Continuum Mechanics and Transport Phenomena
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Lecture - 113
Simplification of Differential Energy Balance

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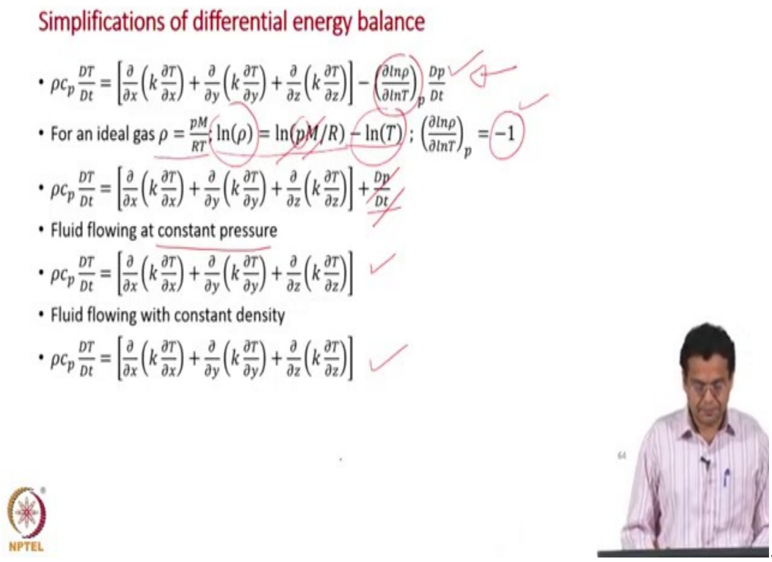

We will look at different levels of simplifications, of the different energy balance equation, to forms which are usually used. So, let us start the equation, which I have seen in the previous slide.

$$\rho c_p \frac{DT}{Dt} = \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] - \left(\frac{\partial \ln(\rho)}{\partial \ln(T)}\right)_p \frac{Dp}{Dt}$$

Now, let us say we have an ideal gas, let us see how the above equation gets simplified. For an ideal gas we know

$$\rho = \frac{pM}{RT}$$

Now, let us take logarithm on both the sides.

$$\ln(\rho) = \ln\left(\frac{pM}{R}\right) - \ln(T)$$

And,

$$\left(\frac{\partial \ln(\rho)}{\partial \ln(T)}\right)_p = -1$$

So, we can just substitute that here, it becomes still simpler.

$$\rho c_p \frac{DT}{Dt} = \left[\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)\right] + \frac{Dp}{Dt}$$

So, this is the differential energy balance equation for an ideal gas. Now, let us say we have a fluid flowing at constant pressure, what happens? $\frac{Dp}{Dt} = 0$, so the last term vanishes when, I say constant pressure, pressure is not a function of time and space.

$$\rho c_p \frac{DT}{Dt} = \left[\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)\right]$$

Remember it is $\frac{Dp}{Dt}$, it has variation of pressure with time and spatial variation also. So, pressure is just a constant, it is not varying both with time and spatial location. And, because pressure is constant the last term drops out. Let us go back to the first equation, in this equation if you consider density as constant for example, let us say water is flowing. The effect of temperature on density of water is not significant. So, the term, $\left(\frac{\partial \ln(\rho)}{\partial \ln(T)}\right)_p$ will not be significant or if it is independent then it is 0. So, once again that term drops out. In this case because pressure is constant, in this equation because density is constant, both result in the same equation.

$$\rho c_p \frac{DT}{Dt} = \left[\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)\right]$$

