Fluid and Particle Mechanics
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Lecture - 66
Boundary Layer - Momentum Integral Analysis - 1

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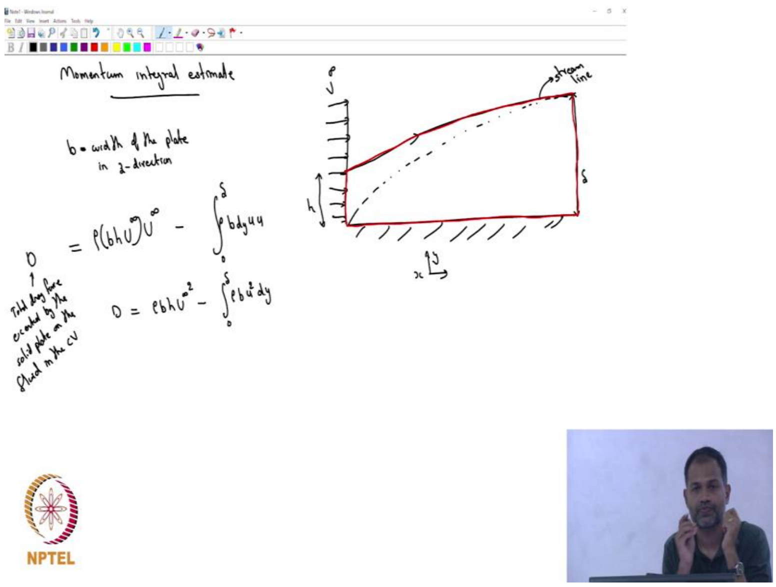

So, this thing is called momentum integral estimate, we will do an integral analysis on the boundary layer, get some you know insight into it, we will do a little bit of differential analysis and get a little insight into it. So, let us first do the integral version, my solid plate, my boundary layer x y delta the boundary layer thickness, the uniform velocity u infinity. I am going to consider a streamline that is going to reach this point so, that is going to reach this point let us say that is a streamline; so, rewrite a stream line.

So, what is a streamline? If I draw tangents to the streamline I will get velocity vector. Can there be flow across a streamline? No, there will not be a flow across a streamline because by definition the tangent is the direction in which velocity is so, there cannot be a flow across the stream. So, let us say this height is h, the height on the right hand side is delta and let us say b is the width of the plate in z direction ok. So, I am looking at x y plane is z direction is the plane per line perpendicular to the screen that you are looking at. So, b let us say is the width in that particular direction which let us imagine to be very large.

And, I want to write down a force balance to this control volume which is this that; so, that is the control volume that I have. So, on the left hand side it is bounded by the uniform velocity profile up to height of h. On the right hand side the height is delta the boundary layer thickness, on top it is bounded by a streamline, on the bottom it is the plate; that is the volume that I am going to look at and I am going to write down a force balance. The control volume is clear? So, what do we write? For when the flux in minus out is equal to total forces, the only force so, we are writing it down on the fluid ok.

So, the solid is going to exert some retarding forces, some drag force on the fluid that is flowing right. So, let me call that force as D. So, what is D? Total drag force exerted by the solid plate on the fluid in the control volume. And, that should be equal to the flux in minus flux out yeah, you have some flux in what the flux in is where is the flux. So, there is only two places where the fluid is coming and leaving right. The fluid is coming from the left hand side through a height of h, the fluid is leaving through the height delta. Since, I have chosen a streamline there is no fluid flow across the streamline.

So, there is no fluid flow through the top part ok, bottom part anyway it is a solid plate so, there is no fluid flow through the plate. So, I have to only worry about the fluid that is entering through h and the fluid that is leaving through delta ok. So, what will be the momentum of the fluid that is entering through left hand side h? We will say that so, b into h is the cross sectional area times U infinity is the flow rate times rho is going to be my mass flow rate times U infinity is going to be my momentum flow rate.

So, that is going to be in the plus x direction minus what is exiting will be if I do the same thing except that. So, here you know on the left hand side the flow is uniform, but on the right hand side the flow is not right there is a profile. So, I cannot do just like that I need to write down an integral.

So, if I take a small section I will say let us say b d y is my cross section times let us call u as the velocity. So, that is my volumetric flow rate times row is my mass flow rate times u is my momentum flux integrated from 0 to delta would be the flow rate or the momentum flux out ok. So, that is going to be in minus out is equal to the total force be exerted or experienced by the control volume fluid that is ok.

