Continuum Mechanics and Transport Phenomena
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Lecture – 111
Deferential Energy Balance – Part 3

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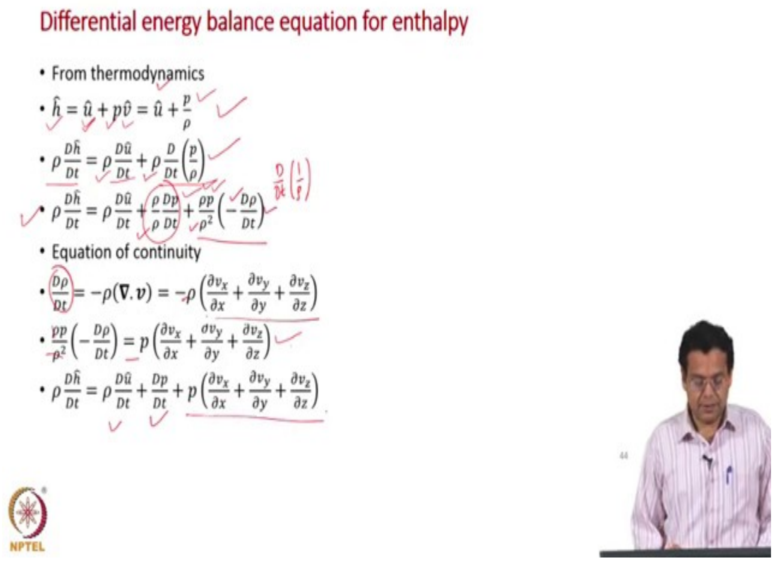

Now, let us proceed to the derivation of the difference energy balance equation, in terms of enthalpy and then later on in terms of temperature. Now, we start with the relationship from thermodynamics. This is a well-known relationship, which relates enthalpy to internal energy,

$$\hat{h} = \hat{u} + p\,\hat{v} = \hat{u} + \frac{p}{\rho}$$

I like to mention $\hat{h}$ represents specific enthalpy. So, it is on a unit mass basis of course, internal energy $\hat{u}$, we know it is per unit mass and then pressure and volume $\hat{v}$ is also per unit mass basis.

What we will do is express this relationship in terms of substantial derivative or saying it the other way. We take the substantial derivative of this equation, both on the left hand side and the right hand side, that is what you have done here

$$\rho\frac{D\hat{h}}{Dt} = \rho\frac{D\hat{u}}{Dt} + \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right)$$

These two terms are simple, let us see how to evaluate the last term. Just use product rule. So,

$$\rho \frac{D\hat{h}}{Dt} = \rho \frac{D\hat{u}}{Dt} + \frac{\rho}{\rho}\frac{Dp}{Dt} + \frac{\rho p}{\rho^2}\left(\frac{-D\rho}{Dt}\right)$$

Now, to proceed further we will use the equation of continuity from a material particle view point,

$$\frac{D\rho}{Dt} = -\rho(\nabla . v) = -\rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

So, we will substitute for $\frac{D\rho}{Dt}$, in this last term. So, I have got

$$\frac{\rho p}{\rho^2}\left(\frac{-D\rho}{Dt}\right) = p\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

So, let us substitute in this equation,

$$\rho \frac{D\hat{h}}{Dt} = \rho \frac{D\hat{u}}{Dt} + \frac{Dp}{Dt} + p\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

