Continuum Mechanics and Transport Phenomena
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Lecture – 110
Differential Energy Balance – Part 2

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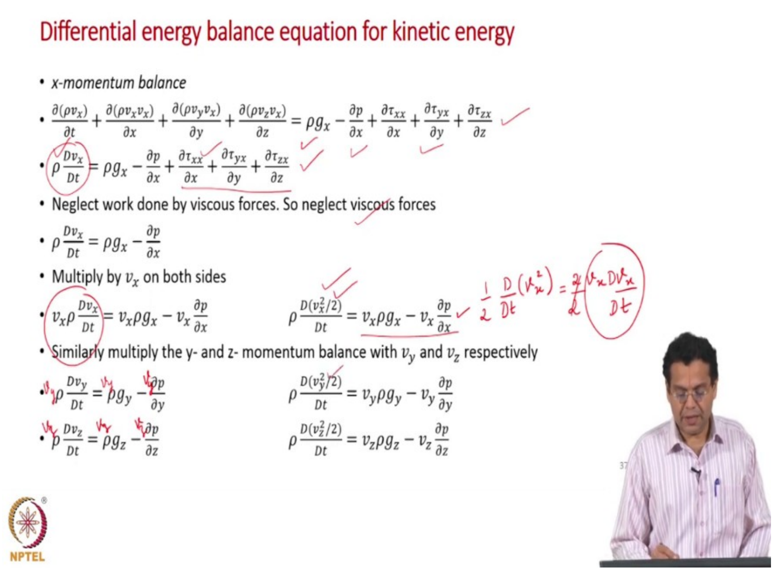

We have got one equation the Differential Energy Balance equation for internal energy and kinetic energy. What will now do is we will derive a separate differential energy balance equation for kinetic energy. That so that we can subtract that equation from this equation and. So, we are proceeding towards deriving a differential energy balance equation for kinetic energy. I said separate equation the reason is that, this equation I would say comes parallelly not sequentially from the previous equation.

Which means that, the starting point for this equation is different. The starting point is the linear momentum balance. So, let us write the x component of the linear momentum balance equation. And let us see, how do we derive their differential energy balance for kinetic energy from the linear momentum balance equation.

$$\frac{\partial(\rho v_x)}{\partial t}+\frac{\partial(\rho v_x v_x)}{\partial x}+\frac{\partial(\rho v_y v_x)}{\partial y}+\frac{\partial(\rho v_z v_x)}{\partial z}=\rho g_x-\frac{\partial p}{\partial x}+\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)$$

So, that is the linear momentum balance equation in the x direction; that transient term, the convection term. And, then we have the body first term, surface force due to pressure and viscous stresses. Now, as I told you the derivation of differential energy balance equation is easier, if we express the equations in terms of substantial derivative or from the material particle viewpoint. So, we will express the left hand side in terms of the substantial derivative of velocity multiply by density, which we have already done, right hand side remains same.

$$\rho \frac{D v_x}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right)$$

So, linear momentum balance equation form a Eulerian view point. This is from a Lagrangian view point. We have just now discuss that we have neglected the work done by viscous forces. So, in the linear momentum balance equation also to proceed further, we will neglect the viscous force on the right hand side. And keep only the body force and the surface force due to pressure. So, that it is in line with our assumption.

So, let us write down that linear momentum balance equation, excluding the viscous forces on the right hand side and that is nothing, but our Euler equation.

$$\rho \frac{D v_x}{Dt} = \rho g_x - \frac{\partial p}{\partial x}$$

So, we have taken the linear momentum balance equation expressed in material particle viewpoint, neglected viscous forces and we have got the Euler equation. Now, what we will do is multiply by v_x on both sides we will understand why do we do that shortly. So, we have

$$v_x \rho \frac{D v_x}{Dt} = v_x \rho g_x - v_x \frac{\partial p}{\partial x}$$

Now, the left hand side of this equation can be expressed as

$$\rho \frac{D\left(\frac{v_x^2}{2}\right)}{Dt} = v_x \rho g_x - v_x \frac{\partial p}{\partial x}$$

