Fluid and Particle Mechanics
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Lecture - 51
Laminar and Turbulent Flows - 2

Let us start with continuity equation ok. Let us apply the continuity equation for a turbulent flow we will just take to Cartesian coordinates.

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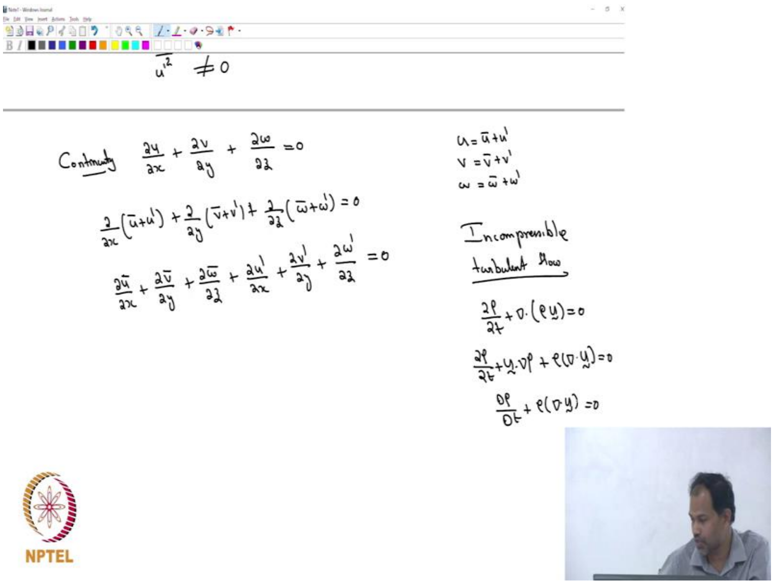

So, I know that my velocity field should satisfy this equation so; that means, my uvw should satisfy this equation. I am going to now split that velocity field into a mean and a fluctuating component. So, I am going to substitute u is equal to u mean plus u prime v equal to v mean plus v prime w is equal to w mean plus w prime so, that I can know whether I can deal with mean values.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$u = \bar{u} + u'; v = \bar{v} + v'; w = \bar{w} + w'$$

Del by del x of u plus u prime plus del by del y of v plus v prime plus del by del z of w plus w prime is equal to 0. And split that dou u by dou x plus dou v by dou y plus dou w by dou z plus dou u prime by dou x plus dou v prime by dou y plus dou w prime by dou z is equal to sorry u bar v bar w bar u bar v bar w bar. I am actually splitting it into mean and fluctuation. See what I know is that the exact value satisfy the governing equation and I am substituting that exact value as mean plus fluctuation and just separated it out doubts.

$$\frac{\partial(\bar{u}+u')}{\partial x}+\frac{\partial(\bar{v}+v')}{\partial y}+\frac{\partial(\bar{w}+w')}{\partial z}=0$$

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}+\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

Yes.

That is incompressible the fluid flow can still be incompressible. What is that? You can have a density fluctuation we are assuming that the flow is incompressible at the moment. In principle, you can have a density fluctuation also though for most of the fluids x; I mean if you deal with gas you will have to worry about otherwise you will not have to worry about hm. So, a fluctuating velocity does not mean it is a incompressible sorry it is compressible a fluctuating velocity does not mean it is compressible ok.

You can have a compressible you can have an incompressible turbulent flow and that is what we are looking at. So, maybe I should write it down.

Yeah, one second yeah.

So, we did when we dealt with continuity equation right we showed that. So, what does it come out to be d rho now del rho by del t plus divergence of rho u equal to 0 that is the full equation. So, we could split this as del rho by del t plus u dot grad rho plus rho into divergence of u equal to 0 so; that means, d rho by dt plus rho into divergence of u is equal to 0 and we said if you are actually following a fluid element and if the fluid element actually has a constant density, it will be an incompressible flow.

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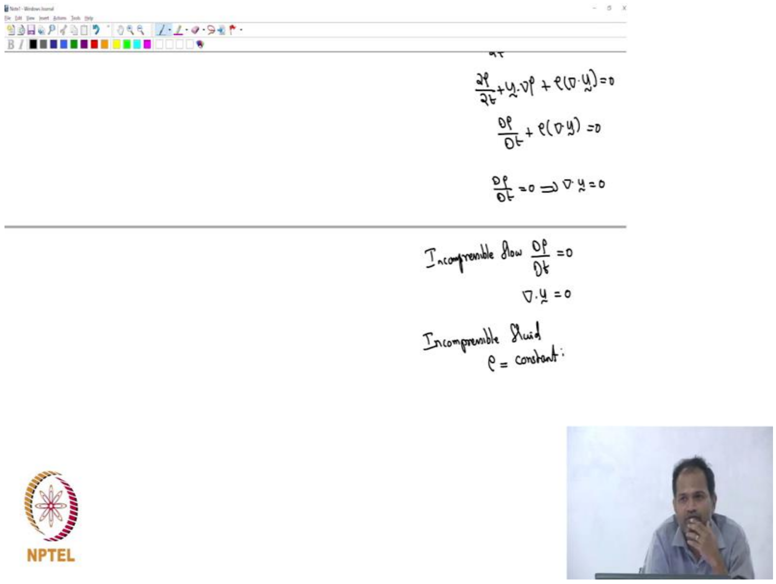

Because if d rho by dt equal to 0, then continuity equation is just divergence of u equal to 0 ok.

