Fluid and Particle Mechanics
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Lecture - 68
Boundary Layer - Differential Approach

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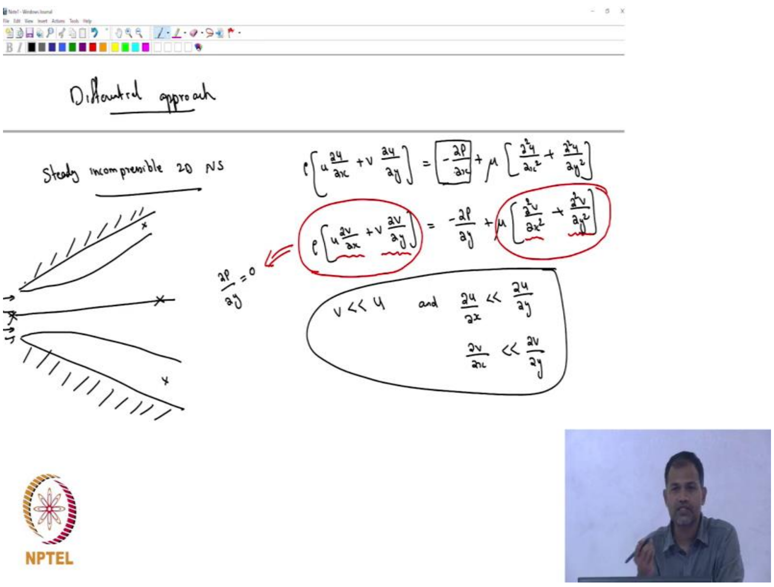

We will start down by writing down a Steady Incompressible 2D Navier Stokes Equations. So, the x-momentum equation, so I will throw away the unsteady term u dou u by dou x plus v dou u by dou y equal to minus dou p by dou x plus mu del square u by del x square plus del square u by del y square dou u dou v by dou x plus v dou v by dou y equal to minus del p by del y plus mu del square v by del x square plus del square v by del y square ok.

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

$$\rho\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{\partial p}{\partial y}+\mu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]$$

So, this is these are equations that you should have really solved to get velocity profile inside the boundary layer. I am going to make some assumptions which is basically on the fact that if you go to higher and higher Reynolds number the boundary layer is suppose to be very thin ok, its going to get smaller and smaller and smaller and also you know that the largest velocity is in the u direction.

In other words the v velocity is going to be much smaller than u. So, I will say if my v is much smaller than u and if I say my dou u by dou x is much smaller than dou u by dou y because dou u y is the small number dou u by dou y is going to change from 0 to some finite number in a small thickness, but dou u by dou x is what the change in u along x which is changing very slowly ok.

So, therefore, this approximation of dou u dou x is smaller than dou u by dou y is good for boundary layer also dou v by dou x is much smaller than dou v by dou by ok. So, these three approximations are found to be good for a boundary layer if the boundary layer is very thin. And if I use that and if I look at my y momentum equation I find that this quantity as a dou v dou x in it this quantity as a v times dou v by dou y, this is also proportional to v this is also you know proportional to v.

In other words this equation this left hand side or I am drawing it in this circle are going to be extremely small quantities for a boundary layer. Now, you have to trust me there because one has to actually do a formal way of doing it and sort of going to tell that is what is going to hold for the boundary layer ok. And the reason I want to tell it is because if you look at this entire equation many of these terms will come out to be so small that you essentially ended with and with this expression that dou p by dou y is going to be 0 ok.

Or in other words pressure variation across the boundary layer is a small number or its actually negligibly small. And so, in other words therefore, pressure variation is only in the x direction ok, its not there in the y direction inside a boundary layer or to put it physically if you have let us say we have a diverging channel and let us say you have a fluid that is coming and a boundary layer is growing. So, this is boundary layer ok; so that was our.

So, let us say we have two plates diverging and there is a fluid that is coming and let us say boundary layer starts growing on the solid wall. Because you have fluid coming and is you know going inside a diverging channel, what happens to the velocity? Velocity will decrease right. So, that would mean pressure would increase.

So, you can really apply Bernoulli's equation and let us say calculate pressure here and pressure here you can use the same pressure here also because the pressure variation inside a boundary layer is going to across the boundary layer is so small. The pressure variation along the boundary layer is then determined by what is happening outside.

