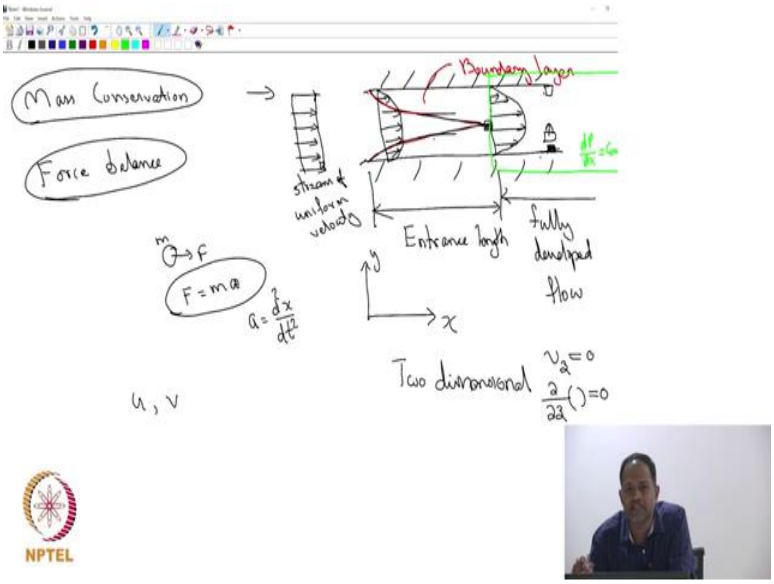
Fluid Mechanics
Prof. Sumesh P. Thampi
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture – 08
Solutions of Navier Stokes – 1

Look at some simple situations to start with so, that you all get into the flow. So, the first thing that we are going to do today is to look at an exact Solution of Navier Stokes equations.

(Refer Slide Time: 00:29)



So, what you have got is fluid between two parallel plates ok. So, what you have got is two plates ok; two solid plates and then there is fluid in between ok. And then we are saying that there is a fluid flow that is going to come from this side and it is going to enter this channel and then go and come out of this channel ok. And let us say that the fluid that is going to come from the left side of the screen is going to be a uniform flow, or the velocity profile is going to be a uniform; velocity profile is uniform. So, what do I mean by velocity profile? Can somebody say what is velocity profile?

So, velocity variation ok; so, when you want to plot velocity as a function of distance that is what you call as velocity profile. So, when I say the velocity profile is uniform or the velocity is uniform, the way I could represent it is that for example, I will say that the velocity which basically says velocity is a vector. So, it has two things. It has got a magnitude, it has got a direction neither of them are changing in space. So, there is a fluid that is coming from the left with a constant velocity. So, the typical way to do it is actually that way ok.

So, two parallel lines with there is nothing changing in between its approaching the plate. And what would happen when this fluid touches the tip of the plate; tip of the channel? So, I am talking about this region. So, if I think about a fluid element that is there, it is going to come like that and touch the tip. What would you expect to happen? It would slow down right. So, it would slow down so; that means, the velocity right there is going to go to 0; the velocity right there is going to go to 0, but the velocity anywhere else would remain as whatever it had come with right.

So, then so, you could say that if I take a little bit ahead in this channel. So, the fluid is coming from this side so, if I take let us say you know a point there; this point would be at what velocity? With the velocity would be same as the velocity with which it came or the velocity would be smaller.

Smaller right. So, you could say that if I take this point and let us say this point the velocity profile would look something like that. So, what do I mean by that? So, here in fact, I should draw it a little bigger. So, that is going to be my velocity profile. So, the region which is very close to this wall, the fluid is going to see the effect of this lower wall and is going to be slowing down. The fluid above you know in this region is going to see the effect of this wall and is going to slow down, but in between it may not have.

So, if I say that this is what it is then as you know it progresses this region where the influence of the wall is seen becomes thinner and thinner ok. And finally, so this region basically will thin, thin, thin and at some point it will disappear or in other words the even a fluid element like this which is at the center of the channel would have seen the effect of two walls. Now, therefore, it would develop a velocity profile from then onwards the velocity profile would be slightly different like that.

