Fluid and particle mechanics
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Lecture - 07
Bernoulli's Equation

So, in the last few classes we have been trying to take an approach based on differential equations to understand fluid motion. We only derived the governing equations, one of them the fact that you know mass is conserved there is a mass balance; we express that as a differential equation. The second one was the momentum balance or force is equal to mass into acceleration, we represented that fact also as a differential equation.

Now, we should learn how to use these equations for the kind of problems that we want to analyze, but before that there is one more you know balance that we often use and that is about energy right the energy is conserved or there is an energy balance. So, one can use that fact also and write down an equation, that can be quite useful. Now the we will not do it as rigorously as we did for the mass balance and for the momentum balance, we are going to only do it so, that it looks very similar to something that you are already familiar with and that is Bernoulli's equation ok.

So, we are going to see how energy conservation is nothing, but similar to what you already know the Bernoulli's equation and how does it get modified really for a real fluid flow.

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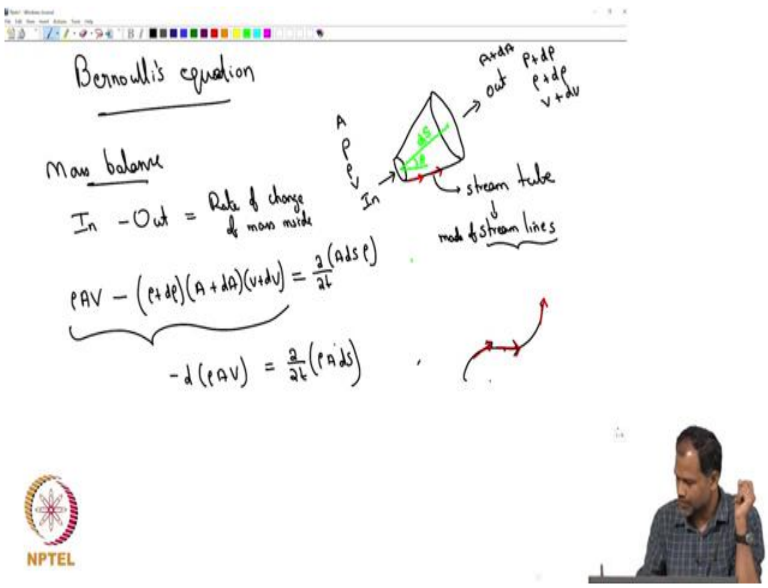

So, we are going to now look at Bernoulli's equation we will try to derive that ok. So, let us say you have some flow field so; that means, there is a velocity, there is a pressure and let us also in general assumed that you know the density is also changing. So, that we don't have to restrict ourselves for incompressible fluid flows and let us say we will consider a differential element.

So, for this particular one we are going to really write down just energy balance as a 1D equation ok. So, we will not consider therefore a cubic element, we will consider something simpler let us say a fluid element which is made of which is actually a stream tube I will explain what is a stream tube. Let us say that is a stream tube stream tube is something that is made up of stream lines. So, you take the surface of the stream tube, the surface of the stream tube is made up of stream lines are made of stream lines what are stream lines?

Stream lines. So, if you know a stream line and if you draw a tangent on this on the stream line, that is actually going to give you the velocity field. So, let us draw some stream line let us say that is a stream line, if you take a particular point and let us say a tangent there, that is actually going to give you a velocity field. If I take another point and draw a tangent there. So, let me draw the tangent of the different color red ok. So, that is going to give me the velocity there if I have here that is going to give me the velocity field here ok. So, stream line is essentially connected by those velocity vectors. So, if there is a change in the direction of the flow stream line will show that. So, you can draw stream lines at every point in the flow field. So, stream tube is basically a surface made up of streamlines. So, for us; that means, that if this is what the stream tube is if I take any point and draw a tangent that is basically going to be the direction in which the fluid is flowing. There cannot be a fluid motion that is perpendicular because by definition stream line is the one which is drawn such that the velocity is tangential to it ok.

In other words if you consider a stream tube because there you cannot that is made up of stream lines, there cannot be a flow from In or Out. So, there is a flow that is going to go In and there is a flow that is Out. So, this is in that is OUT. So, this is like really a tube and then there is fluid that is coming from the left side and then the fluid is going to go outside, but on the surface because its made up of stream tube there is no flow through the surface. So, its the stream tube is as if its like a rigid object for that matter because there is no flow that is going to go through the stream tube, but its really made up of fluid element ok.