If pressure was changing with respect to y then you would have to worry about how does the pressure change, but if you want to only determine pressure inside the boundary layer you can actually calculate it from the numbers that is not outside. I have given a very sort of a hand waving arguments if you do not believe it just do not worry it take for granted that pressure variation across the boundary layer is going to be so small. Therefore you can actually determine the pressure variation inside a boundary layer just by knowing the pressure outside. This is.

In laminar meaning laminar boundary layer.

No, so, there we are basically saying that the pressure variation is only in a x direction right.

No there we can actually show that right. So, there what happens is if you looked at v velocity all of them will identically go to 0 and therefore, we will see the dou p by dou y is 0 and therefore, p is not a function of y. Now, I know that its a v is not 0. So, none of the terms are actually 0, but they are all small.

So, in other words the flow is going to be very similar to what we will have in your pipe flow ok, v component is very small that will basically end up with a similar situation where p with y variation is going to be small. So, the situation is similar; so you can see the analogy already that its because of the reason that v is very small that you ending up with a such a conclusion. Questions?

So, that would mean that this quantity dou p by dou x is simply dt by dx.

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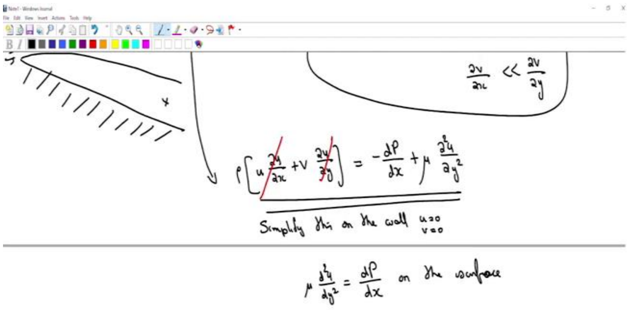

So, I will write down my x momentum equation again; my x momentum equation is going to be rho times u dou u by dou x plus v dou u by dou y is equal to minus dp by dx plus mu del square u by del x square sorry del x square u by del y square.

$$v \ll u; \frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y}; \frac{\partial v}{\partial x} \ll \frac{\partial v}{\partial y}$$

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{dp}{dx} + \mu\frac{\partial^2 u}{\partial y^2}$$

So, I am not writing del square u by del x square, but its del square u by del x square is also expected very small and the only reason I want to write it down is because I want to write this equation on the surface of a or on the wall I want to simplify this on the wall. So, what is u on the wall? 0. What is v on the wall? 0. So, u is 0, v is 0 and therefore, if I actually look at my x momentum equations these things go away on the surface of the wall and therefore, I actually have mu v square u by d y square is equal to dp by dx on the surface ok.

$$0 = -\frac{dp}{dx} + \mu\frac{\partial^2 u}{\partial y^2}$$

All of you agree pressure is decreasing with x. So, what is the situation that I am going to look at? I am looking at this kind of a diverging channel no I do not want to do that I want to start with the other one let us look at converging channel. So, velocity decreases pressure increases; so dp by dx is greater than 0 ok.

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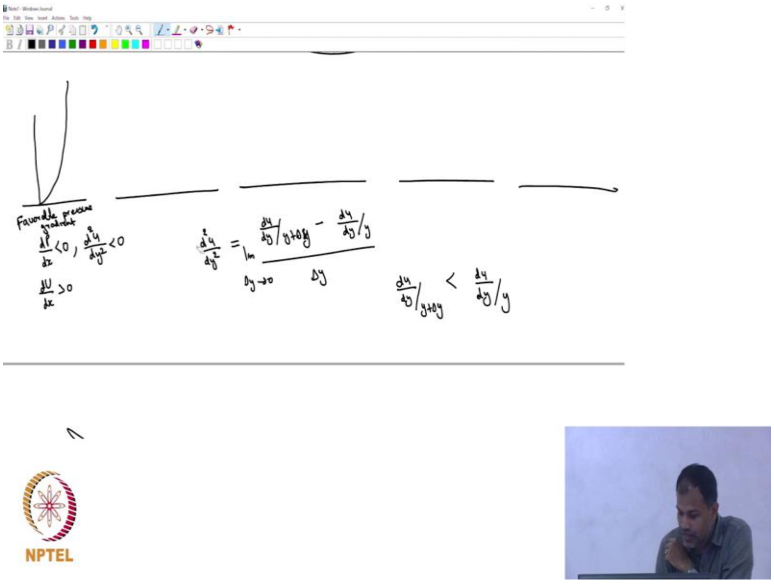

So, let us say we have a converging channel in the converging channel velocity decreases that would mean pressure increases. So, if I have a boundary layer.