You see what I am saying. So, the fluid which has come with a constant velocity enters and then the fluid experiences the friction from the wall ok. And how do you quantify it in terms of any of the fluid properties?

Student: Viscosity.

Viscosity ok. So, you could say that because the fluid is viscous. It has a viscosity, it is going to be seeing the friction from the wall and it will slow down and that slowing down is what is going to result in this particular profile. Is that clear? So, there is; so you could say so, this is a stream of uniform velocity, this region where the fluid was developing its velocity field ok. So, it entered like that and it finally, became like this that region is known as entrance length ok.

So, that is the entrance length and the region from here onwards is known as fully developed flow. Is that clear, any ∂bts? This I have drawn this line right, I did not say anything about that. What does that line indicate? That is really a boundary of the boundary in which the fluid is getting affected by the solid ok. There is a name for that and that is known as boundary layer and we will talk about boundary layer in detail at a later point. So, right now what we are going to do is we are going to look at only the fully developed flow.

So, we will define a coordinate system, let us say that is your x direction, that is your y direction and z is perpendicular to the screen and we will assume that your flow is right now two dimensional. What is what do I mean by the flow being two dimensional? So, there is no velocity perpendicular to the screen that is in the z direction and there is no variation in the z direction. So, what do I mean by that? v is that the z component of velocity is 0, there is nothing that is going to change in the z direction ok.

So, I do not care about my z direction anymore I am going to restrict all my analysis in two dimensions, I only have an x component and the y component alright. So, now, the question is if I want to find out let us say this profile, how do I proceed and you also you told me that you know that you have to solve Navier Stokes equations to find out what this velocity profile going to be. So, let us just do that. So, Navier Stokes equations consists of two parts ok. The first is about mass conservation which is what you called as continuity equation on that day ok; the second is about a force balance ok.

So, you have two components; one is mass conservation and then you have a force balance. So, the mass conservation ok; you know that it is one of the fundamental principles and then whatever flow whatever mechanics that you are going to talk about, it should be obeying the mass conservation. So, somebody that they said it is A1 V1=A2 V2 right so; that means, whatever fluid is going to enter your domain has to get out of the domain, there is no escape unless there is something else going on like you know some reaction some fission fusion and so, on the mass conservation is always going to be obeyed ok.

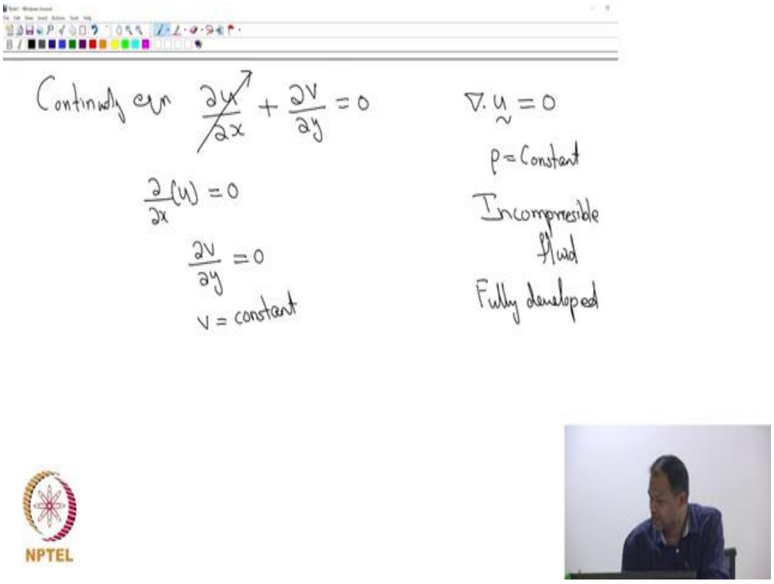
So, it is a one fundamental principle and it has to be obeyed by the fluid flow. So, whatever you do, you have to respect that it is going to obey the mass conservation. Second is that you are going to talk about something that is moving. Why does it move? Why would the fluid that is coming from this side is going to move to the other side? Because you must have applied some force. Unless you have applied a force, it is not going to move right and the simple way you would have known that is equal in terms of Newton's second law. What is Newton's second law? Force is equal to mass into acceleration that is all what we write in the force balance.