So, that is the stream tube that we are considering. Let us say this side the pressure is p density is ρ velocity is v and outside its P plus dp is the pressure ρ plus d ρ is the density v plus d v is the velocity and there is a change in area. So, there could be a change in area because fluid may not just go straight fluid probably would be expanding and flowing like that the fluid might be flowing like that. So, you could you have a inlet area let us say A and let us say the outlet area is A plus dA. So, this is the system under consideration.

$$In: P, \rho, V, A$$

$$Out: (P + dP), (\rho + d\rho), (V + dV), (A + dA)$$

And what we will do is we'll write down a mass balance and a momentum balance for this particular fluid element that we have considered. So, we already know how to do it, we can say that the mass In minus mass Out should be equal to rate of change of mass in the system. So, if you think about mass balance. So, mass In minus mass Out has to be equal to rate of change of mass inside right. So, what is the mass that is flowing into the system that is going to be volumetric flow rate times density, but volumetric flow rate is nothing, but the area times velocity. So, ρ times A times V is the mass that is growing In minus the mass that is coming Out is going to be ρ plus d ρ times A plus d A times v plus dv and that has to be equal to the rate of change of mass which is ∂ by ∂ t of the mass in that area.

So, let us say the, so, sorry in that volume. So, the volume is given by some area times particular length. So, let us mark the length also, let us say from the center to here the length is given by d s and also let us say that this is actually at an angle θ with respect to the horizontal. So, the rate, so, that the volume of this is going to be approximately equal to A times d s times density is going to be the rate of change of mass in the system. So, this is nothing, but really change in a ρ into A V. So, let me rewrite it as:

$$\rho AV - (\rho + d\rho)(V + dV)(A + dA) = \frac{\partial}{\partial t}(\rho AdS)$$

$$d(\rho AV) = \frac{\partial}{\partial t}(\rho AdS)$$

$$\frac{\partial \rho}{\partial t} AdS = -d(\rho V)A$$

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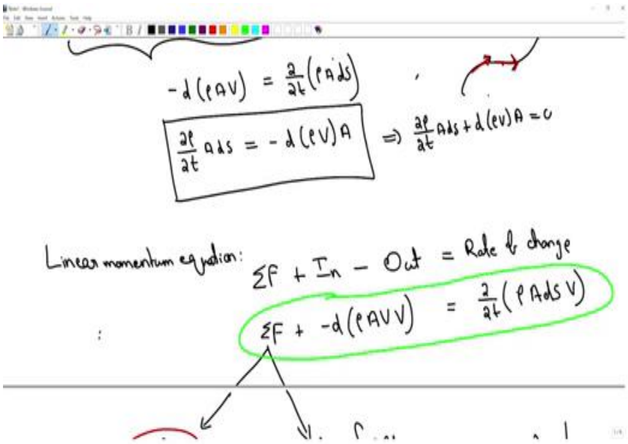

So, A is a particular area that you have considered and d is the total length of the differential length of the fluid element that you have considered ok. So, that is what the mass conservation equation gives you.

So, now we will write down the linear momentum equation. So, linear momentum equation is nothing but again momentum In minus momentum Out has to be equal to the rate of change of momentum, but we also know that the rate of change of momentum can come from forces. So, we could write that In momentum minus Out momentum has to be equal to rate of change, but that rate of change can also come from all the forces acting in the system. Remember there is a fact that we used to write down the momentum conservation equation.

$$Linear\ momentum\ equation: \sum F + In - Out = Rate\ of\ change$$

So, for this particular fluid element that we are looking at ok. So, there is a force there is a force that is coming from pressure, there is gravity that is acting and then there could be viscous forces and then we can also calculate what is the momentum that is going In what is the momentum that is going Out ok. So, let us start with the right hand side the rate of change of momentum can be written as ∂ by ∂ t of momentum of the fluid. So, that is mass ok.