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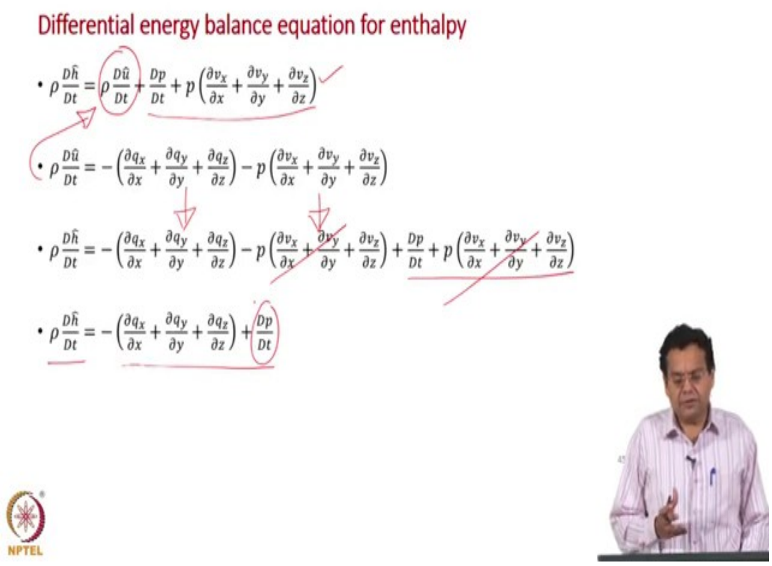

So, let us write down the equation which are derived for the substantial derivative of enthalpy.

$$\rho\frac{D\hat{h}}{Dt}=\rho\frac{D\hat{u}}{Dt}+\frac{Dp}{Dt}+p\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)$$

Now, we have already derived the differential energy balance equation for internal energy and that is what is being written now.

$$\rho\frac{D(\hat{u})}{Dt}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)-p\left[\frac{\partial(v_x)}{\partial x}+\frac{\partial(v_y)}{\partial y}+\frac{\partial(v_z)}{\partial z}\right]$$

So, that we can substitute this on the right hand side so, let us substitute

$$\rho\frac{D\hat{h}}{Dt}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)-p\left[\frac{\partial(v_x)}{\partial x}+\frac{\partial(v_y)}{\partial y}+\frac{\partial(v_z)}{\partial z}\right]+\frac{Dp}{Dt}+p\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)$$

And, the two terms cancel each other, leaving us with

$$\rho\frac{D\hat{h}}{Dt}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)+\frac{Dp}{Dt}$$

The first time we are coming across substantial derivative of pressure. Also like to mention, we discussed about substantial derivative in the very beginning of the course. There it would have appeared as if we are suddenly discussing a new derivative of course, we discuss the physical significance but, now look at the utility of that substance derivative.

We are using the substance derivative very frequently especially in the derivation of the difference energy balance equation. Not alone that when we introduce we said, we have substantial derivative of velocity, temperature, concentration etcetera. But, now look at the derivations we have substance derivative of all the energy terms, we have for internal energy, we have for enthalpy, we have also got for a pressure.

So, significance is same the variable for which the substance derivative is expressed that changes. So, it does have wide application, in terms of derivation and the corresponding physical significance. So, now, we will be really convinced the discussion on substance derivative in the beginning part of the course.

We have been looking at limited scope; you have come across only $\frac{D}{Dt}$ of density for the continuity equation, $\frac{D}{Dt}$ of velocity, we have come across. But, now the derivation of energy balance, we have come across substantial derivative of different variables energy, pressure etcetera.

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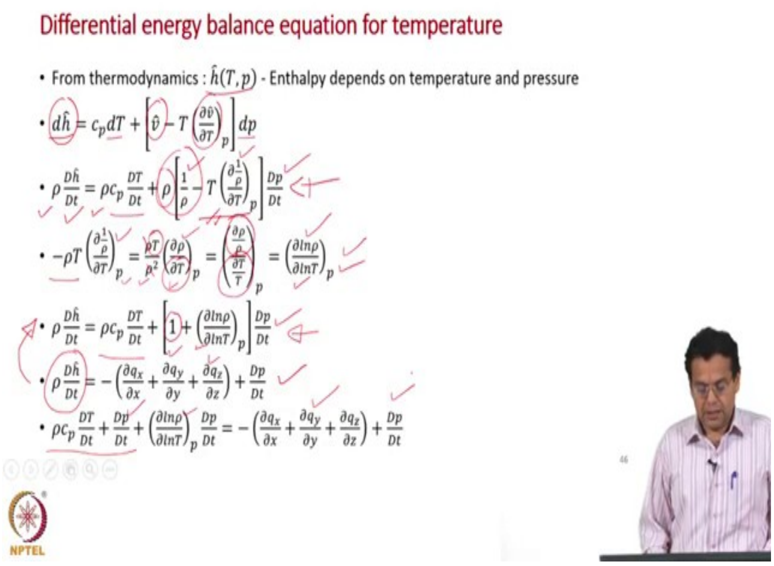