So, we are saying that the fluid is you know obeying its mass conservation, one of the fundamental principles that it has to respect, second is that it is obeying Newton's law which is force is equal to mass into acceleration. And all that you have done in your continuum mechanics is you have written force is equal to mass into acceleration in terms of a differential equation. So, force is equal to mass into acceleration was really a particle and you are applying a force. And if the particle had a mass m, you would have written that force as:

$$F = ma = m.\frac{d^2x}{dt^2}$$

Now, when we talk about fluid, the fluid is there everywhere in the domain. We can't talk about such particles or otherwise, we need to do this force is equal to mass into acceleration for every particle in the fluid and things might be changing in x direction; things might be changing in y direction. So, by writing the force balance in a differential form, you have really generalized this equation. So, all you are doing is mass conservation and force balance. Is that clear? So, now, let us write down the equations in the differential form ok.

(Refer Slide Time: 12:16)



So, continuity equation all of you wrote it down the other day. So, let us write it again continuity equation and as we said, we are going to deal with two dimensions.

So, we will only worry about the x component and the y component. Let us say u and v are the x component of velocity and the y component of velocity respectively, then continuity equation says:

$$Continuity\ equation: \nabla . \vec{V} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

So, that day somebody interpreted divergence as you know something a net, you know flux of something through ok. So, the fact that there is no net flux is what is giving rise to divergence of u being 0 so; that means, we are assuming something and we are assuming that what property is constant.

Student: density.

Density is constant so; that means, we have already assumed that we are going to look at the constant density fluid which you also call as incompressible fluid.

So; that means, in general if you want to talk about a system in which the density is changing; for example, you want to model a reactor and then the reaction is happening so;

that means, at every point in space the density is going to change, then you have to worry about a general equation or it could be that the temperature is varying. Let us say you are talking about a heat exchanger and you are looking at a heat the fluid which is going through one of the tubes in heat exchanger. The temperature is going to change along the length of the tube; that means, the density is going to change and you have to worry about the general form of this equation which of course, you know what it was.

So, right now we are restricting ourselves to a simple situation. So, we have $\nabla . u$ is equal to 0 so, that is the case. Now, we are only going to look at this fully developed part ok, we are not going to right now worry about what is happening in the entrance length. So, by definition fully developed flow is something in which things are not changing in the x direction. All of you agree with that that is by definition. So, we are going to incompressible flow and fully developed.

$$Fully\ developed\ flow\colon \frac{\partial u}{\partial x} = 0$$

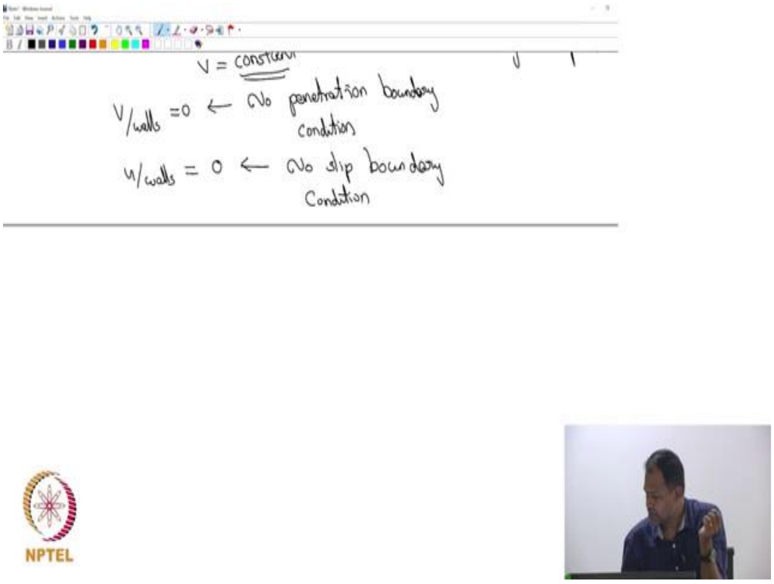
So from my continuity equation:

