coincides with any path line. Path line for a particle at this location at time $t=0$ will coincide with the streamline passing through $(2,8)$. So, we are asked to show that equation of the path line for a particle at $(x_0, y_0)=(2,8)$ at $t=0$ is a same as the equation of the streamline.

Now, we are also asked to find out what is the velocity of the particle. It is starting at the initial location, it as the velocity, it travels along the path line, and after some $t=6$ second it has a velocity, we are asked to find out what is its velocity.

Solution: (Refer Slide Time: 04:03)



So, let us go ahead. Let us first see how to calculate the streamlines. The velocity field is given to us,

$$v_x = Ax\,;\, v_y = -Ay$$

Now, we have seen the equation for streamline is that

$$\frac{dy}{dx} = \frac{v_y}{v_x}$$

The $\frac{dy}{dx}$ is the slope of the stream line, $\frac{v_y}{v_x}$ is the tangent of the angle made by the velocity vector with the x axis. This condition came up because streamline is a line for which if you draw a tangent that represent the velocity. Now, just do a simple rearrangement, so that you can integrate. So,

$$\frac{dy}{v_y} = \frac{dx}{v_x}$$

The reason for writing this is if you substitute for v_y by v_x you can integrate. Now, I substitute for $v_y = -Ay$ and substitute for $v_x = Ax$.

$$\frac{dy}{-Ay} = \frac{dx}{Ax}$$

Now, we integrate, if you integrate you get,

$$\ln(y) = -\ln(x) + lnC$$

Now, this equation can be represented as

$$xy = C$$

Now, $xy = C$ in this particular case represents a hyperbolic equation and that is what represented as streamlines here. Now, this C differs from one streamline to the other streamline. How does it get determined? It gets determined based on the point through which the streamline passes, that is why we were given find the equation of a stream line passing through a point $(2,8)$.

Now, so plot a family of streamlines. Remember when need a several streamlines we can draw tangents to all the streamlines at different locations, get the velocity field in the entire region velocity vectors in the entire region. So, plot of family of streamlines by varying parameters C. So, for one streamline you have one value of C.

Now, C depends on the point through which the streamline passes. In this particular case $(2,8)$. So, for stream line passing through $(2,8)$

$$C = x_0\, y_0 = 2 x 8 = 16$$

So, the equation of streamline passing through $(2,8)$ is

$$xy = 16$$

This is the general expression for stream setup streamlines. All the streamlines for this flow field are represented by this general equation, $xy = C$. Based on the point is specify or based on which streamline you want, how do you specify? You specify a point on the streamline then this constant keeps changing, and, this center one is the streamline $xy = 16$.

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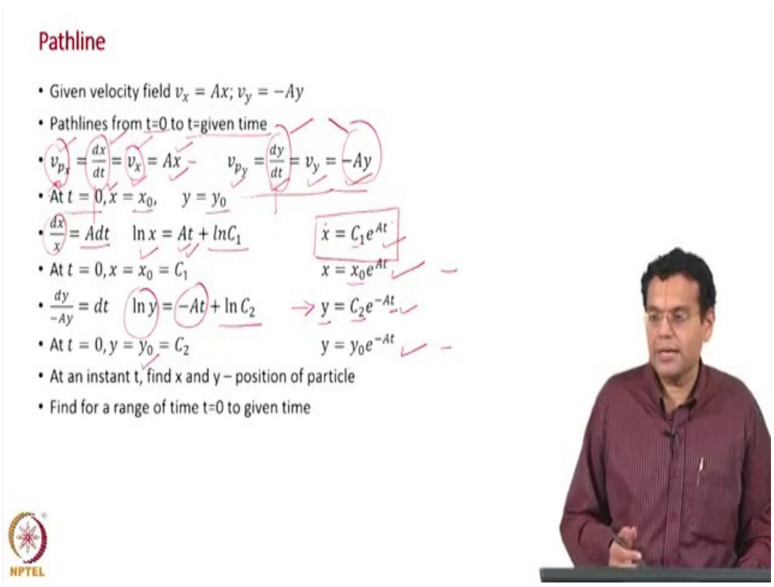

Now, how do we calculate a path line? Now, whatever integration we have seen earlier we will become clear now. We will start with the velocity field, given velocity field

$$v_x = Ax\, ; v_y = -Ay$$

Now, path lines are the path followed by fluid particles and from a time $t=0$ to given time let us say 0 to 2 seconds, 0 to 5 seconds etcetera.

Now, the velocity of the particle in the x direction is given by the rate of change of its x coordinate. Now, remember we are use this principle that at a particular point it is instant the velocity of the fluid particle is same as that of the local velocity given by the velocity field, relating the Lagrangian view point to the Eulerian view point. That is why

$$v_{p_x}=\frac{dx}{dt}=v_x=Ax\,;v_{p_y}=\frac{dy}{dt}=v_y=-Ay$$

When I say this it is velocity of the particle in the x direction is equal to the velocity of the flow field in the x direction. So, by equating this we use our principle which I use several times, equating velocity particle to the velocity field at that location at that instant.