Now, we are towards the last step, we are going to express the energy balance equation in terms of temperature. Let us do that once again, we are going to use the thermodynamic relationship. What is that? From thermodynamics we know that for a pure species, the enthalpy is a function of temperature and pressure, $\hat{h}(T, p)$.

$$d\hat{h} = c_p dT + \left[\hat{v} - T\left(\frac{\partial \hat{v}}{\partial T}\right)_p\right] dp$$

And this equation, we will derive in a thermodynamics course when we derive, when we discuss fundamental property relations, where we discuss Maxwell relations etcetera, we discuss this derive this relationship.

What does it tell you? It relates the differential change in enthalpy, to the differential change in temperature and the differential change in pressure. What is utility? This expression can be used to find out change in enthalpy, if you are given change in temperature and pressure, that is a use of this equation. Now, once again we will express this equation in terms of substantial derivative, let us do that and multiplied by density.

$$\rho\frac{D\hat{h}}{Dt}=\rho c_p\frac{DT}{Dt}+\rho\left[\frac{1}{\rho}-T\left(\frac{\partial\left(\frac{1}{\rho}\right)}{\partial T}\right)_p\right]\frac{Dp}{Dt}$$

So, the thermodynamic relationship has been expressed in terms of substantial derivative number 1, number 2 the specific volume has been expressed in terms of density, we have multiplied by density both the left and right hand side. Now, let us take the last term multiplied by rho and simplify that. So, we have got

$$-\rho T\left(\frac{\partial\left(\frac{1}{\rho}\right)}{\partial T}\right)_p=\frac{\rho T}{\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_p=\left(\frac{\frac{\partial\rho}{\rho}}{\frac{\partial T}{T}}\right)_p=\left(\frac{\partial\ln(\rho)}{\partial\ln(T)}\right)_p$$

So, this term tells about term effect of temperature on density but, it is not $\frac{\partial\rho}{\partial T}$ tells about change in $\ln(\rho)$ for change in $\ln(T)$. Now, let us substitute that in this equation, left hand side we have rho substantial derivative of enthalpy.

$$\rho\frac{D\hat{h}}{Dt}=\rho c_p\frac{DT}{Dt}+\left[1+\left(\frac{\partial\ln(\rho)}{\partial\ln(T)}\right)_p\right]\frac{Dp}{Dt}$$

The previous step we have derived the energy balance equation in terms of enthalpy.

$$\rho\frac{D\hat{h}}{Dt}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)+\frac{Dp}{Dt}$$

So, in the left hand side, we will substitute the substantial derivative of enthalpy, in terms of a the thermodynamics relationship, which we have seen now and that is what we will do now.

$$\rho c_p\frac{DT}{Dt}+\left[1+\left(\frac{\partial\ln(\rho)}{\partial\ln(T)}\right)_p\right]\frac{Dp}{Dt}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)+\frac{Dp}{Dt}$$

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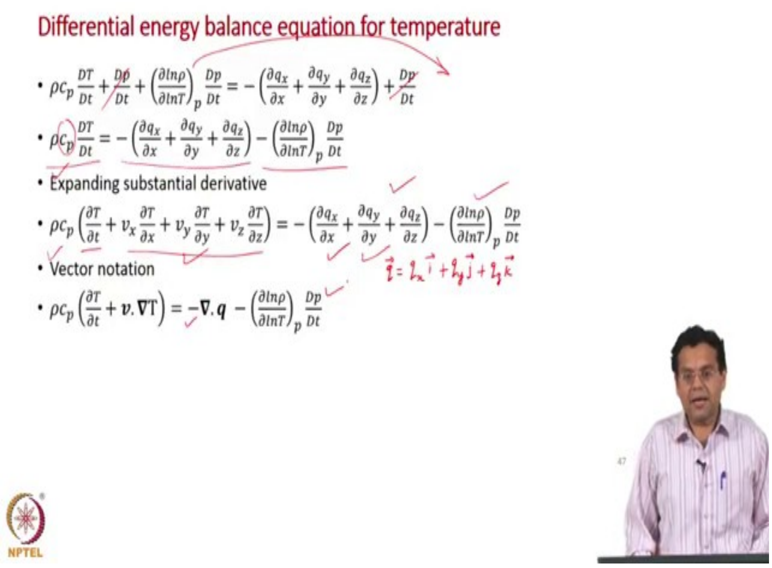