So, that is then density into volume times v. So, that is the rate of change of momentum in that volume, that is equal to the differential of the momentum in minus a momentum out. So, just like the mass conservation we can write it as:

$$\sum F - d(\rho AVV) = \frac{\partial}{\partial t}(\rho A\, dS\, V)$$

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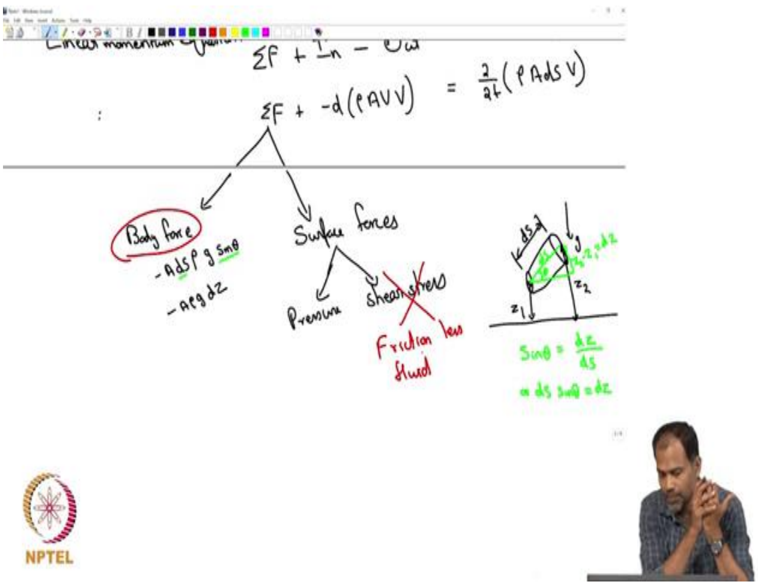

Now, here and we need to write down what are the forces one obvious force is the body force ok. So, the body force which is coming from gravity. So, what will be the body force is nothing, but just the weight of the system. So, here we have A as the inlet area d s as is the length and what and then gravity is acting downwards only thing is that the. So, gravity is acting downwards, but the element is actually you know up like that in at a particular angle. So, we need to find out the component of gravity that is acting along the d s direction ok.

So, this is the direction that we are trying to see now, its the direction ok. So, we are writing down the force balance in that direction. So, in the direction of that pink arrow. So, we have to find out what is the way what is the component of gravity, that is acting along it and that is going to be nothing, but so, because gravity is acting downward ok. So, that is the direction in which gravity is acting. So, its going to be g sin θ that is going to act in that particular direction. So, body force is going to be area times length, A ds is going to give you the volume times ρ the volume times density that is the mass times g sin θ and its acting.

So, g is acting downwards. So, its minus g sin θ, because its acting at an angle θ with respect to the horizontal now. So, let me make a smaller diagram here. So, that is the direction in which g is acting, that is a kind of fluid that we were really looking at and we said that is nothing, but d s right the length of the element. Let us say this in edge from a particular base is at a height is z 1 and out edge is at a height is at z2.

So, if I make a triangle there with that angle θ. So, this is ds this is z 2 minus is z 1 let me call it d z then sin θ is equal to d z divided by d s or d s sin θ is simply d z. So, I can use that d s sin θ here and write this as minus A ρ g times dz ok. So, that is what the body force is.

Now, we have got surface forces and we know that surface forces are two types; one is pressure the other is shear stress ok. What we are going to do so we are going to neglect shear stresses. So, we know that shear stresses is basically arising from viscosity or it is due to friction. So, by really neglecting shear stresses we are talking about frictionless fluid. So, if there is a fluid which was not having any viscosity does not feel any friction, that was the case then there is no shear stresses and that is the case that we are want to look at and then we will correct it later on how to account for friction. So, the only force that we are this surface was that we have worried about really is the pressure force.

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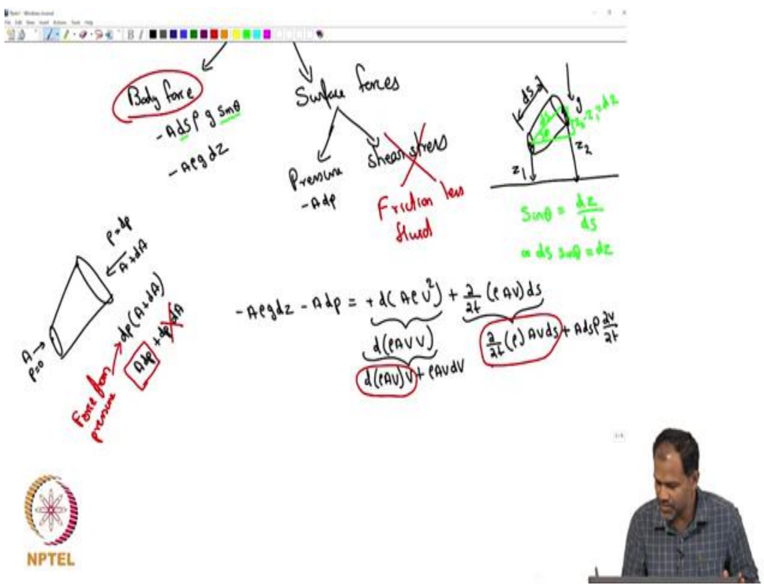