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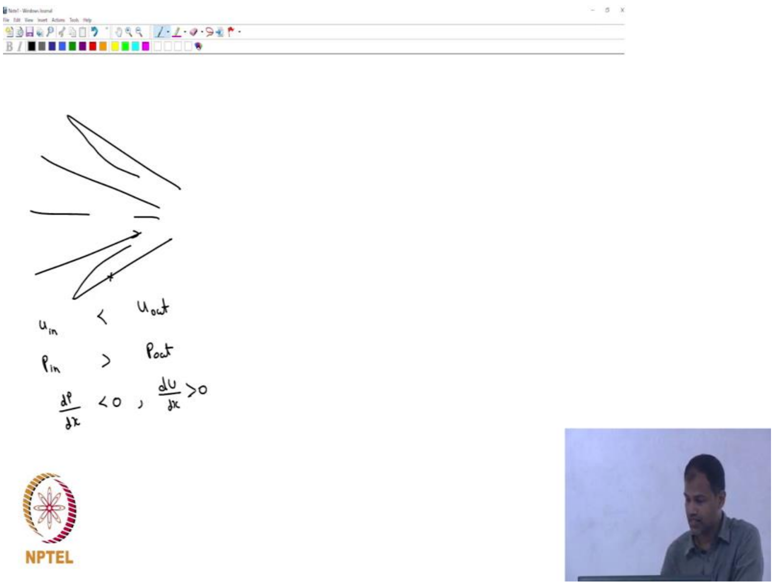

So, let us say you know you have a kind of a converging channel there is a fluid that is coming and going. So, if there is a growth of boundary layer like that I am basically looking at you know a point somewhere there and trying to write down what is it what is the differential form telling me. So, I have taken a situation where dp by dx is greater than 0, you have question.

So, let us say you have. So, far we have been talking about a flat plate and a fluid that is coming right. Now, and we said of course, the outside the boundary layer the fluid is moving with a constant velocity. Now, I'm expanding that case to which the outside fluid is also either accelerating or decelerating.

So, the u infinity that I have been having I am think I am look at it as an accelerating fluid or a decelerating fluid. So, it will accelerate or decelerate I mean to accelerate to decelerate I am trying to put in a converging or a diverging channel. So, I have a uniform flow that comes and then it is seeing a converging section. So, it's basically going to enter the converging section while it is enter entering the converging section it's basically going to see it is going to develop a boundary layer as if it's like to going to have an entrance length ok. So, its going to its going to develop a boundary layer and I want to I also said look if I look at the pressure gradient outside that is going to be same as the pressure gradient inside also ok.

The dp by dx inside the boundary layer is same as dp by dx outside the boundary layer. So, if I simply apply my Bernoulli's equation which is the fact that the velocity is increasing I would see that my pressure is decreasing if it's a converging otherwise its increasing in a diverging channel ok. So, just by looking at whether u infinity is increasing or decreasing I am able to tell what is going to happen for the pressure inside the boundary layer and by looking at that I am actually going in to now predict what is going to be the flow profile exactly on the wall.

So, what am I am going to first look at cases where it is a converging channel. So, u in is smaller than u out. So, that would mean p in is greater than p out and therefore, dp by dx should be less than 0, correct in a converging channel sorry we made mistake dp by dx is less than 0 is the first situation that we are look at which also would mean du by dx.

So, u is increasing with sorry u e u is increasing with x; so d u by dx is u is increasing with x du by dx is greater than 0 ok. So, we said on the surface this is the expression that should hold ok. So, if I have a situation in which dp by dx is less than 0 my d square u by dy square should also be negative. So, if my d square dy square is negative can we see, can we tell the what the velocity profile should be? Should it be like that, should it be like this, should it be like that, three different type is should it be like this should it be like that or should it be like this.

Remember this expression this expression that we have written down is valid only at the wall. So, I can only tell how it should be at the wall, but by looking at this where I have dp by dx less than 0; that means, d square u by dy square less than 0 whether this green or blue or this magenta will satisfy d square u by dy square less than 0. Green well blue cant because blue means its a straight line. So, the second derivative should go to 0. So, blue is ruled out now green or magenta.