Let us write it neatly D is equal to rho b h U infinity square minus integral 0 to delta rho b u square dy. Let us also write down a mass conservation principle fluid in is equal to fluid out.

$$D = \rho b h U_\infty^2 - \int_0^\delta \rho b u^2 \, dy$$

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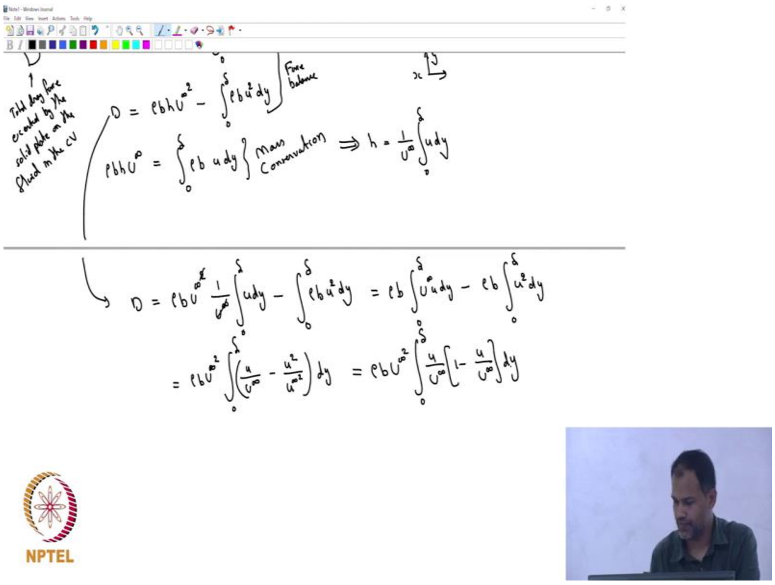

So, what is the fluid in? Rho b h U infinity should be equal to integral of rho b y u dy, sorry rho b dy times u so, that is correct. So, remember this is my force balance, this is my conservation of mass. Any doubts on the equations? Total drag force is node 0. Why is the total drag force 0?

That is what we have called as D so, this oh this is D ok, D it is not 0. I will try to write it better maybe yeah, I mean our interest is to calculate really that force and connect it with the thickness ok. So, mass conservation if you simplify what do you get? h so, rho goes b b goes h is equal to 1 by U infinity integral u d y ok, take that and substitute into your force balance D is equal to rho b U infinity square, instead of h I will write 1 by U infinity integral u dy minus integral 0 to delta rho b u square dy, get one of them.

So, that is equal to rho b so, this limit is also 0 to delta right. So, rho b integral 0 to delta U infinity u dy minus rho b integral 0 to delta u square dy is equal to rho b U infinity square into integral. No, we do it yeah that is fine I think U infinity square u by U infinity minus u square by u infinity square d y 0 to delta, is this right ok. So, that is really the drag force yeah, I think that is fine.

$$\rho b h U_\infty = \int_0^\delta \rho b u \, dy \rightarrow h = \frac{1}{U_\infty} \int_0^\delta u \, dy$$

$$D = \rho b U_\infty^2 \int_0^\delta \frac{u}{U_\infty} \left[1 - \frac{u}{U_\infty}\right] dy = \rho b U_\infty^2 \theta$$

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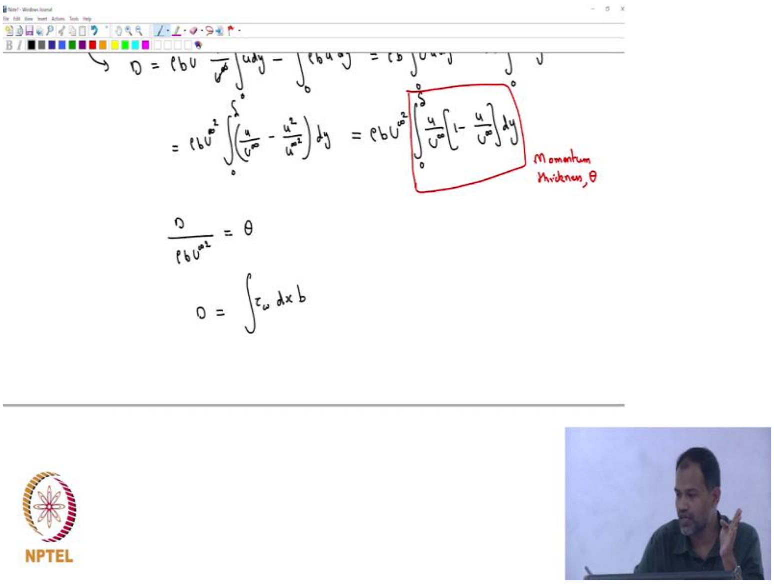

Now, what we are going to do is that we are going to give this part a name and that is momentum thickness theta ok, it is really a measure of drag force ok. So, if you really look at it D divided by rho b U infinity square is equal to theta, because that integral is theta ok. So, that is nothing, but just a measure of total drag exerted by the plate on the fluid. Now so, this is how we have gotten an expression for the drag force on the fluid, but you also know it can be evaluated from your shear stress ok.