So, this is an incompressible fluid incompressible flow. Did we discuss this incompressible flow versus incompressible fluid ok? So, look. So, the what I have written on the right hand side is the continuity equation right. So, del rho by del t plus del dot rho u equal to 0. Now I am expanding the second term which is divergence of the rho u and I am assuming that rho is not a constant, then what happens is that del rho by del t plus u dot grad rho plus rho into divergence of u is equal to 0. So, that is just the product rule of differentiation except that the differentiation is now a divergence ok.

And the if I combine the first two terms, I get a d rho by dt. What is d rho by dt? The substantial derivative of rho plus rho into divergence of u is equal to 0. What does substantial derivative indicate?

Following a fluid particle ok. So, if I am following a fluid particle that is what d rho by dt says that and if I am seeing a change in the density of that fluid particle, then that is what is d that is what d rho by dt will capture, but if I am following a fluid particle and if its density does not change, then d rho by dt is going to be 0. And so, for a incompressible flow. So, for an incompressible flow d rho by dt is 0 and therefore, the continuity equation is just divergence of u is equal to 0. But if you want to talk about an incompressible fluid, then you would simply say rho is a constant ok. But either way you are going to get divergence of u is equal to 0.

$$Incompressible\ flow: \frac{D\rho}{Dt} = \nabla\vec{u} = 0$$

$$Incompressible\ fluid: \rho = constant$$

Anyway that is not important if you didnt really follow the point is the divergence of u is equal to 0 is for a incompressible fluid and incompressible flow ok. So, we are dealing with continuity, we have written down in the Cartesian coordinates and we have split it into mean values and the fluctuations ok. Now, what we are going to do is we are going to integrate this equation ok.

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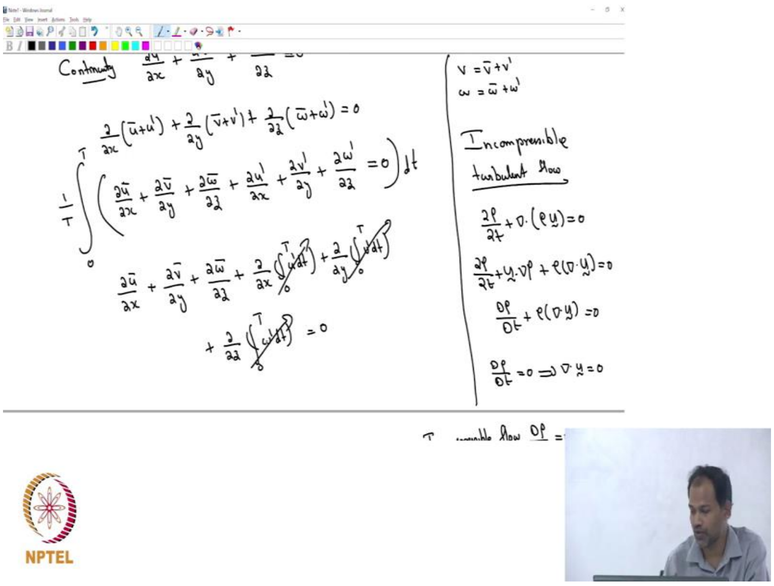

This entire equation we will integrate from 0 to t dt, I am allowed to do that ok where t is some time that I have taken so, that I can average out during that time and therefore, all the fluctuations therefore, you know whatever comes out I can put to 0. So, that is the idea. So, if I do that and then I can do a 1 by T of course, so the first term dou u bar by dou x is that function of time? No. So, I can just do the integration ok. So, it basically gives you dou u bar by dou x that is actually a constant as far as a integration is concerned.

Next term is also a constant next term is also a constant plus u prime u prime is definitely a function of time, but I interchange my integration and differentiation I will first do the integration and then do the differentiation. So, I will write this as rather del by del x of integral 0 to T u prime dt plus del by del y of integral 0 to T v prime dt plus.