So, when you multiply the equation by v_x on both sides, the left hand side can be expressed as shown above. Now, let us repeat the exercise for the y direction and z direction, multiplying by the respective velocity components. So,

$$\rho \frac{D\left(\frac{v_y^2}{2}\right)}{Dt} = v_y \rho g_y - v_y \frac{\partial p}{\partial y}$$

Similarly the z direction as well,

$$\rho \frac{D\left(\frac{v_z^2}{2}\right)}{Dt} = v_z \rho g_z - v_z \frac{\partial p}{\partial z}$$

So, it is multiplied by respective velocities.

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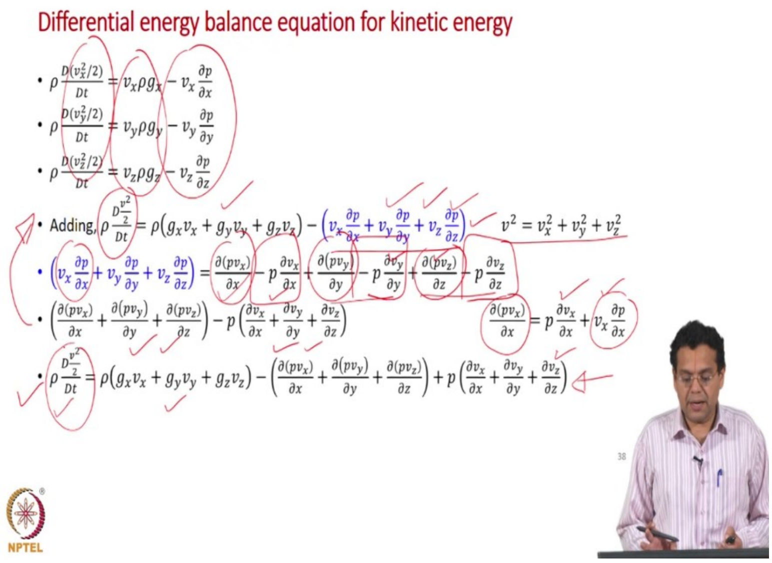

So, now let us write down the equations, which you have derived in the previous slide in all the three directions.

$$\rho \frac{D\left(\frac{v_x^2}{2}\right)}{Dt} = v_x \rho g_x - v_x \frac{\partial p}{\partial x}$$

$$\rho \frac{D\left(\frac{v_y^2}{2}\right)}{Dt} = v_y \rho g_y - v_y \frac{\partial p}{\partial y}$$

$$\rho \frac{D\left(\frac{v_z^2}{2}\right)}{Dt} = v_z \rho g_z - v_z \frac{\partial p}{\partial z}$$

Now, you can only start recognizing that $\frac{v_x^2}{2}$ is one part of the kinetic energy. So, if you sum all this equations what we have is $\frac{v_x^2}{2} + \frac{v_y^2}{2} + \frac{v_z^2}{2}$ which is nothing, but $\frac{v^2}{2}$; we have a half there. So, its means we have got the kinetic energy on the left hand side. That is why, we multiplied every we multiplied each equation with the respective to velocity component.

So, if you sum the left hand side becomes

$$\rho \frac{D\left(\frac{v^2}{2}\right)}{Dt} = \rho\left(g_x v_x + g_y v_y + g_z v_z\right) - \left(v_x \frac{\partial p}{\partial x} + v_y \frac{\partial p}{\partial y} + v_z \frac{\partial p}{\partial z}\right)$$

That is what our objective was. Our objective was to get a separate equation for the kinetic energy and so we have got a now a differential energy balance equation for kinetic energy.