In other words you know that D is nothing, but tau w if let us say tau w is the wall shear stress exerted on a small element d x and b integrated. So, the drag force the total drag force should be an integral of the wall shear stress right. The force exerted by the plate on the fluid is through the viscous stresses so, that is nothing, but what you call as the shear stress; so, that I have this integrated. So, why have I done the integration because, it need not be a constant as we saw earlier.

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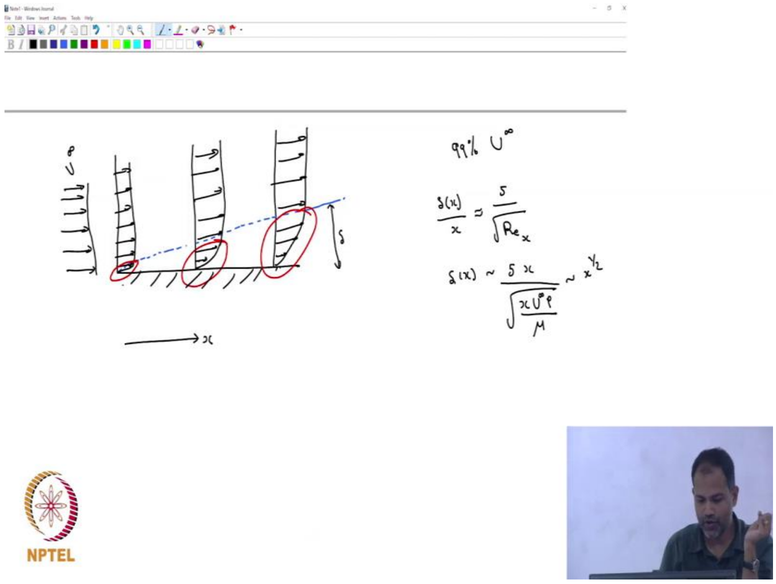

So, if you look at you know the first, second, third and so on the fluid slope is going to be different ok. So, the shear stress is going to be a function of x, shear stress will not be a constant along x. So, I said if I take a small area of length dx and width b so, it is a b dx and then on that tau w is acting. So, tau w dx is going to be the force on that small element and if I integrate it over all my length up to x I will get the total force ok.

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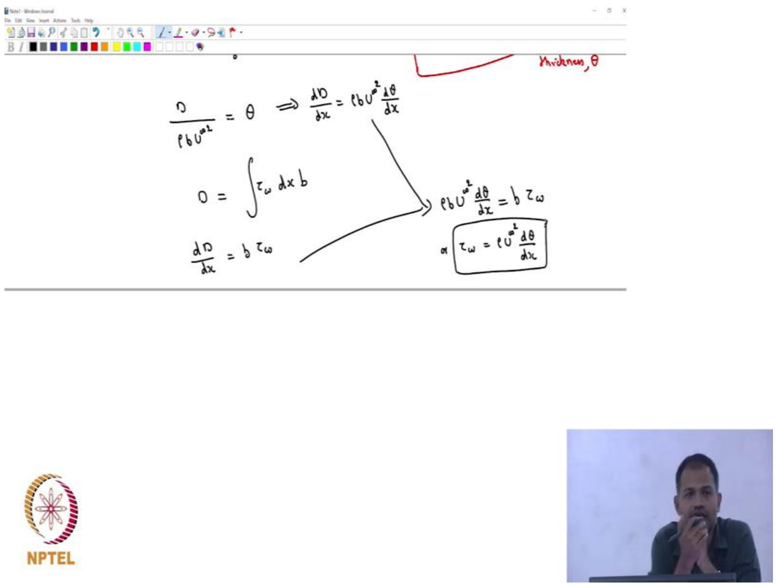

So, that means if I do a d D by d x I am going to get b times tau w or in other words shear stress is nothing, but the derivative of the total drag force with respect to x. I could also write d D by dx is equal to rho b U infinity square d theta by dx right where I had defined it theta some integral. And therefore, if I use these two what I see rho b U infinity square d theta by d x is really b times tau w or tau w is equal to rho U infinity square d theta by dx ok.

$$\frac{dD}{dx} = \rho b U_\infty^2 \frac{d\theta}{dx}$$

$$D = \int \tau_w \, b \, dx$$

$$\frac{dD}{dx} = b\tau_w$$

$$\tau_w = \rho U_\infty^2 \frac{d\theta}{dx}$$

So, it is really rewriting the wall shear stress in terms of some derivative, it is just for manipulation alright. Now, so far we have not assumed anything about the flow profile inside the boundary layer ok. So, we have done an integral estimate; so, this should be true whether the flow is laminar or turbulent ok.