$$\frac{\partial v}{\partial y} = 0 \rightarrow v = constant$$

What is v? The velocity of the fluid perpendicular to the plates right. So, the velocity of a fluid any fluid element that is perpendicular to the plate ok; now and we are saying that is a constant. So, does; that means, that if I talk about if the fluid element let say fluid element here, the velocity in this direction is what we called as v and if that is constant everywhere in the domain, then what should that constant be? Why should that be 0? Because there is nothing that can come from the wall to outside.

So, any fluid element that you are going to look at on the wall should also have that constant velocity and we know that that velocity is 0 at this wall. We know that velocity is 0 at that wall; that means, the velocity there cannot be a velocity that is normal or that is normal to the walls ok. So, what is that boundary condition? So, that is really a boundary condition and that boundary condition is called.

(Refer Slide Time: 16:52)



So, the fact that v at walls is equal to 0 is called no penetration boundary condition. So, this would change, let us say if you were talking about a porous plate or let us say you want to do a filtration setup and you want to talk about fluid that is flowing on a you know a membrane, ok. Then there will be things that are coming from one side to another and then this condition will break down. So, why do you need a boundary condition? Because we are solving differential equations which will be when where and you need to so, you will be ending up with constants; just like this constant that we ended up with. And we can find those constants with the help of boundary conditions ok.

Now, there will therefore, be a boundary condition about u also right. What is the, what is the value of u? So, what is u? The component of velocity parallel to the wall. What would that be? So, why is that 0?

So, any fluid element that you are going to again look here; so, if you look at this fluid element the fluid element will essentially be sticking to this wall that fluid element is not going to go anywhere, you know that that is a physical situation. So, this is no slip boundary condition is just a mathematical representation of saying that u the component of velocity parallel to the wall is 0 and you call it no slip boundary condition. Alright, any questions?

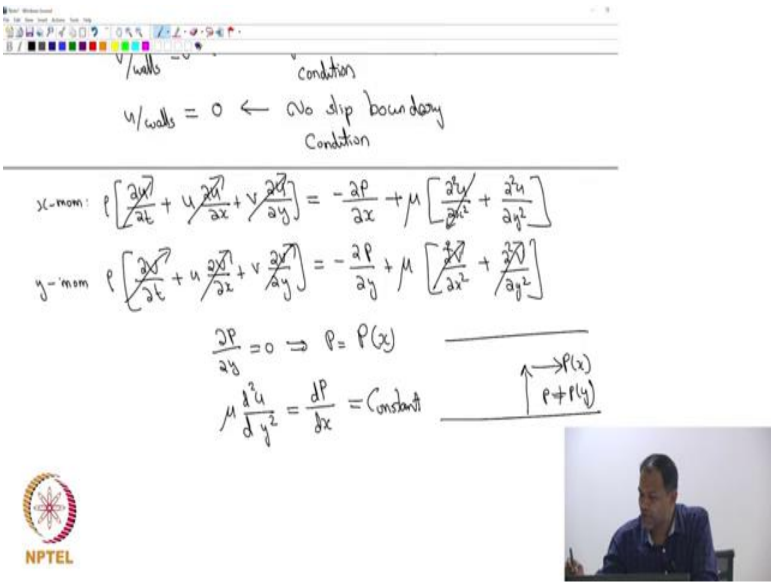
$$v_{wall} = 0\ (No\ penetration\ BC)$$

$$u_{wall} = 0 \; (No\ slip\ BC)$$

So, we have now learned something that the velocity normal to the channel is 0. So; that means, there is only one particular component of velocity that we need to worry about and that is the u velocity ok. There is just one component and of course, that is what we plotted also, that there is only one component of velocity that is parallel to the channel wall and we want to now find out that and to find out that. So, this much we have learned only from the mass conservation or the continuity equation.

So, now, let us take you know utilize the force balance and see what more we can get. So, we are going to write.