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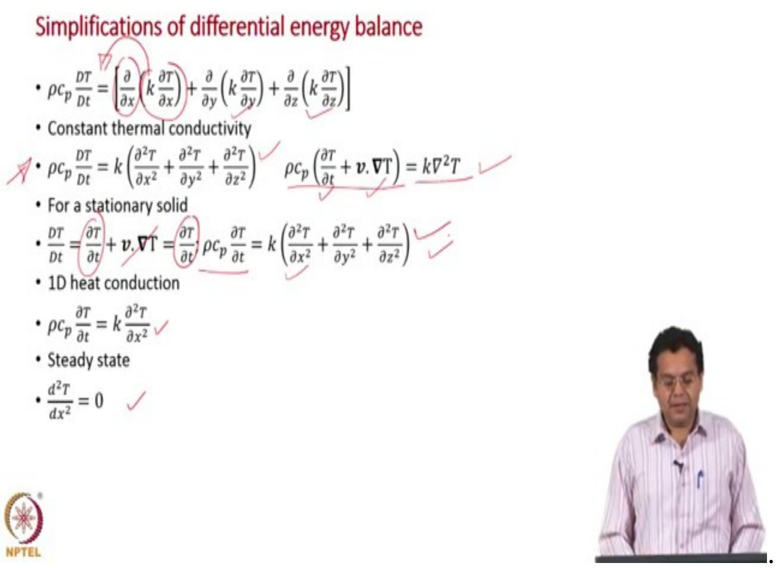

Now, let us simplify further, this is the equation which I have seen in the previous slide.

$$\rho c_p \frac{DT}{Dt} = \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]$$

At this stage, we have k inside the spatial derivative. So, thermal conductivity can change with spatial location, if you assume thermal conductivity to be a constant. Why should k change with spatial location, thermal conductivity as you have seen is a function of temperature.

So, if you have a solid object with different temperatures then, k can vary with spatial location also because, it is depends on temperature. If you assume k to be constant then, you can take this out of the spatial derivative and that becomes a second order derivative.

$$\rho c_p \frac{DT}{Dt} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

Left hand side, I have expressed the substantial derivative in terms of the local component and the convective component. Right hand side, I have expressed in terms of the Laplacian of temperature.

$$\rho c_p \left(\frac{\partial T}{\partial t} + v.\nabla T\right) = k\nabla^2 T$$

So, this is the better representation of this equation. Now, let us say we have a solid and then a stationary solid now, what happens for a stationary solid. The substantial derivative of temperature which has two components, only the first component will be remain, the second component which represents convection will become 0, $v.\nabla T = 0$.

So, $\frac{DT}{Dt}$ will become $\frac{\partial T}{\partial t}$ and that is what is shown here.

$$\rho c_p \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

Now, suppose if it is 1 dimensional heat conduction in the solid,

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

Further if it is steady state,

$$\frac{\partial^2 T}{\partial x^2} = 0$$

If you look at this equation should be really happy at it, the way in which we started taking all the energy terms in to account and work done term, heat input term. Several levels of simplification, you get a very simple nice looking equation. So, if you have a solid 1D, if you have solid 1D and then transient is also there and solid, 3D transient.

So, that is why the in one sense the discussion energy balance is simpler than the than the discussion on momentum transport or fluid mechanics. One big difference is fortunately q is the vector, the heat flux is the vector, it is the big simplification for us, τ was a tensor and many terms drop out in the energy balance equation, the terms which contribute are only few or one of them, I would say.

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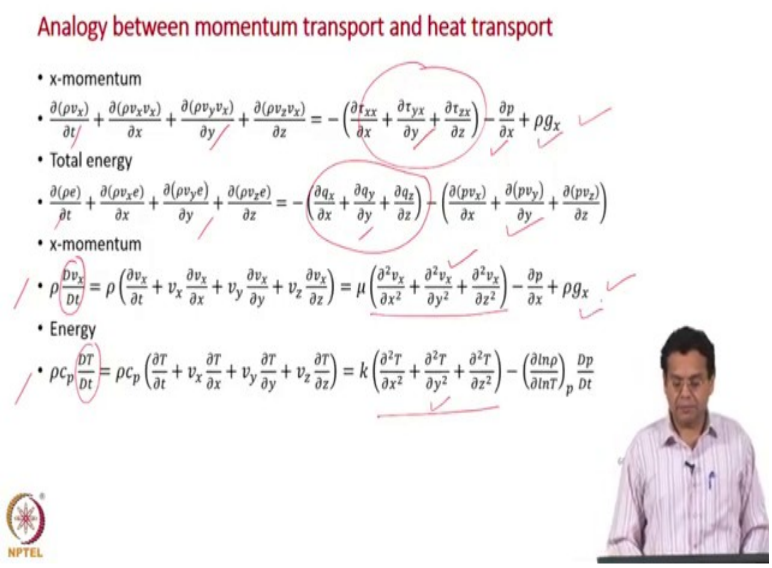