Now, we have got two ordinary differential equations

$$\frac{dx}{dt}=Ax;\frac{dy}{dt}=-Ay$$

Now, to solve them we need initial conditions, at time $t=0$ the particle is at x_0 and y_0.

$$x=x_0\,,y=y_0 at\,t=0$$

To solve the differential equations first order differential equation we need one initial condition for this and one initial condition for this equation that is what we are given in terms of the initial position of the particle.

Now, we will integrate this equation; we will integrate this equation, do a variable rearrangement,

$$\frac{dx}{x}=Adt$$

Integrate, you get

$$lnx=At+\ln C_1\,;x=C_1e^{At}$$

So, x is the coordinate of the particle as a function of time. C_1 is a constant yet be determined,.

Now, we will use the initial condition at time $t=0$ we have $x=x_0$. So, substitute

$$x = x_0 e^{At}$$

So, this is an equation with C_1 evaluated now as x_0. So, when $t=0$ you will get the initial x coordinate. So, this gives you as a function of time what does x coordinate of the particle.

Now, let us do this for y direction also. So, I take this differential equation,

$$\frac{dy}{-Ay} = dt$$

After integration you get,

$$\ln y = -At + \ln C_2;\ y = C_2 e^{-At}$$

Let evaluate the constant C_2, once again use the initial condition at $t=0$ the $y=y_0$, so substitute

$$y = y_0 e^{-At}$$

This is the equation which gives you the y coordinate to the particle as a function of time. These two put together constitute the path line for the particle. So, at an instant t, find x and y the position of the particle. So, if you give time I can find out x and find out what is y. And for a range of time to the given time you can find out all the x and y coordinates and then plot the path line.

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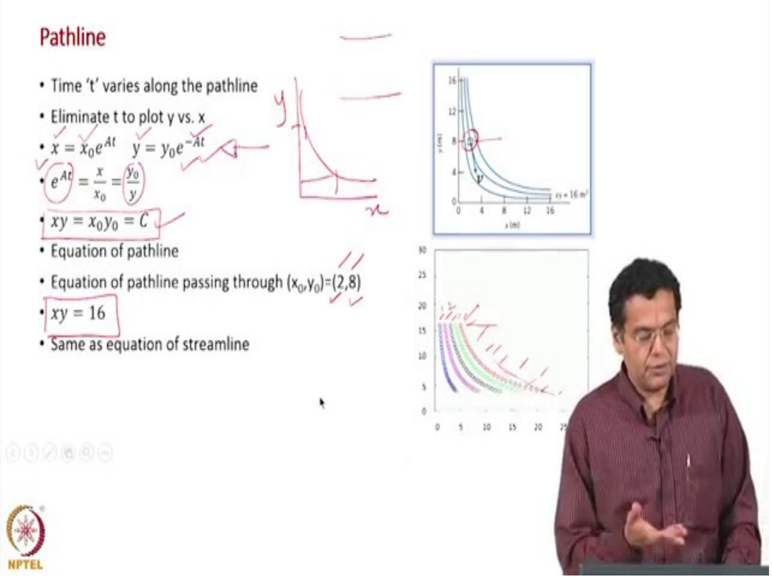

The time t varies along the path line. So, and so we can eliminate the time t to plot y versus x ; so, how do you do that? We have got the equation for x, we got the equation for y, and eliminating t means that let us say expressed t in terms of x and substitute in second equation

$$x=x_0 e^{At}; y=y_0 e^{-At}$$

$$e^{At}=\frac{x}{x_0}=\frac{y_0}{y}$$

So, from this equation you can write

$$xy=x_0 y_0=C$$

So, what is the use of this equation? If you give me time I can find out what is the x coordinate of the particle, I can find out what is the y coordinate of the particle. What is use of the last equation? If you give me the x axis and y axis I will straight away plot the path line. At every time I find out what is x what is y, that is utility of these two expressions. The last equation straight away gives me what is the equation of the path line in x, y coordinate.

What they are asked is the equation of path line passing through (2.8). The question was prove that the equation of path line for a particle which starts at (2.8) at $t=0$, it is same as that of the streamline. So, equation of path line passing through (2.8) which means that the particle is at (2.8) at $t=0$. So,

$$x_0=2; y_0=8$$

So,

$$xy=16$$

This is the equation of path line, which was also the same as the equation of the streamline, same as equation of streamline. We have seen that for steady flow. We have also demonstration through flow between parallel plates that path lines and streamlines coincide for steady flow. You have seen another example now, where path line coincides with streamline in terms of equation as well.

