Fluid mechanics
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Lecture - 06
Equation of linear momentum 2

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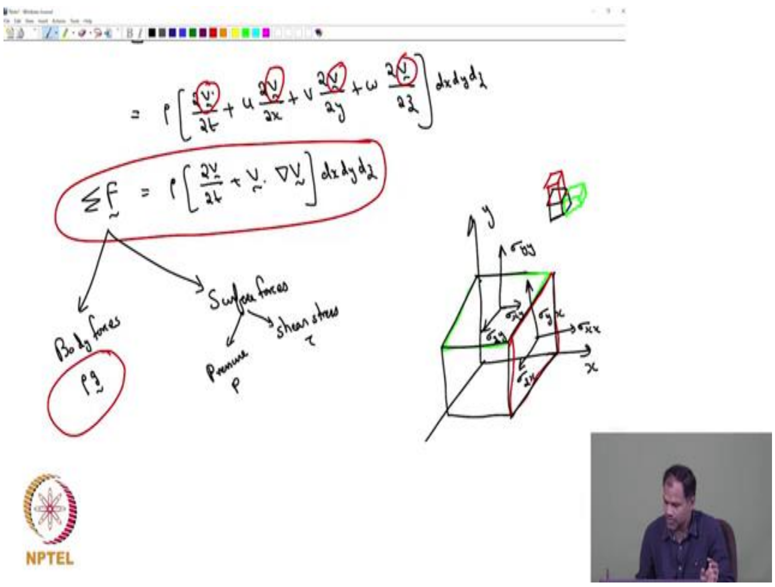

So, the surface force is typically represented using a second order tensor, which is called as σ_{ij}. σ_{ij} is actually a force acting in i direction on an area directed in j direction. So, you can always define a vector normal to any area that you consider and let us say j is the direction and i is the direction of the force ok. So, that is how you will represent your surface forces for example, if you talk about σ x y ok; that means, you have a force which is acting in the x direction, but it is on an area that is perpendicular to y direction. So, that is the idea of defining it. So, if you want to mark here, let us say in our diagram we want to represent σ.

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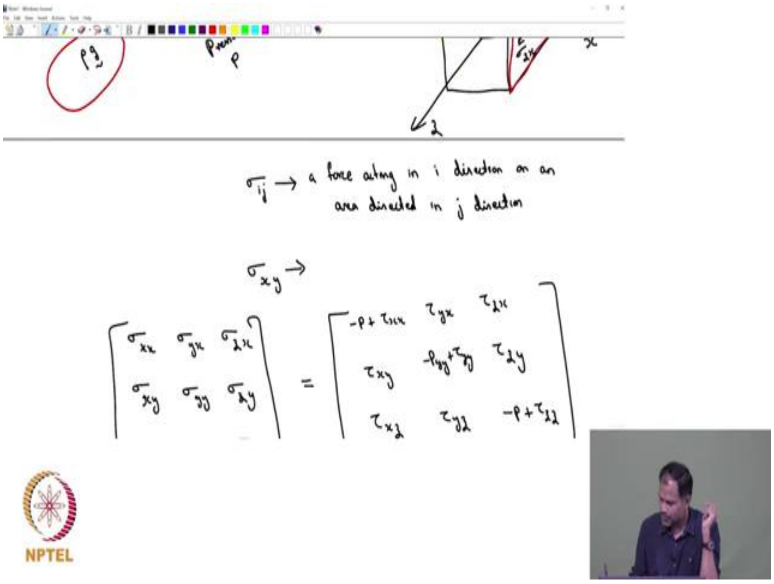

So, remember. So, how many components of σ would be there? There will be nine components. There will be nine components of the stress and is often conveniently represented as a matrix and that can be simplified or that basically constitutes two parts this is the total stress, that stress can be either due to pressure or due to shear stresses ok.