So, let us look at the analogy between momentum transport and heat transport in terms of the conservation equations. You have already seen these two equations and discussed the analogy.

$$\frac{\partial(\rho v_x)}{\partial t}+\frac{\partial(\rho v_x v_x)}{\partial x}+\frac{\partial(\rho v_y v_x)}{\partial y}+\frac{\partial(\rho v_z v_x)}{\partial z}=-\mu\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)-\frac{\partial p}{\partial x}+\rho g_x$$

$$\frac{\partial(\rho e)}{\partial t}+\frac{\partial(\rho v_x e)}{\partial x}+\frac{\partial(\rho v_y e)}{\partial y}+\frac{\partial(\rho v_z e)}{\partial z}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)-\left(\frac{\partial(p v_x)}{\partial x}+\frac{\partial(p v_y)}{\partial y}+\frac{\partial(p v_z)}{\partial z}\right)$$

Same equations we will write again now, in terms of the measurable variables.

$$\rho\frac{Dv_x}{Dt}=\rho\left[\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right]=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)$$

This is the linear momentum balance; this is the Navier-Stokes equation. Substitute at for τ, the molecular momentum flux in terms of the velocity gradient using the Newton's law of viscosity.

Let us write the energy equation once again in terms of measurables,

$$\rho c_p\frac{DT}{Dt}=\rho c_p\left(\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)-\left(\frac{\partial\ln(\rho)}{\partial\ln(T)}\right)_p\frac{Dp}{Dt}$$

I have expressed the heat flux or the components of heat flux, in terms of temperature gradient, using the Fourier's law of heat conduction and return this equation. So, now, the equations are analogous, we have the transient terms and convective transport terms, molecular transport terms and then surface force due to pressure, body force rate of work done by pressure force. And, if you look at this form of the conservation equation, once again they are analogous.

Left hand side, they are written in terms of the substantial derivative. And, right hand side we have Laplacian of velocity here, Laplacian of temperature of course, other terms are also there. Of course, we should know that, three such equations are there for the momentum, energy balance only this one equation alone.

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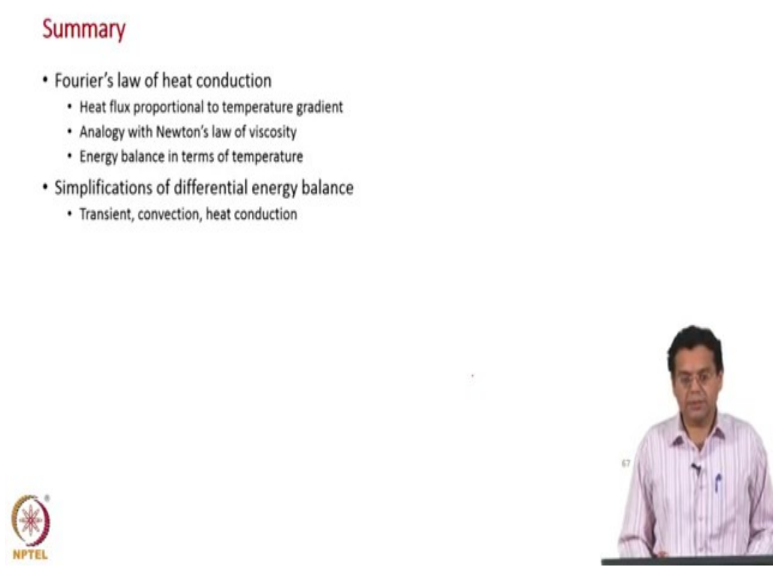

Let us summarize this lecture, we have discussed the Fourier's law of heat conduction, which relates heat flux to temperature gradient, we have also discussed the analogy between Fourier's law of heat conduction and Newton's law of viscosity. In 1D they are analogous; in 3D they are not analogous.

And, using the Fourier's law of heat conduction, we express the energy balance completely in terms of temperature. We also looked at the simplifications of the differential energy balance equation. And finally, the terms that were left out were the transient term, the convection term and the heat conduction term. Only those were the terms finally, which remained in the differential energy balance equation.