Now, let us simplify this and just rewriting the same equation here that is a left hand side and that is a right hand side. The right hand side is same as what we have seen earlier, in the differential energy balance equation for enthalpy.

$$\rho c_p \frac{DT}{Dt} + \frac{Dp}{Dt} + \left(\frac{\partial \ln(\rho)}{\partial \ln(T)}\right)_p \frac{Dp}{Dt} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_x}{\partial y} + \frac{\partial q_x}{\partial z}\right) + \frac{Dp}{Dt}$$

So, now, $\frac{Dp}{Dt}$ cancels on both sides and then we will express the equation in terms of $\frac{DT}{Dt}$ that has been our long standing objective. To get a differential energy balance equation in terms of temperature, that is what we have done sequentially.

$$\rho c_p \frac{DT}{Dt} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_x}{\partial y} + \frac{\partial q_x}{\partial z}\right) - \left(\frac{\partial \ln(\rho)}{\partial \ln(T)}\right)_p \frac{Dp}{Dt}$$

Also like to mention, this C_p is in terms of mass units, is on a mass basis. So, it is SI unit is Joule per kg per Kelvin.

Let us expand the substantial derivative, we already have derived the energy balance equation, we will by expanding the substantial derivative, we have the local term and the convection term.

$$\rho c_p\left(\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z}\right)=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)-\left(\frac{\partial \ln(\rho)}{\partial \ln(T)}\right)_p\frac{Dp}{Dt}$$

When we first time, when we introduce substantial derivative, we discussed in terms of velocity and in terms of temperature. Now, you really seen application or you have seen a conservation equation, where the $\frac{DT}{Dt}$ occurs.

Of course, you can also put an vector notation. So,

$$\rho c_p\left(\frac{\partial T}{\partial t}+v.\nabla T\right)=-\nabla.q-\left(\frac{\partial \ln(\rho)}{\partial \ln(T)}\right)_p\frac{Dp}{Dt}$$

So, we have finally, achieved our objective of deriving a differential energy balance equation, in terms of temperature. Now, we are in a situation similar to what we were after deriving the linear momentum balance equation, what is that situation let us see.

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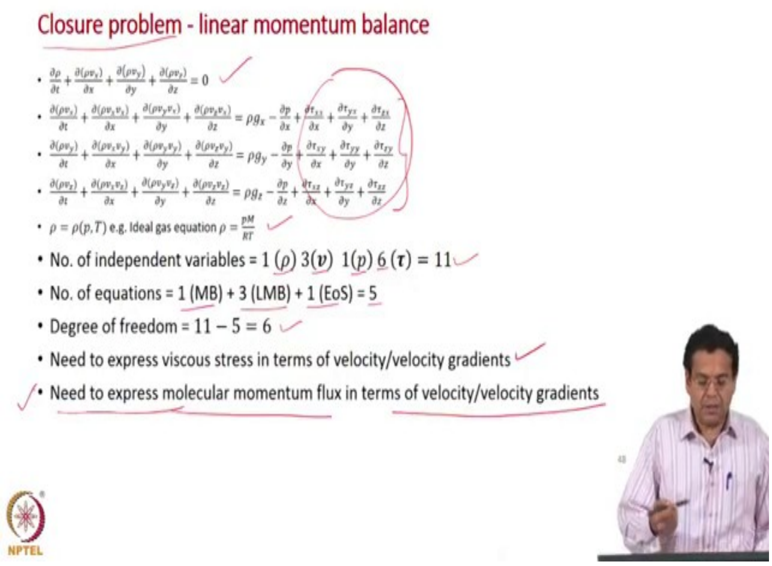

Let us have a recall slide. This is the slide which we discussed after deriving the linear momentum balance equation, we listed the continuity equation, we listed all the linear momentum balance equations in the three directions. We also express the equation of state as density as a function of pressure and temperature.