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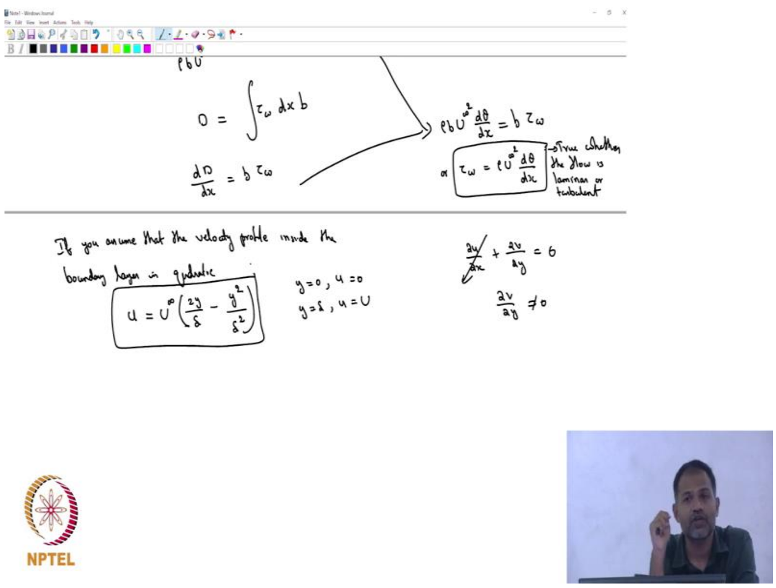

So, this should be true whether the flow is laminar or turbulent, yeah steady state. Well, what is steady state? Yes, it is in terms of the incoming fluid, it is a steady thing yes.

Yes, but turbulent flow you have to be careful, you must be talking about an averaged quantity.

Why should it be a straight line?

No in fact, the path of the particle will basically go a little up oh ok, I cannot think about any other explanation that why would it go up, but the fact is the, oh yeah one thing is this continuity would say that ok. So, if you look at continuity dou u by dou x plus dou v by dou y is equal to 0, we know that dou u by dou x is not 0 so that means, that dou v by dou y is not 0 ok. So that means, v can be function of y ok;

so, the v basically will be a function of y; that means, that at every point the fluid will not only have an x velocity, it will also have a y velocity. And, if it does not have that y velocity, it will have to violate the continuity equation or in other words it is basically a consequence of mass conservation.

What is that?

No, we know this this dou u but, this mass conservation is true whichever you do right whether it is laminar turbulent, whether you talk about average fluctuations and so on. So, it will be true in general ok. So, the v velocity develops because it is because the mass has to be conserved.

So, it must be have a better explanation in terms of the total flow rate, but I cannot think of anything as such ok. So in fact, that is what we had used whenever we talked about our fully developed flow. When we talked about a fully developed flow we said our dou u by dou x is 0 and therefore, dou v by dou y is equal to 0 ok.

And therefore, v is 0 at some place and v will be 0 everywhere ok. So, we have gotten an integral form, now what we are going to do is that we are going to assume that the velocity profile in the boundary layer is quadratic. Now, if you ask why it is? There is no a priori reason, but what we want good see is that it basically going to serve well. So, if you assume that the velocity profile inside the boundary layer is quadratic, in other words you would assume that u is given by U infinity times 2 y by delta minus y square by delta square.

$$u = U_\infty \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2} \right)$$

So, we are actually hypothesizing this form and this form one way of thinking it is that yeah, it is quadratic in y ok. Now, this is because in pipe flow we know that it is quadratic so, that is one of the reason. If you look at y is equal to 0, what is u? u is 0 ok, if you look at it y is equal to delta what is u? U so, this profile essentially satisfies the boundary condition. In the sense that this profile satisfies the fact that it is no slip at the so, on the solid plate it actually gets a velocity y; it could have gone linearly, it could have gone quadratically and so on.

In fact, one would be able to you know write down the full equation you will be able to solve it and when you solve it you will find that the profile is very close to parabolic. In reality the profile only v deviates a little bit from parabolic, but we do not want to go through the entire calculation of doing it. So, we are basically taking a simpler approach in a in which we are assuming that the velocity profile is parabolic.