So, let me make a diagram again to tell, what is the best way to do it. So, that is our diagram that we are looking at, we said there is an inlet pressure and then there is an outlet pressure. So, p there is a p that is acting on the inlet and there is a p plus ∂ta p that is acting at the outlet, but we know that its only the difference in pressure that is going to create a force and therefore, let us say that the force at the inlet or the pressure here is 0 and the pressure at the other side is simply dp because dp is the one that is going to really constitute the pressure.

Here we have an area A and here we have an area A plus dA ok. So, the net force acting is simply going to be dp times A plus d A in this manner right. But the point is that if you write it as A times d p plus dp times d A and we are looking at really a differential element where any of these differentials actually the values go to 0 where dp goes to 0 dA goes to 0. So, now, this quantity is going to be much smaller than A dp or in other words the effective pressure force that is going to act is going to come mainly from A dp.

So, look at this when I write down the force this is the force arising from pressure, force from pressure is what I have calculated which is basically based on the inlet side and the outlet side, but there is actually pressure contribution that is going to come also from the periphery from the lateral sides, I have neglected it because that is again going to be a small number and you could actually show that that is going to be much smaller compared to the contribution that I have identified here ok.

But let us not worry about how to show that at the moment let us see that its A into dp is the net force that is going to be acted and that is going to act in the direction opposite of your d s and therefore, the pressure contribution is going to be nothing, but minus A into dp.

So, putting all of them together, we have minus A ρ g d z that is a force. So, where am I substituting I am substituting it here and substituting it into that expression. I have the force I have the change in the momentum and the rate of change. So,

$$-A\rho g\, dz - AdP = d(\rho AV^2) + \frac{\partial}{\partial t}(\rho AV)dS$$

Now,

$$d(\rho AV^2) = d(\rho AV)V + \rho AVdV$$

$$\frac{\partial}{\partial t}(\rho AV)dS = \frac{\partial \rho}{\partial t}AVdS + AdS\frac{\partial V}{\partial t}$$

If you look at the continuity equation:

$$A\rho\left(dS\frac{\partial V}{\partial t} + VdV + \frac{dP}{\rho} + gdz\right) = 0$$

$$\frac{\partial V}{\partial t}dS + VdV + \frac{dP}{\rho} + gdz = 0$$

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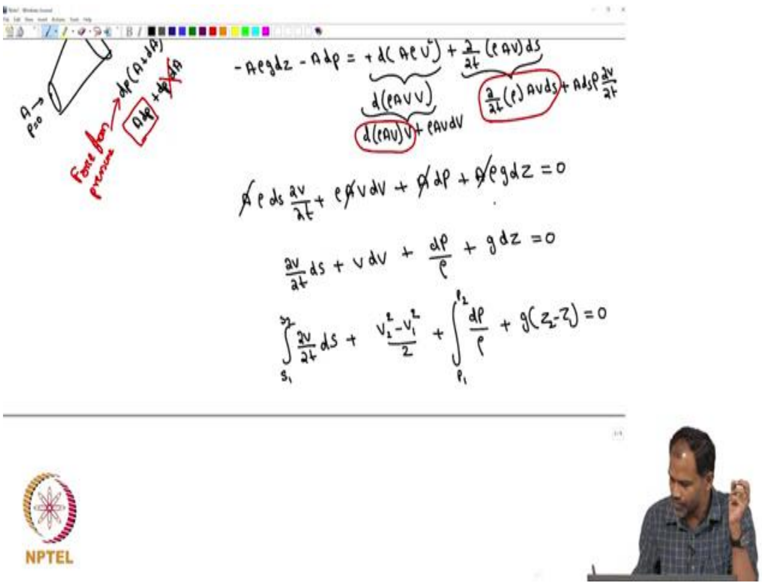

So, this is really an unsteady Bernoulli's equation written in a differential form. On integration we have,

$$\int_{S_1}^{S_2}\frac{\partial V}{\partial t}dS + \frac{V_2^2 - V_1^2}{2} + \int_{P_1}^{P_2}\frac{dP}{\rho} + g(z_2 - z_1) = 0$$