So, we said there are two kinds of surface forces pressure and stress shear stress. So, the pressure is typically denoted by P the shear stress is let us say we will use τ to represent that.

$$\begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -P + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -P + \tau_{yy} & \tau_{zx} \\ \tau_{xz} & \tau_{yz} & -P + \tau_{zz} \end{bmatrix}$$

So, this is if you want to write your surface stresses as pressure and shear stresses viscous stresses separated. So, I can also show it schematically. So, for example, let us look at σ_{xx}. Its a force that is acting in x direction on a plane in x direction. So, if you looked at σ_{xx} would actually be this. Now you I can also define of this. So, the area that I am looking at is this the one which is perpendicular to x direction ok. So, I can define two more forces I can define a force that is acting in y direction on this area x. So, a force that is acting in x direction, so, acting in y direction on this area x that I can write it as σ y x, I can talk about another stress that is acting in z direction.

So, that will be σ z x. Similarly if I consider another point let us say that this on this plane. So, that plane is a plane that is perpendicular to y. So, if I define a σ yy, that will be a force that is acting in y direction on the area y, I can define a force that is acting in x direction on this plane y. So, that we will write it as σ x y or I can write down a force that is acting in the z direction. So, that will be σ z y ok. So, these are the various components of forces or various components of this matrix the stress that we are talking about.

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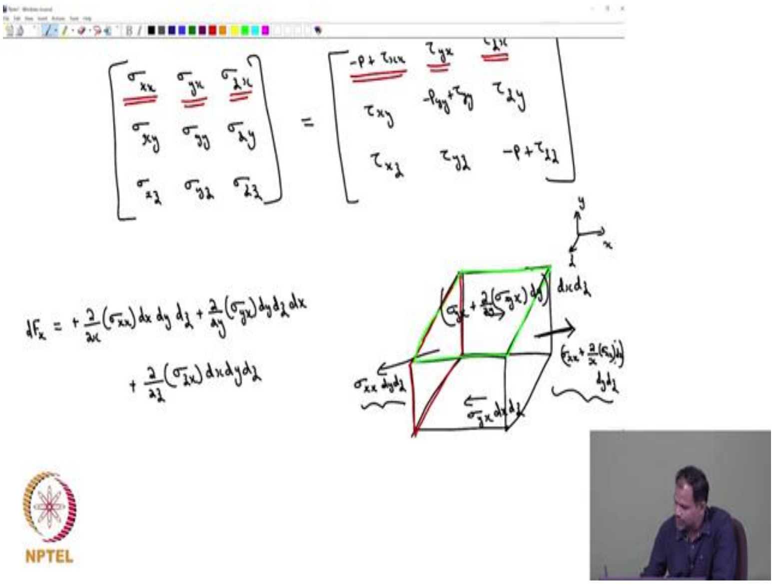

Now, what we really need is that, we need to write down how these stresses come as part of the force that we are talking about, because that is the expression that we are really looking for. In other words we need to write down the stresses as forces. So, we have already done that in the context of a equilibrium system a fluid which was under you know equilibrium which was not flowing a static fluid and there we found that the force is essentially written as gradient of pressure. So, we found that its not pressure that is really important, its a differences in pressure that is what is going to be important its the gradient of pressure that generates a force. Similarly, we need to we will give something similar here we need to find out how is stress going to be reflected as force.

So, let us make another diagram let us consider a fluid element or your cell notation x y z, let us look at one of the faces say the red face ok. So, there I have σ x x. So, σ xx is acting on that side ok. So, if σ xx is acting on this side, on the other side we can say that is the force this stress is going to be σ x x plus ∂ by ∂ x of σ xx or if you want to talk about the force is going to be multiplied by dy d z and on this side as well its going to be multiplied by dy dz. In other words the forces that is going to come in x direction from this particular component, so, difference ok.

The difference between the force acting on the left side and the force acting on the right side is the force that is acting in x direction, :

$$dF_x = \frac{\partial}{\partial x}(\sigma_{xx})dx\,dy\,dz + \frac{\partial}{\partial y}(\sigma_{yx})dx\,dy\,dz + \frac{\partial}{\partial z}(\sigma_{zx})dx\,dy\,dz$$

$$F_x = \left[\frac{\partial}{\partial x}(-P + \tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})\right]dx\,dy\,dz$$

So, that is the total force that is going to be acting on this fluid element ok. So, the forces defined positive in the positive x direction and therefore, the net force that is acting in the plus x direction is given by that extra contribution. So, that is the force in x direction. Similarly, you can derive a force in y direction and you can derive a force in z direction as well.