Now, to proceed further and to make our derivation convenient, we will take this term and express as a sum of two terms. What do we do? Take the last term in right hand side and we will express in a form which will be convenient for further steps in the derivation. Let us do that and then I will explain what is that we have done.

$$\left(v_x \frac{\partial p}{\partial x} + v_y \frac{\partial p}{\partial y} + v_z \frac{\partial p}{\partial z}\right) = \frac{\partial\left(p v_x\right)}{\partial x} - p \frac{\partial\left(v_x\right)}{\partial x} + \frac{\partial\left(p v_y\right)}{\partial y} - p \frac{\partial\left(v_y\right)}{\partial y} + \frac{\partial\left(p v_z\right)}{\partial z} - p \frac{\partial\left(v_z\right)}{\partial z}$$

$$\text{¿} \left[\frac{\partial\left(p v_x\right)}{\partial x} + \frac{\partial\left(p v_y\right)}{\partial y} + \frac{\partial\left(p v_z\right)}{\partial z}\right] - p\left[\frac{\partial\left(v_x\right)}{\partial x} + \frac{\partial\left(v_y\right)}{\partial y} + \frac{\partial\left(v_z\right)}{\partial z}\right]$$

Of course should be obvious to you should look at it. What is that we have done now? Use the product rule, how to use the product rule? We know that

$$\frac{\partial\left(p v_x\right)}{\partial x} = p \frac{\partial\left(v_x\right)}{\partial x} + v_x \frac{\partial\left(p\right)}{\partial x}$$

What we have on the left hand side is $v_x \frac{\partial(p)}{\partial x}$. This is the term on the left hand side and that we expressed as

$$v_x \frac{\partial(p)}{\partial x} = \frac{\partial(p v_x)}{\partial x} - p \frac{\partial(v_x)}{\partial x}$$

That is all we have done. And similarly y direction; similarly z direction and now, we have grouped the terms here and then grouped rests of the terms as a second term. So, let us substitute them back here of course, we have minus here.

$$\rho \frac{D\left(\frac{v^2}{2}\right)}{Dt} = \rho\left(g_x v_x + g_y v_y + g_z v_z\right) - \left[\frac{\partial(p v_x)}{\partial x} + \frac{\partial(p v_y)}{\partial y} + \frac{\partial(p v_z)}{\partial z}\right] + p\left[\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z}\right]$$

So, left hand side is of course same, right hand side first term is same but right hand side instead of second term, we have now got two set of terms. So, just substituting this on the right hand side of the this equation.

So, now this is the differential energy balance equation for kinetic energy. Now, what we will do is look at the physical significance of each of the terms here.

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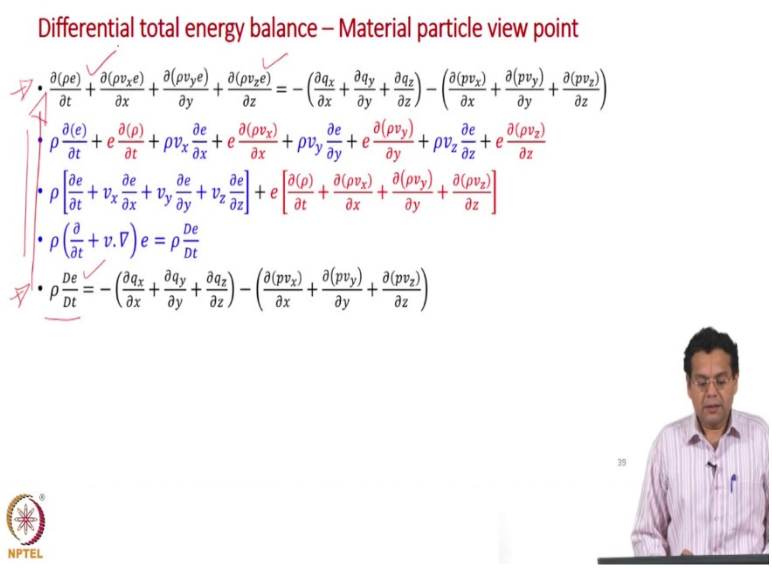