Then, we did a degree of freedom analysis, listed the number of independent variables, density is 1 variable, velocity components is 3 variables, pressure is 1 variable and then we have 6 independent components of the viscous stress tensor. So, they are another 6 variables and gave as 11 independent variables. Number of equations 1 mass balance, 3 linear momentum balance, 1 equation of state and which was number of equations was 5.

So, when you subtract we were left with 6 degrees of freedom (DOF = 11 – 5 = 6) and we concluded that the components of the viscous stress is are unknowns and to close the system of equations. We need to express those components in terms of velocity; velocity gradients and we call this as a closure problem. Now, we are exactly in the same situation now, later on, when we started the transfer phenomena part of the course. And, and we discussed momentum transport, we interpreted τ as molecular momentum flux.

So, if you want to restate this from that view point, we can say that need to express molecular momentum flux, in terms of velocity gradients, both are two different viewpoints. The second view point will be more useful because, we are discussing about heat flux.

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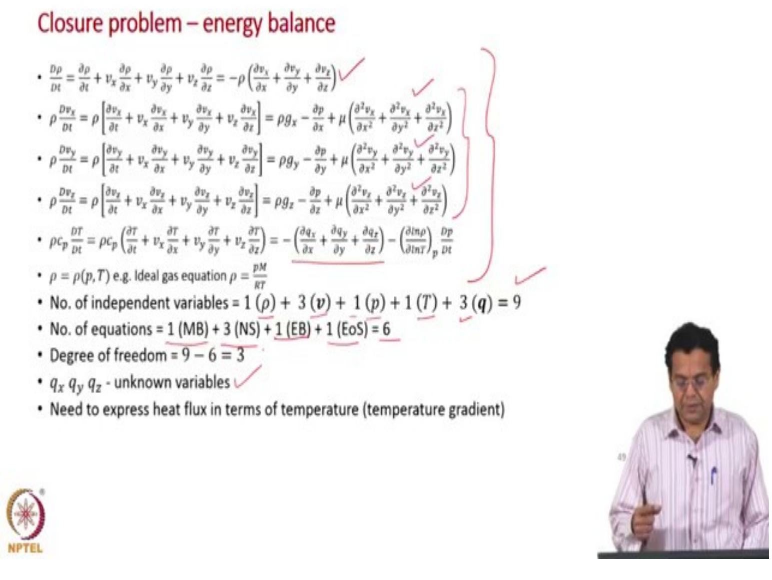

Now, we are in the similar situation now, let us see why is it. So, let us list down all the conservation equations, which are derived so far. We will express from a material particle view point.

$$\frac{D\rho}{Dt}=\frac{\partial \rho}{\partial t}+v_x\frac{\partial \rho}{\partial x}+v_y\frac{\partial \rho}{\partial y}+v_z\frac{\partial \rho}{\partial z}=-\rho\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)$$

$$\rho\frac{Dv_x}{Dt}=\rho\left[\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right]=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)$$

$$\rho\frac{Dv_y}{Dt}=\rho\left[\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right]=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)$$

$$\rho\frac{Dv_z}{Dt}=\rho\left[\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right]=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)$$

So, that is the equation of continuity, the linear momentum balance equation in the x, y, z direction, written in terms of substantial derivative number 1, number 2 right hand side, I have already substituted for the viscous stresses or the molecular momentum flux using the Newton's law of viscosity.