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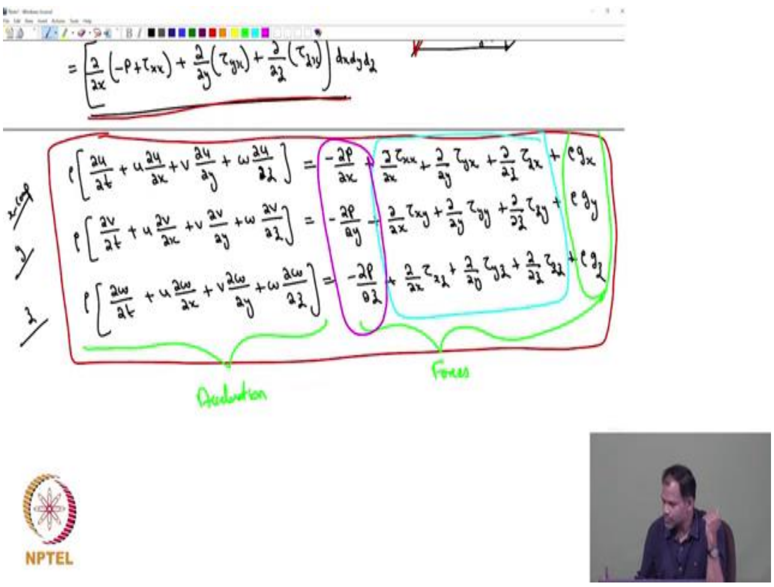

In other words this is going to be the force density this is going to be the force density that we really need in x direction in the expression that we had derived here. So, that was the expression that we had derived here. So, there is a ρ g that we have we have it from the body force and then we have a shear stress term that is going to come here. So, we can now use all this fact and simplify and write down our momentum equation as:

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial P}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+\rho g_x$$

So, there is a full expression for the momentum conservation in x direction. Similarly we can write down the y part:

$$\rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]=-\frac{\partial P}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+\rho g_y$$

$$\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]=-\frac{\partial P}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}+\rho g_z$$

So, this is what the equation of linear momentum written for the fluid element in the Eulerian frame of reference. It just looks complicated, but actually it's not that is ok. So, the left hand side if you look at that is nothing, but the substantial derivative the substantial derivative of velocity and the substantial derivative of velocity is nothing, but acceleration. So, the left hand side really represents the acceleration and if the left hand side is acceleration the right hand side should be force remember we have written it for unit mass ok. So, this side is going to be nothing, but total forces.

So, the forces consist of two parts, one is that you have got gravity forces and the other is that you have got viscous forces and then we have got pressure forces. So, in each equation acceleration is equal to pressure plus viscous forces plus gravitational forces that is all it is; now it's convenient. So, remember we did not write what τ is we just defined τ as a shear stress, we defined it as a surface force and we said that is what is acting on the fluid element, but we need to further simplify it in order to you know use it further ok. Because, if you write it in this particular form there are too many unknowns and what are the unknowns? For example, velocities are unknown the way its written pressure we do not know, shear stress we do not know that is where Newton's law of equation comes into picture.

Newton's law of viscosity comes into the picture we can use Newton's law of viscosity to write stress in terms of velocity gradients ok. So, stress if you write it in terms of velocity gradients the unknowns the stress being unknowns will go away and instead we are going to get velocities. So, we can try to do that now.