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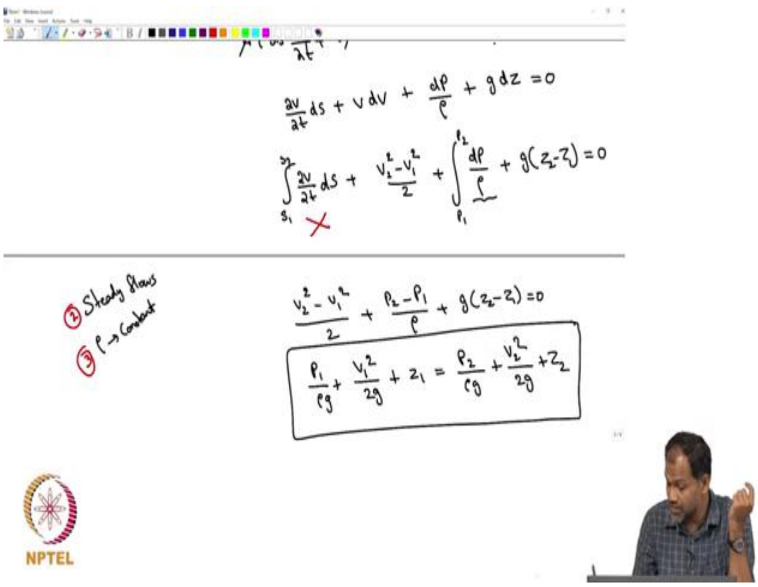

Now if I take the assumption that let us say were worried about steady state flows or steady flows where time dependency is not there, then I will not consider this term and will also say that ρ; ρ is a constant.

$$For\ steady\ flows, \rho \rightarrow constant$$

$$\frac{V_2^2 - V_1^2}{2} + \frac{P_2 - P_1}{\rho} + g(z_2 - z_1) = 0$$

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

Now let me just tell you the assumptions that went in deriving that equation, because that tells you when you should be using Bernoulli's equation ok. So, we started of course, with a simple you know differential element, we are actually doing here in a 1 D fashion we are just worried about the direction which we called as d s we calculated the mass balance we calculated the momentum balance and the first assumption that we made is about the frictionless fluid we said the fluid has no friction ok.

So, that is one that is important because then; that means, Bernoulli's equation should be applied only when your fluid has really no friction. Now the second assumption that we made is that it is applicable only for steady flows, third is that we are assuming that the density is constant ok. So, these are the three cases these are the three assumptions that have really gone in in deriving the Bernoulli's equation now.

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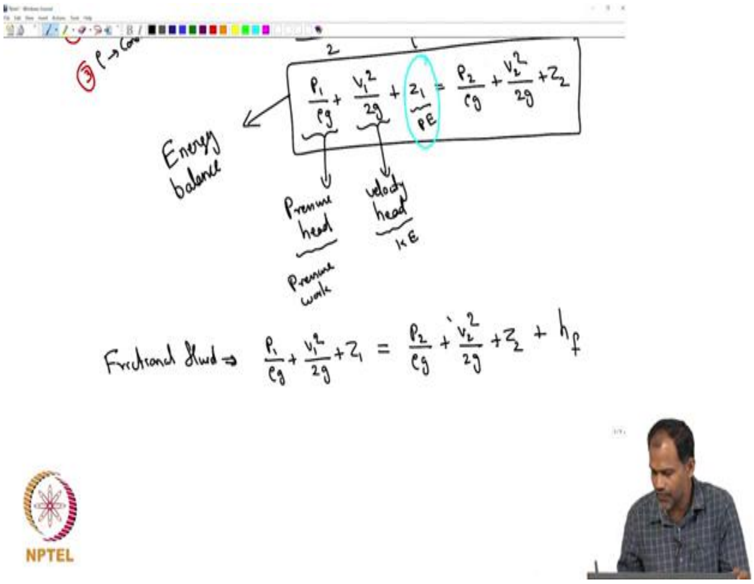

So, let us see what this means or let me just give you this. So, pressure this first term is typically called pressure head ok. This is called velocity head and then of course, then height some height ahead. So, head really means some height. So, why is it called before that. So, what are these terms ok? So, this is really nothing, but the pressure work the work this is the work is done well. So, by the pressure right. So, whenever the fluid is moving pressure is a force that is acting and there is the fluid is moving. So, there is a 4 times velocity that gives rise to a work in the work or the fluid. So, pressure p this first term really represents that work that is done on the fluid.