To help us to look at the physical significance of each term we will re-express this from a Eulerian view point just to help us a recall slide which we just now did.

$$\frac{\partial(\rho e)}{\partial t}+\frac{\partial(\rho v_x e)}{\partial x}+\frac{\partial(\rho v_y e)}{\partial y}+\frac{\partial(\rho v_z e)}{\partial z}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)-\left(\frac{\partial(p v_x)}{\partial x}+\frac{\partial(p v_y)}{\partial y}+\frac{\partial(p v_z)}{\partial z}\right)$$

What we did was, we had derived the differential energy balance from a control volume point of view or a Eulerian point of view.

$$\rho\frac{De}{Dt}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)-\left(\frac{\partial(p v_x)}{\partial x}+\frac{\partial(p v_y)}{\partial y}+\frac{\partial(p v_z)}{\partial z}\right)$$

So, now the left hand side was from a Eulerian point of view, we did a series of steps and expressed in terms of the substantial derivative. The situation what we have right now is the reverse of that, we have already the differential energy balance equation, the left side of that in terms of substantial derivative. So, we like to go back here and express in terms of the Eulerian view point.

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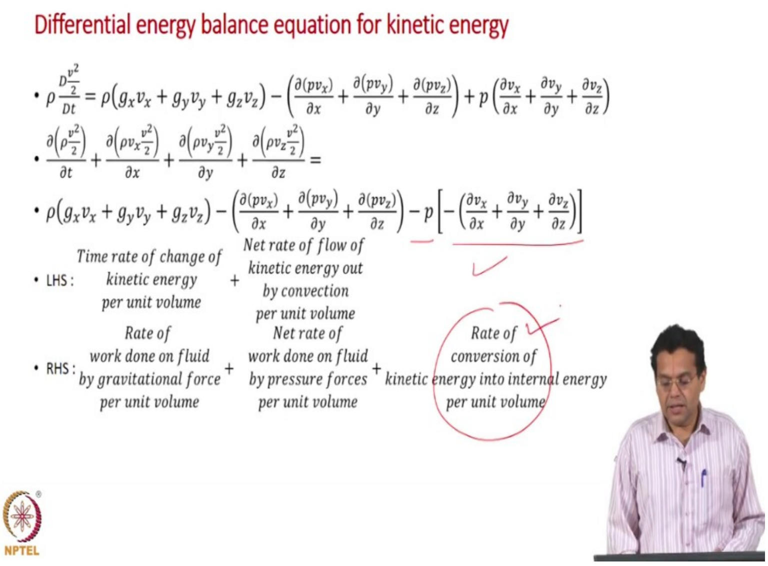

So, let us look at the next slide then you will understand.

$$\rho\frac{D\left(\frac{v^2}{2}\right)}{Dt}=\rho\left(g_x v_x+g_y v_y+g_z v_z\right)-\left[\frac{\partial(p v_x)}{\partial x}+\frac{\partial(p v_y)}{\partial y}+\frac{\partial(p v_z)}{\partial z}\right]+p\left[\frac{\partial(v_x)}{\partial x}+\frac{\partial(v_y)}{\partial y}+\frac{\partial(v_z)}{\partial z}\right]$$

That is the differential energy balance equation for kinetic energy, the left hand side is in terms of the substance derivative. We expresses the left hand side in terms of the transient and then the convection terms.

$$\frac{\partial\left(\rho \frac{v^2}{2}\right)}{\partial t}+\frac{\partial\left(\rho v_x \frac{v^2}{2}\right)}{\partial x}+\frac{\partial\left(\rho v_y \frac{v^2}{2}\right)}{\partial y}+\frac{\partial\left(\rho v_z \frac{v^2}{2}\right)}{\partial z}=\rho\left(g_x v_x+g_y v_y+g_z v_z\right)-\left[\frac{\partial\left(p v_x\right)}{\partial x}+\frac{\partial\left(p v_y\right)}{\partial y}+\frac{\partial\left(p v_z\right)}{\partial z}\right]+p\left[\frac{\partial\left(v_x\right)}{\partial x}+\dots\right.$$

That is a transient term and then the convection term. So, this equation the first equation is from a material particle viewpoint a Lagrangian viewpoint.