(Refer Slide Time: 19:43)



Momentum equations:

$$x\ momentum: \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$y\ momentum: \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial P}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

So, we know that v is 0. So, that makes our y momentum equation very simple almost everything goes away except pressure because we have not said anything about the pressure. And the reason why we did not say anything about the pressure is because we did not say anything about the force so far because pressure is nothing, but force per unit area right. So, it will come only in a force balance, all our analysis was about continuity equation that is why pressure never came into the picture. So,

$$\frac{\partial P}{\partial y} = 0 \rightarrow P = P(x)$$

So, in our channel pressure is not going to change in that direction, pressure can change in this direction. So, pressure is a function of x; pressure is not a function of y. So, like pressure. so, pressure is not a function of y ok. So, that is done. Now so, that was the use of writing down the y momentum equation even though the v component itself was 0 still you should have analyzed it because it is not just about velocity, it is also about knowing pressure. So, when we say that you know you want to solve a flow problem, the thing is that you are actually trying to find out both velocity field and pressure field ok.

So, when you say you want to talk about fluid flow in some you know, in some domain in some geometry anywhere and you want to talk about whether you know everything about the flow. The description becomes complete when you talk about both pressure and velocity, because both pressure and velocity can change with the space and time. Was there a question? Any questions? So, yeah so, we have now figured out that pressure is only a function of x. Now let us see what does x momentum equation tell you. So, we have a fully developed flow right, that is what we said. So, would u be function of time?

No, u is not a function of time. How about u do u by ∂ x? It is 0 because ∂ u by ∂ x is 0, u is not changing with x v ∂ u by ∂ y. It is 0 no because v is 0 ok. So, v is 0. So, that is also gone and actually the simplification that you see here is very important because if you look at this entire equation, these are the terms that are like product terms, u into u its like u square this is like u into v ok. These terms are really non-linear terms and you know how difficult it gets when you have to solve non-linear equation. Do you know any method of solving non-linear differential equations? Everything that you would have learned would be about linear differential equations ok.

So, you can use all of them when you have so, we you really have governing equations as linear differential equations ok. So, whenever you have a system in which these non-linear terms go away, you can actually expect to solve things analytically otherwise it becomes difficult ok. So, this problem is a nice problem because this has already thrown away all the non-linear terms. And therefore, you can expect to get linear equations and therefore, you should be able to know you are you should be able to utilize your mathematics to solve the system. Is that clear? Ok. So, that is there, ∂ p by ∂ x anyway we do not know anything, but we know that P is a function of x. So, we have that then d square u by dx square 0. So, that is also gone d square u by dy square, it is non-zero and that is what externally we want to find out.

So, we find that:

$$\mu \frac{d^2u}{dy^2} = \frac{dP}{dx}$$

Now, if you look at this so, what does this tell you? Can you say what this could be? Anything more? Can this be a function of so, if I write this f should I write it as a function of x or y? Or x? So, this part says that u is not a function of x, u is only a function of y. This part says that P is only a function of x, not a function of y. So, the most I can expect is this is a function of x, this is a function of y; a function of x can be equal to a function of y only if that is a constant. So, there is nothing possible like f of x comma y, the best that you can say is that this is going to be a constant in this system. Agreed?

$$\mu \frac{d^2u}{dy^2} = \frac{dP}{dx} = constant$$

So, that is going to be useful because now I can integrate this equation ok. Now what is it saying? dP/dx is a constant so; that means, if I go back to our original system ok, we are looking at regions from here onwards ok, that is the region of interest; in that region dP/dx is a constant ok. So, P is going to change along the x direction and it’s only going into whatever variation is there, that variation is going to be constant. Which side will be pressure more? Pressure will be more on this side or that side? Pressure will be more on this side because that is why fluid is moving.

So, you have applied pressure on this side and then that fluid in between experiences that pressure and is going to go there. So, this is where pressure is going to be larger inlet pressure is going to be larger than the outlet pressure and it is just going to vary linearly. In fact, because dP/dx is a constant; that means, P is a linear function of x, P will just change linearly with x. Clear?