How do you know that?

So, let us say d square u by d y square I will write using finite difference of approximation du by dy at y plus delta y minus du by dy at y divided by delta y right in the limit of delta y going to 0, I want d square u by d by square to be less than 0 so; that means, du by dy at y plus delta y should be smaller than du by dy at y.

So, du by dy is the slope and that slope should be largest at the wall; so that satisfied by the magenta line right. Remember u is on the x axis y is on the y axis ok. So, when I say du by dy you should really be looking at the slope like that ok. So, its highest like this; so its a magenta line that is going to be satisfying this situation d square u by dy square less than 0 ok.

So, I can immediately say that therefore, the green cannot be and it should be something like the magenta ok. So, this is what the boundary velocity profile should be if I say dp by dx is less than 0 ok. So, what should happen is that the profile would and that is what we had we have drawn and that is what we had analyzed also. So, its just we going to be something of that is sort.

Yes.

Cannot be a du by dx is you are arguing du by dx is not small I am saying that du by dx is going to be smaller than du by dy u changes from 0 to capital u in a small distance of y, while u changing very slowly along the x direction.

So, it has to be a sort of a slowly converging I it cannot be like that it has to be something where this approximation holds. So, that is because we cannot solve actual equation. So, we are saying can be actually simplify it is in some way. So, that we can get some qualitative arguments and then which might true hold in general equations and that is what we are trying to aim at. So, this situation is called a favorable pressure gradient, we will see why its favorable later let us call that favorable pressure gradient.

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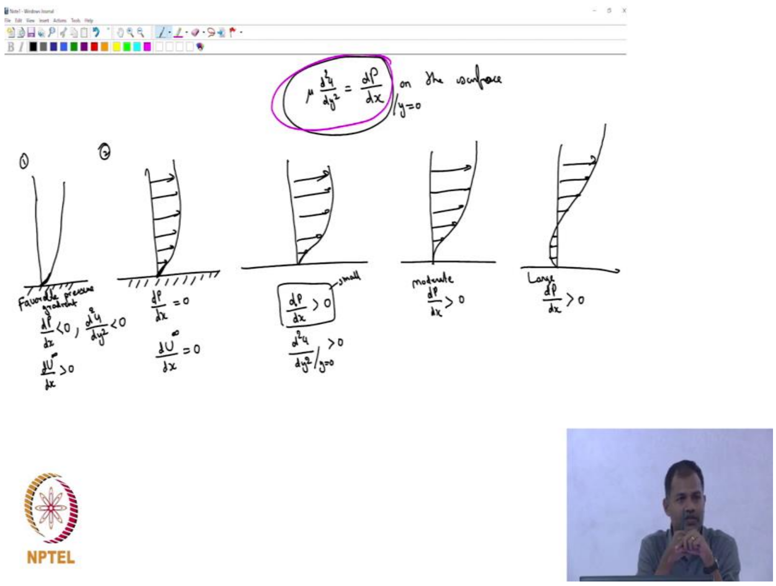

So, the next one we want to analyze is dp by dx is equal to 0. So, that is case 1, case 2, dp by dx is equal to 0 what would be the profile? We know dp by dx is given by. So, d square u by d y square the curvature is 0, so the velocity at the wall should be linear ok. So, it should be should start linear and but then we do not know what is going to happen something of that sort that is going to be if dp by dx is equal to 0.

And if dp by dx is equal to 0 what would you say about d u by d x the outside velocity d u infinity if you like du infinity by dx is also 0 ok. So, outside fluid is know not changing its velocity in any fashion pressure is also not changing.

So, see here if you look at this would be linear, but then we do not know because this equation holds true only on the surface right this equation is not completely general we put we got it when we put u is equal to 0 v is equal to 0. In the conservation equation in the full we put u equal to 0 v equal to 0 threw away these terms and therefore, we go out this. So, that can true hold only on the surface of the wall.

Yes.

This is only for the thin boundary layer yes if the. So, because we are saying that the this dou dp by dx right you can say that its only a function of outside when the boundary layer is so thin. So, that we can neglect you know all x all x gradient and so on and typically boundary layers are thin like you know it may be like a millimeter or smaller things like that.

On the surface.