Second equation which is being written now is in terms of the Eulerian viewpoint. What is that we have done I like to quickly repeat it. We have got the equation for kinetic energy with the left hand side in terms of substantial derivative. What we are doing right now is just to get a physical interpretation for the for this differential energy balance equation that is only objective.

Now, earlier when we did for the total energy balance equation, the left hand side was from a Eulerian point of view. We express that in terms of Lagrangian point of view. What did what is that we have right now? We have already the substance derivative of kinetic energy per unit mass, we like to express back in this form.

This once again I want to emphasis, this is done just to look at the physical significance of all the terms in the different energy balance equation for kinetic energy. We will not carry forward derivation using this form of the equation. As I told you, the convenient form for derivation is only the Lagrangian form of equation.

Now, look at the significance of each term left hand side the terms are familiar to us. It is time rate of change of kinetic energy per unit volume. We have come across earlier mass, momentum then total energy. Now, we have come across kinetic energy.

What is the next set of three terms represent? They are the convection terms. So, they represent net rate of flow of kinetic energy out by convection per unit volume. We have seen this earlier for mass, momentum total energy now analogous meaning for kinetic energy. Now, let us look at the right hand side, we have already seen the significance of this in detail; tells rate of work done on fluid by gravitational force per unit volume. Now, we have also seen the significance of the second last term that is net rate of work done on fluid by pressure process per unit volume.

Now the last term is something which we have not come across, we will just write the significance and then few slides later we will justify the significances. It just tells you that rate of conversion of kinetic energy into internal per unit volume. We will few slides later we will discuss, why this term represents this physical significance which is conversion of kinetic energy into internal energy.

What this equation tells you that the kinetic energy can change because of all these factors. That is the overall concept which we can take from this equation. So, the terms are all analogous to any conservation equation, the transient term, the convection term on the left hand side and then we have rate of work done by gravitational force, pressure force and there is a term which tells about the conversion of kinetic energy to internal energy, which we will discuss little later.

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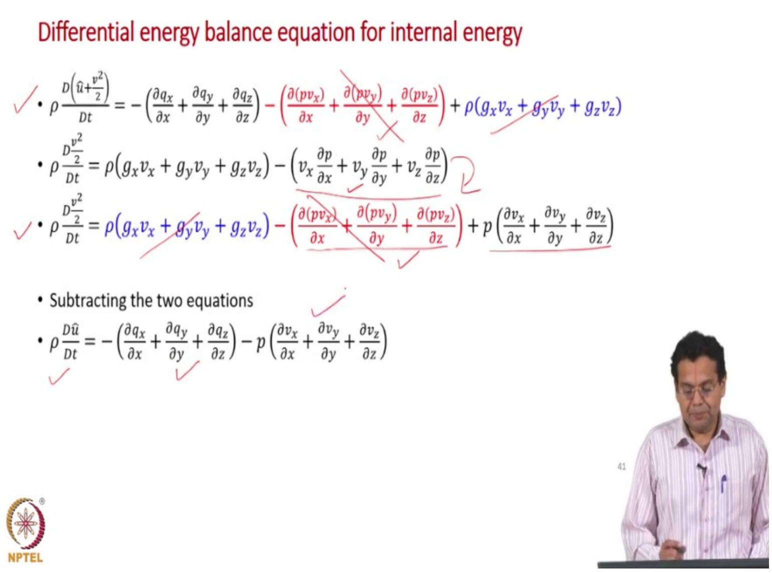

We have derived equation for internal energy and kinetic energy, which we derived from the total energy balance. And starting from the linear momentum balance, we have derived equation for kinetic energy. So, now, becomes very simple for us to almost write down the energy balance equation for internal energy.