And, so what is shown here is the Navier-Stokes equation, not the linear momentum balance equation. And, that is the energy balance equation which you derived just now in terms of temperature;

$$\rho c_p\frac{DT}{Dt}=\rho c_p\left(\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z}\right)=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)-\left(\frac{\partial \ln(\rho)}{\partial \ln(T)}\right)_p\frac{Dp}{Dt}$$

We also require an equation of state expressing ρ in terms of pressure and temperature.

$$\rho=\rho(p,T)=\frac{pM}{RT}$$

So, these are all the equations which we have now. Let us do a degree of freedom analysis, what are the number of independent variables, like earlier we have

$$\textit{No of independent variables}=1(\rho)+3(v)+1(p)+1(T)+1(q)=9$$

ρ as 1 variable, 3 velocity components adds to 3 variables, pressure is 1 variable. Earlier we had the tau by 6 variable because, I have expressed in terms of the Newton's law of viscosity, they are not the variables now. And, what are the other variables, the energy balance has temperature unit so, that is another variable. Energy balance has these components of heat flux vector due to conduction and they add 3 components q_x, q_y, q_z, resulting in 9 independent variables.

$$No\ of\ equations = 1(MB) + 3(NS) + 1(EB) + 1(EoS) = 6$$

What are the number of equations? 1 mass balance, 3 Navier-Stokes equation ok, not linear momentum balance equation, we have already substituted the Newton’s law of viscosity. 1 energy balance equation, 1 equation of state, resulting in 6 number of equations. Let us subtract find the degree of freedom it is

$$DOF = 9 - 6 = 3$$

Which means that; the 3 variables are unknown, what are the 3 variables. The components of heat flux vector in the x, y, z direction are the unknown variables, q_x, q_y, q_z.

So, analogous statement just like we saw in the previous slide, what is that need to express the heat flux in terms of temperature, temperature gradient. That is why I said in the previous slide, the interpretation of τ in terms of molecular momentum flux is more useful now. And, we said molecular momentum flux has to be expressed in terms of velocity, velocity gradient, analogous statement now is need to express heat flux in terms of temperature, temperature gradient so, very much analogous.

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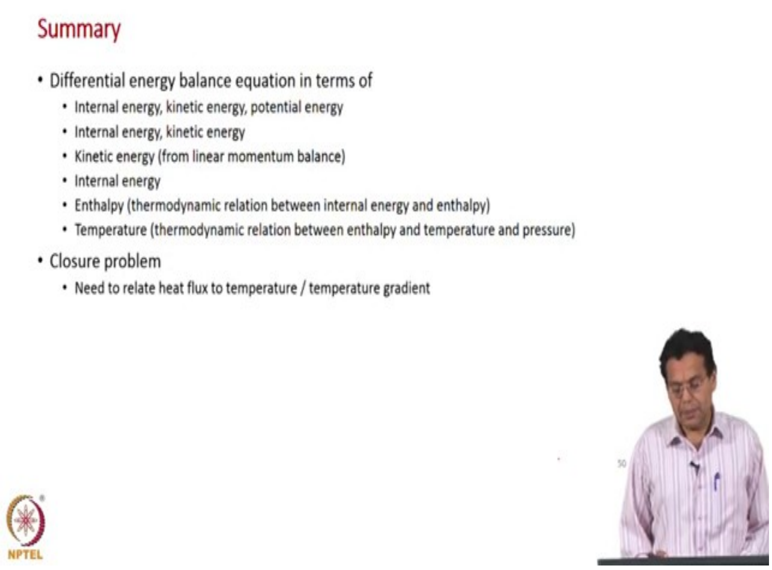

Let us summarize what we have discussed so far. We have derived the differential energy balance equation through a series of stages, first we expressed in terms of internal energy, plus kinetic energy, plus potential energy, which is what we discussed in the previous lecture. In this lecture, we took the effect of gravity from potential energy on the left hand side to work done on the right hand side.

So, that we could get a differential energy balance equation in terms of internal energy and kinetic energy, independently we derived an equation for kinetic energy from linear momentum balance equation. Subtracted these two and got a differential energy balance equation for internal energy. Used equation use thermodynamics relationship between internal energy and enthalpy and got equation for enthalpy.

And, then once again used the thermodynamics relationship between enthalpy and temperature and pressure and finally, arrived at the equation finally, arrived at the differential energy balance equation in terms of temperature. Towards end the closure problem, what is that need to relate heat flux to temperature, temperature gradient.