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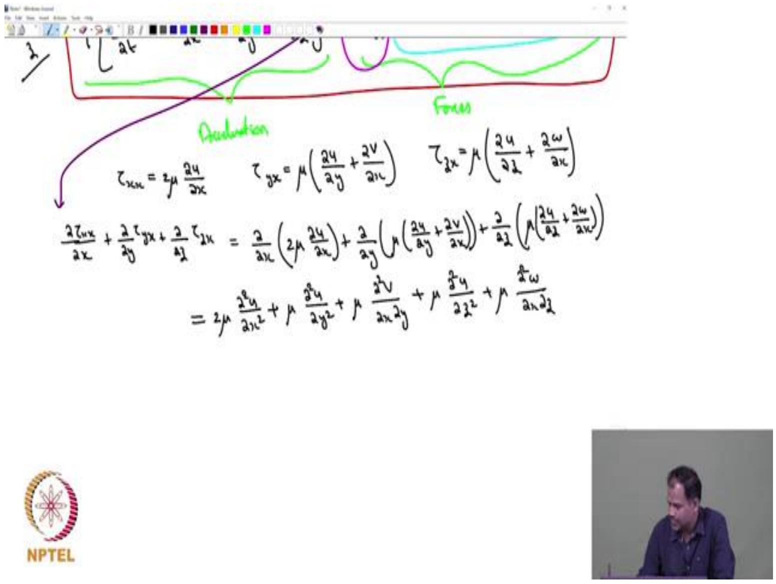

So,

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x};\ \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right); \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Remember, when we derived the Newton’s law of viscosity we simply connected stress with rate of strain here I have written down in a much more general fashion, where each of the components of the shear stress is related to what you call as the velocity how it is connected to the velocity gradient ok.

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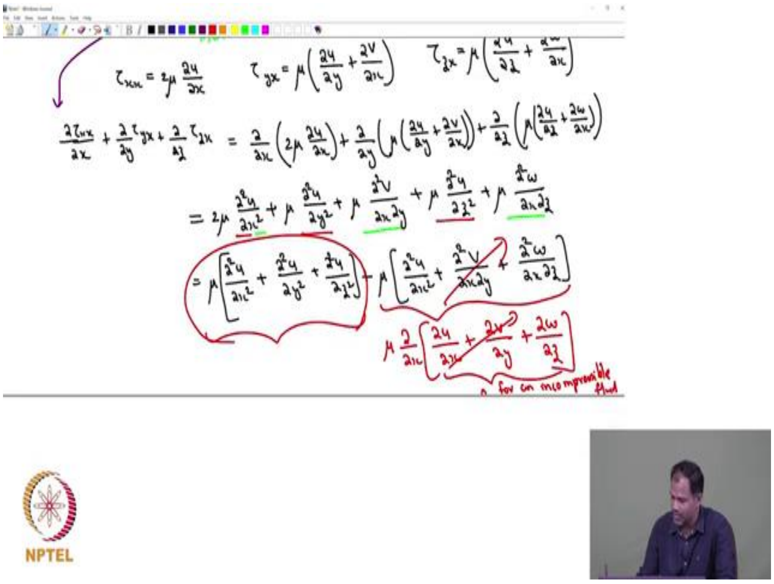

And this can be further simplified.

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \frac{\partial}{\partial x}\left(2\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right)$$

$$= 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x\, \partial y} + \mu \frac{\partial^2 u}{\partial z^2} + \mu \frac{\partial^2 w}{\partial x\, \partial z}$$

$$= \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\, \partial y} + \frac{\partial^2 w}{\partial x\, \partial z}\right]$$

$$= \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \mu \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]$$

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + 0$$

So, this this entire expression is 0 for an incompressible fluid, that is what we had derived incompressible fluid so; that means, it’s not there or in other words it’s not there. So, what I wanted to tell you is that the right hand side of the equations that we derived essentially simplifies into this particular form and therefore, I can substitute that back into the equations that we derived. So, the one that you are seeing in this purple color we can now replace. And if you replace it and we can do that for other equations as well and then the equations simplify further.