So, let us do that here simple subtraction. So, we are back to the Lagrangian viewpoint.

$$\rho \frac{D\left(\hat{u}+\frac{v^2}{2}\right)}{Dt} = -\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right) - \left(\frac{\partial (p\,v_x)}{\partial x}+\frac{\partial (p\,v_y)}{\partial y}+\frac{\partial (p\,v_z)}{\partial z}\right) + \rho\left(g_x v_x + g_y v_y + g_z v_z\right)$$

So, all our left hand sides once again will be in terms of $\frac{D}{Dt}$. Only just two slides in between to discuss the significance of different terms in the kinetic energy equation, we went back to the Eulerian description ok. That is the differential energy balance equation for internal and kinetic energy.

$$\rho\frac{D\left(\frac{v^2}{2}\right)}{Dt}=\rho(g_x v_x+g_y v_y+g_z v_z)-\left(v_x\frac{\partial p}{\partial x}+v_y\frac{\partial p}{\partial y}+v_z\frac{\partial p}{\partial z}\right)$$

And this is the equation which we obtained for the kinetic energy from the linear momentum balance equation. And then we took the last term and expressed as sum of two sets of terms. And now, we will understand why we really did that.

$$\rho\frac{D\left(\frac{v^2}{2}\right)}{Dt}=\rho(g_x v_x+g_y v_y+g_z v_z)-\left[\frac{\partial(p v_x)}{\partial x}+\frac{\partial(p v_y)}{\partial y}+\frac{\partial(p v_z)}{\partial z}\right]+p\left[\frac{\partial(v_x)}{\partial x}+\frac{\partial(v_y)}{\partial y}+\frac{\partial(v_z)}{\partial z}\right]$$

So, let us subtract this equation from this equation, the equation for kinetic energy from the equation for internal and kinetic energy.

$$\rho\frac{D(\hat{u})}{Dt}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)-p\left[\frac{\partial(v_x)}{\partial x}+\frac{\partial(v_y)}{\partial y}+\frac{\partial(v_z)}{\partial z}\right]$$

And so the left hand side we will be left out with the substantial derivative of internal energy of course, multiplied by density. So, now we have got a very simple equation.

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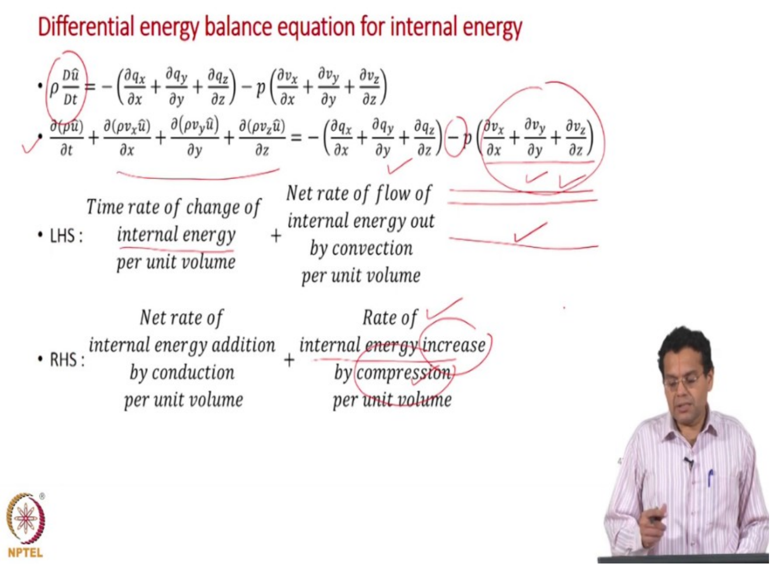