Let us have a demonstration (shown in the above slide image). What is shown here are different fluid particles and then we are going to see the path line followed by each of this fluid particle. Remember we said path line is for a particular fluid particle. So, if you track this you will get one path line, if you track this will get the second path line etcetera. And now, let us run this. So, these are the path lines followed by different fluid particles. This represents their position at time t = 0 and then these are the x, y coordinates at different times. Remember along a path line time varies, t = 0 here and t increases along the path line here. Now, you have seen that of course, the same equation as a streamline.

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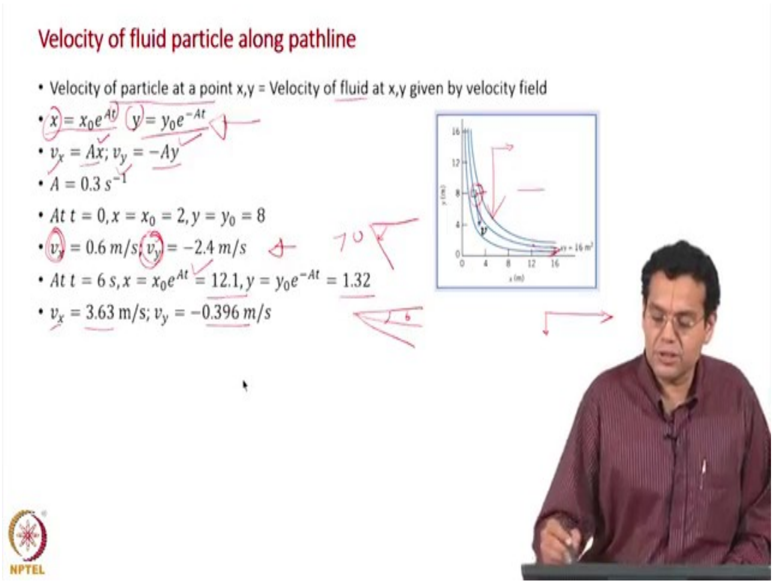

The third part of the question was to find out the velocity of the fluid particle at some time instant along its path line; let us see how do we do that.

$$Velocity\ of\ particle\ at\ a\ point\ x, y = velocity\ of\ fluid\ at\ x, y\ given\ by\ velocity\ field$$

This we have seen several times relating a Lagrangian view point to the Eulerian view point. So, the velocity of the particle is equal to velocity of the fluid. Now, from the path line equations for the individual coordinates, we know what is the position of the particle as a function of time, both the x coordinate and the y coordinate.

$$x = x_0 e^{At}; y = y_0 e^{-At}$$

Now, we know of course, the x component of the velocity field and y component of the velocity field, A is also given to us. At t = 0 of course, we know the portion of the particle is

$(x_0, y_0) = (2,8)$. So, if you substitute the x value here you will get the velocity in the x direction and velocity in the y direction.

$$v_x = Ax = 0.3\,x\,2 = 0.6\frac{m}{s}$$

$$v_y = -Ay = -0.6\,x\,8 = -2.4\frac{m}{s}$$

When I say velocity, two face of interpreting this, velocity of the particle in the x direction, velocity of the particle in the y direction which is same as velocity of the flow field in the x direction, velocity of the flow field in the y direction.

Now, as we have seen to begin with the x velocity is a smaller value in terms of magnitude, 0.6 and the y velocity is larger value of $-¿$2.4 meter per second in terms of magnitude, ok, larger y component and a smaller x components. So, particle here will have these values 0.6 in the x direction and 2.4 in the negative y direction. What does it where as to find out? Let us say the particle travels over 6 seconds, what is its velocity?

$$x = x_0 e^{At}; \; y = y_0 e^{-At}$$

Now, as I told you this set of equations gives you the x coordinate and y coordinate particle as a function of time. So, let us substitute his in this equation t = 6.

$$x = x_0 e^{At} = 12.1$$

$$y = y_0 e^{-At} = 1.32$$

So, this will be the x position of particle 12.1 and the y coordinate to the particle will be 1.32.

Now, we know the position and as usual as we have done earlier we can find out what is the x component of velocity and the y component of velocity. Once again, represents the velocity of the fluid particle and the field as well, velocity field as well.

$$v_x = Ax = 0.6\,x\,12.1 = 3.63\frac{m}{s}$$

$$v_y = -Ay = -0.6\,x\,1.32 = 0.396\frac{m}{s}$$

Now, if you see these values the x component as the large value 3.63 meters per second and y is a small negative value relativity just −¿0.4 meter per second. That is why in terms of angle this the velocity vector here would make for example, in this particular case the velocity vector would make the roughly about 70 degrees or 75 degrees with the x axis, but this velocity vector would just make roughly about 6 degrees. And that is why the particle move moves in this direction here along this direction and here it is moving along this direction.

So, what you have done this to summarize this particular slide as we know the equation x and y coordinates of the particle at different time instance. So, if you know the time you can find out the x and y coordinates, use the velocity field, and use this physical concept of Lagrangian, Eulerian equality, and find out the velocity of the particle.