Ok, represents the potential energy. So, this is really kinetic energy, this is really potential energy and then there is a pressure work. So, its says that if you look at pressure work kinetic energy and potential energy that sum remains constant along any point that you along any streamline that you consider ok. So, if you consider two different points and write down the kinetic energy plus potential energy plus pressure work, if the fluid has no friction and if its steady state and its incompressible, then you can apply Bernoulli's equation and that is why this is actually an energy balance.

So, this equation that we have derived really is an energy balance. So, z 1 has the dimension of length then the v 1 square by 2 g should also have the dimension of length and you can verify that. Similarly, pressure by ρ g will also have the dimension of length ok. So, this equation now is written in such a way that each term the such as pressure head, velocity head all of them have the dimensions of length ok. So, we are really expressing energy in terms of length. So, for example, when you have potential energy and you say the potential energy is really dependent upon the height at a particular height it is, similarly kinetic energy is expressed as a height pressure work is also expressed as a height.

Now, therefore, if you say that your fluid is actually not frictionless, but there is a friction ok. So, if it has friction; that means, that certain amount of energy is lost due to frictional forces because of frictional forces some energy is converted into heat. So, in other words the action of viscosity between fluid layers resulting generation of heat ok. Typically that number is very small and therefore, you will not worry about I mean the number need not be small, but if you actually convert it into temperature that may be very small. So, typically you do not consider temperature variations, but still there is an energy loss ok.

So, we can say that if you had a pressure head a velocity head and then you know there is a potential energy and a part is lost due to friction, then you say will say there is a head loss in the system ok. So, for a frictional fluid you can say that your equation is:

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_f$$

$$where, h_f \rightarrow frictional\ head\ loss$$

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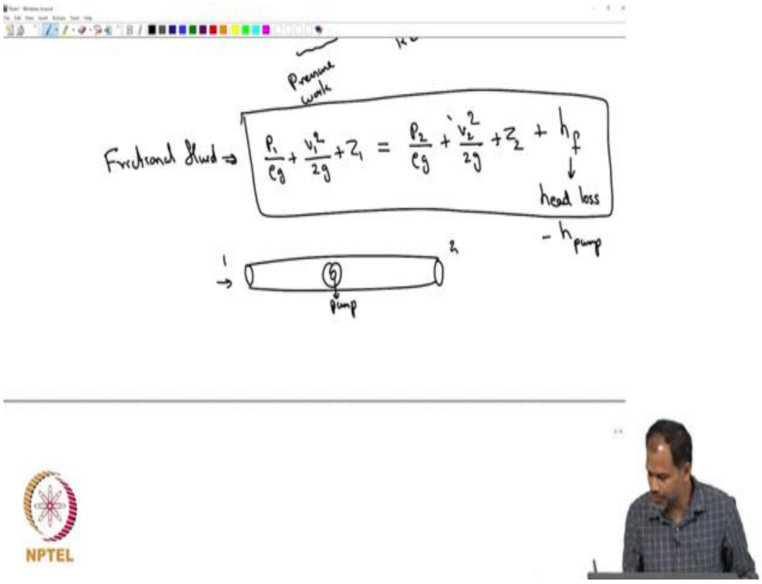

So, that is energy that is lost in the system ok. So, you can see that loss would essentially be a positive quantity because there will always be friction invariably. So, that particular number or the particular quantity that I am comes up will always be positive.

On the other hand let us say if you are applying this equation to a pipe ok a long pipe and then you have fluid in so, which you will call 1 and then you have 2. Then you can actually apply Bernoulli's equation between point 1 and point 2 and you could write down the equation where now you have you can you have to say what is the hf or your idea most of the times would be to find out what is hf. On the other hand let us say if you put a pump some pump in between. So, that is a pump what does the pump do? Pump would give you energy into the system.

So, one way of writing it is that, if you want to write you could say you have an hf then you can say minus h pump because hf is the loss in energy pump is going to give you energy into the system. So, you can again talk about a pump head or head thanks to the pump ok. So, that energy that is input into the system, you can call it a pump head. So, typically we will worry about that equation if you have a pump you know how to add it or if you have something which is extracting energy like for example, a turbine. So, then you have to remove that turbine head ok. So, depending upon what you have in the system, you can keep adding or subtracting different heads. So, that is really the energy balance.