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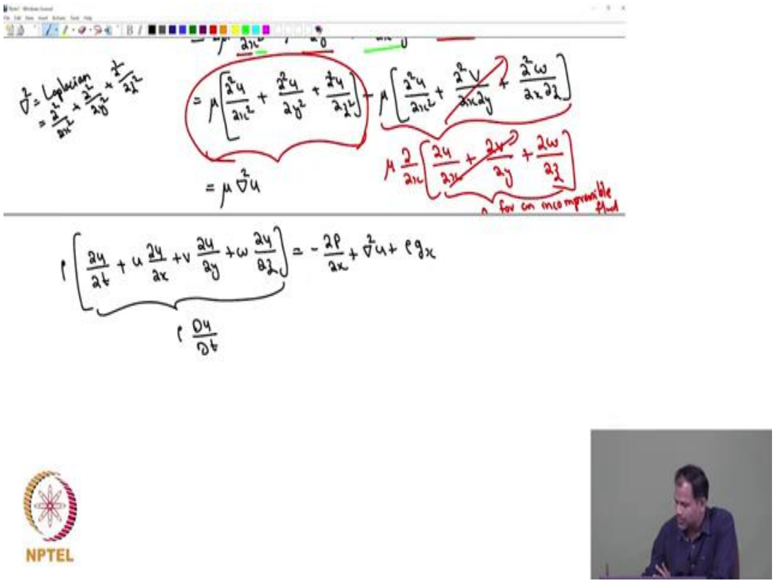

And the simplified equations would be

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=\rho\frac{Du}{Dt}=-\frac{\partial P}{\partial x}+\nabla^2 u+\rho g_x$$

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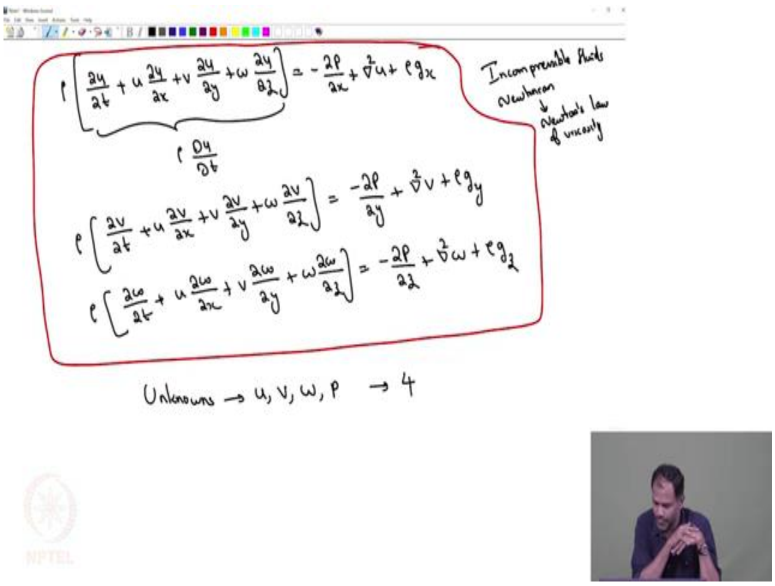

Now, similarly for the y and z momentum equation

$$\rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]=\rho\frac{Dv}{Dt}=-\frac{\partial P}{\partial y}+\nabla^2 v+\rho g_y$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = \rho\frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \nabla^2 w + \rho g_z$$

So, that is the equations of linear momentum expressed as you know only in terms of velocity. So, if you look at now you only have u v w and pressure as the unknowns ok. So, let us see unknowns are u v w and p. So, that is 4 in number how many equations are there? You have three equations here equation of x y z linear momentum and then you also have conservation of mass. So, you will have four equations, four unknowns and given any problem you will be able to solve and find out the solution ok.

The equations look a little more detailed ok, but there are many cases where it is it can be so simplified and solved and we will look at some of the cases later on, which will give us some insight into what does the fluid flow look like in a given situation. So, these equations that we have derived we have assumed Newton's law of viscosity, we have assumed incompressibility. So, this is really applicable for incompressible fluids and we have also used Newtonian assumption in other words we have used Newton's law of viscosity and of course, we are using we have expressed this in Cartesian coordinate system.