Now, let us look at the significance of the different terms in this energy balance equation for internal energy.

$$\rho \frac{D(\hat{u})}{Dt} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_x}{\partial y} + \frac{\partial q_x}{\partial z}\right) - p\left[\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z}\right]$$

Like we did for kinetic energy, we will express the left hand side back in terms of the Eulerian viewpoint. So, that we can explain the significance of each term, with this also we can explain you put in this form because we have explained the significance of all the conservation equations earlier in this form only that is why to be in line with those explanations, we are expressing the left hand side or the entire equation back in Eulerian form.

$$\frac{\partial(\rho\hat{u})}{\partial t} + \frac{\partial(\rho v_x \hat{u})}{\partial x} + \frac{\partial(\rho v_y \hat{u})}{\partial y} + \frac{\partial(\rho v_z \hat{u})}{\partial z} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_x}{\partial y} + \frac{\partial q_x}{\partial z}\right) - p\left[\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z}\right]$$

So, now, let us take the left hand side I think you should be able to write down the significance now very easily. First terms tells about time rate of change of internal energy per unit volume. Next set of three terms represent convection. So, net rate of flow of internal energy out by convection per unit volume. Right hand side, we are familiar with this term. Net rate of internal energy addition by conduction; what it tells you is because of conduction there is increase in internal energy.

Or because of heat input there is increase in internal energy that is very much understandable. Now, let us come to the last term which requires some discussion. You also discuss the importance of this term here and in the kinetic energy equation. What this tells you if you read this significance rate of internal energy increase by compression per unit volume. How do you explain this? For that, we know this term as $\frac{\partial(v_x)}{\partial x}+\frac{\partial(v_y)}{\partial y}+\frac{\partial(v_z)}{\partial z}=\nabla.v$, but remember we also discuss this as a volumetric strain rate.

What does it tell you? Fractional rate of change of volume; so now, let us say there is a decrease in volume what happens this term is negative and this will become positive and that will result in an increase in internal energy that is what this tell you. When you say there is decrease in volume it is which means that we have compression. So, rate of internal energy increase by compression of course per unit volume.

So, just to repeat the last three terms put together represent fractional rate of change of volume. And if there is compression there is decrease in volume that term is negative. So, the entire term becomes positive, it adds to the internal energy and that is why this term tells you rate of internal energy increase by compression per unit volume.

Now, let us look at the corresponding term in the kinetic energy equation, if you look at the two terms here. Here it is $-p(\nabla.v)$ and here it is $-p(-\nabla.v)$. Let us take the same case. Let us say there is decrease in volume $\nabla.v$ term is negative this entire term is positive. Now, what will happen in the right side of the kinetic energy equation it will be negative which means that, the increase in internal energy is because of decrease in the kinetic energy that compression adds to the right hand side of the internal energy equation.

It results in subtraction in the right hand side of kinetic energy equation, which means that there is a conversion of kinetic energy to internal energy. Now, not alone this the other way can also happen. What is that? Let us say there is expansion. What will happen? This $\nabla.v$ will be positive. So, the whole term will be negative in the internal energy equation. So, it will result in reduction in internal energy correspondingly there will be increase in the kinetic energy. So, now, this term becomes very clear rate of conversion of kinetic energy into internal energy per unit volume.

We can explain the significance only after deriving the equation for internal energy that is why we postponed the explanation for that particular term. Also like to mention that because you can have either compression or expansion and so there could be conversion; conversion of these two energies in both the directions. That is what we have seeing. It could be internal energy to kinetic energy or it could be kinetic energy internal energy.

So, more formally we call this as reversible conversion of kinetic energy to internal energy. So, what you have seen now is arrived at the differential energy balance equation for internal energy, express that in Eulerian form and then looked at the significance of the different terms in that equation for internal energy. And, also specifically one term which appears in the internal energy equation and the kinetic energy equation with opposite sign, we have discussed the physical significance of that term as well.