On the surface y is equal to 0 yes.

On the surface y is equal to 0 u and v are also 0 right.

Yes, so, let me just tell you again. So, what; so see by looking at the v momentum equation whether you believed me or not what I said is that pressure across the boundary layer, pressure in y direction is not changing ok; that means, pressure in x direction is the only thing that is changing. So, if I have a boundary layer and outside the boundary layer if I know how pressure is changing I will know how pressure changes inside the boundary layer also because pressure does not change in the y direction. Is that clear ok?

You have a boundary layer let us say this is the plate you have a boundary layer, the boundary layer is very thin and saying that pressure variation across the boundary layer is very small; that means, if I know pressure gradient outside I will know pressure variation inside also. If it is one atmosphere 1.5 atmosphere, it will be 1 atmosphere and 1.5 atmosphere inside also because I know pressure is not changing across y ok.

Knowing a pressure gradient outside will help me to tell what is pressure gradient inside also in the boundary layer that is a first thing that I said. Second thing that I have said is if I look at the x momentum equation and if I write down the x momentum equation on the wall; on the wall then I know because of no slip and no penetration my u component of velocity is 0 on the wall; my v component of velocity is 0 on the wall and therefore, I am able to simplify my governing equation in to this magenta circle ok, where I have only a viscous term and a pressure term ok.

Now, remember that pressure term dp by dx I say I already know it from the external flow field when I say external this external to the boundary layer. So, by knowing what is the pressure gradient outside I can tell something about my velocity profile on the wall that is where I am doing really ok. By knowing the pressure gradient outside I am able to tell something about by velocity gradient on the wall is that clear. Does that answer your question, what is your question?

Turbulent.

Yes it is only on the surface. So, remember I am drawing it only on the surface ok. So, when I said you know d square u by dy square is less than 0, I only know it at this region when I am saying d square u by dy square is 0 I only know about this the other part I just connected and drew I do not know what it is and I am not going to care about that also at the moment.

This is wall. The other is just if you draw the profile I am just drawing it can 0 distance.

Because I have pressure decreasing, so velocity increasing.

So, this is only up to boundary layer whatever I have drawn is up to. So, delta is going to be much larger than that. So, let us say ok. So, this distance up to which I have drawn is really inside the boundary layer its not even going to connect outside. See I cannot say even anything because I am really looking at wall what is the; what is the governing equation at the wall and making some deductions.

It will be actually continuous because it’s increasing and it will go on then join there and outside then.

Outside again there could be some changes which will be along x direction not along y direction.

This one at y is equal to 0. You have some doubts no ok. So, then if I have a case where dp by dx is less than 0, in fact all three cases are going to be dp by dx greater than 0 ok. What do we know about d square u by d y square at the wall? Should be greater than 0 correct. So; that means, my curvature cannot be like this my curvature should change the sign.

So, initially it was initially it was like this then I said its going to be like this now it has to be like that ok. So, in other words my velocity profile should start doing that ok, the original one of those color green or magenta that we had wrote it down. So, like that and then it would do that sign d square u by d by square should be greater than 0.

So, write it down again using your finite difference and remember u is your x axis, y is your y axis ok. So, its typically opposite to the slope that we talk about, so.

Correct sorry that is right ok. So, this is what the velocity profile going to be if I have. So, when would this happen? This would happen when dp by dx is greater than 0; that means, pressure increases along the x direction that would happen when u decreases along the x direction, u will decrease along the x direction and it is a diverging section ok.

So, if you have a diverging section then you should expect the velocity profile to curve in a different way is that ok. So, if dp by dx is even larger, what would you expect this curvature would be increase at the wall it would be almost become flat and do that. If dp by dx is much larger it would even bulge a more and it would actually do that inside the boundary layer and drawing my u velocity. What does the last figure tell? That on the wall the flow is actually in the opposite direction, if dp by dx is very large.

So, this is you know dp by dx is small let us say this is some moderate and this is some large ok. So, if outside fluid outside the boundary layer if pressure increases and if that pressure increases very large on the wall the fluid will actually start flowing in the opposite direction that is what we actually called flow separation.

So, whenever the talked about you know those diverging sections and so on I always drew fluid flowing in the opposite direction right. So, this because of this particular reason this is called actually boundary layer separation ok. The external pressure gradient will cause a fluid flow to you know flow in the opposite direction on the wall.