Plus del by del z of integral 0 to T w prime dt is equal to 0 and you do know that this is 0 that is 0 that is 0 and therefore you basically get back that dou u bar by dou x plus dou v bar by dou y plus dou w bar by dou z is equal to 0.

$$\frac{1}{T}\int_{0}^{T}\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}+\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)=0$$

$$\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}+\frac{\partial}{\partial x}\left(\int_{0}^{T} u' dt\right)+\frac{\partial}{\partial y}\left(\int_{0}^{T} v' dt\right)+\frac{\partial}{\partial z}\left(\int_{0}^{T} w' dt\right)\right)=0$$

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0$$

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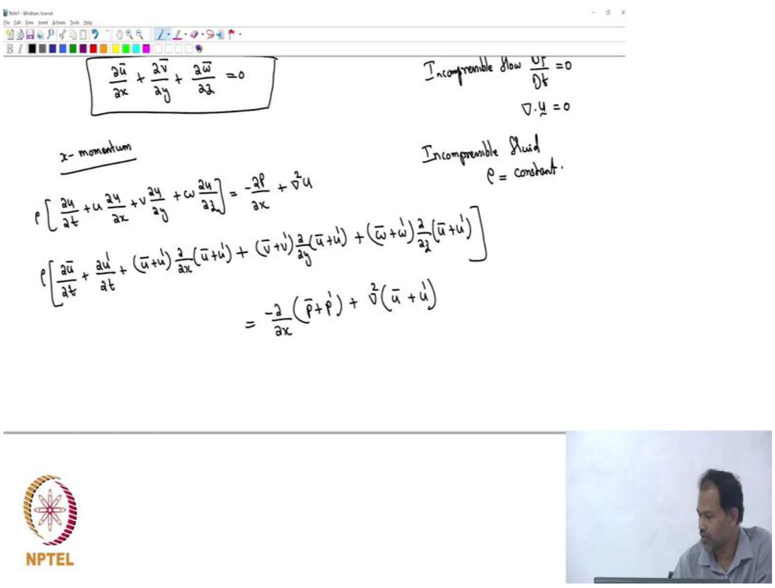

In other words the mean velocity satisfies continuity equation that is what you have proven ok. So, if you if you want to only calculate a mean value of velocity, you can still use the same continuity equation without any change that is clear right. So, let us worry about what is momentum conservation equation going to do ok. So, we will just write down again x momentum and we will just deal with that.

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial p}{\partial x}+\mu\nabla^2 u$$

So, that is my x component of momentum conservation and I am going to substitute u equal to u bar plus u prime plus now in some of the next terms. So, if you substitute u bar plus u prime into del by del x of u bar plus u prime plus the next term is v bar plus v prime times del by del y of u bar plus u prime plus w bar plus w prime times del by del z of u bar plus u prime is equal to minus del by del x of p bar plus p prime plus del square of u bar plus u prime that is what if I simply substitute u is equal to u prime and what happens if we expand ?

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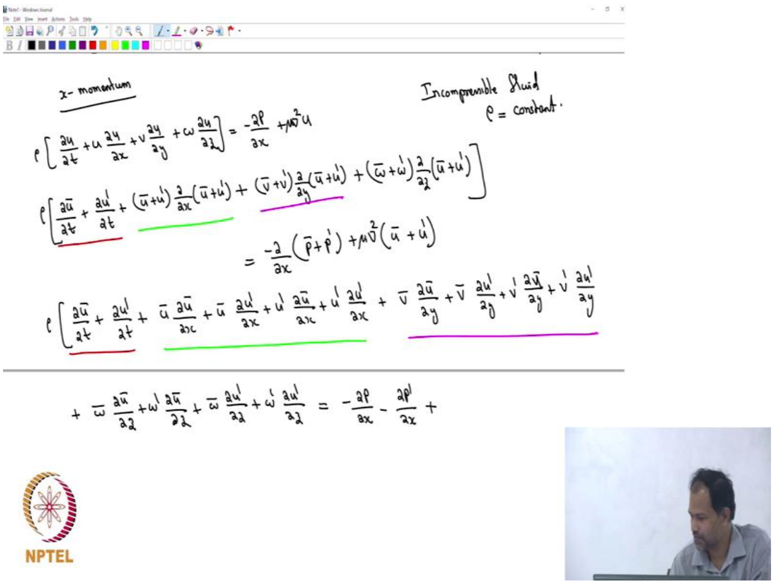

Rho first term dou u bar by dou t plus dou u prime by dou t plus there are four terms that are going to come up now right.

Because its u bar plus u prime times del by del x of u bar plus u times. So, u bar del u bar by del x plus u bar del u prime by del x plus u prime del u bar by del x plus u prime del u prime by del x let us first the four terms. And then there will be four more terms from the next expression v bar dou u bar by dou y plus v bar dou u prime by dou y plus v prime dou v bar by dou dou u bar by dou y plus v prime dou u prime by dou y plus.

So, should I draw things. So, this is that this is that next is oh sorry I did a mistake. So, that is that this is that custom w bar dou u bar by dou y dou z plus w prime dou u bar by u prime by dou z plus w w prime dou u prime by dou z.

I am just expanding ok; did I miss something in the equation this equation? Some important fluid property viscosity.

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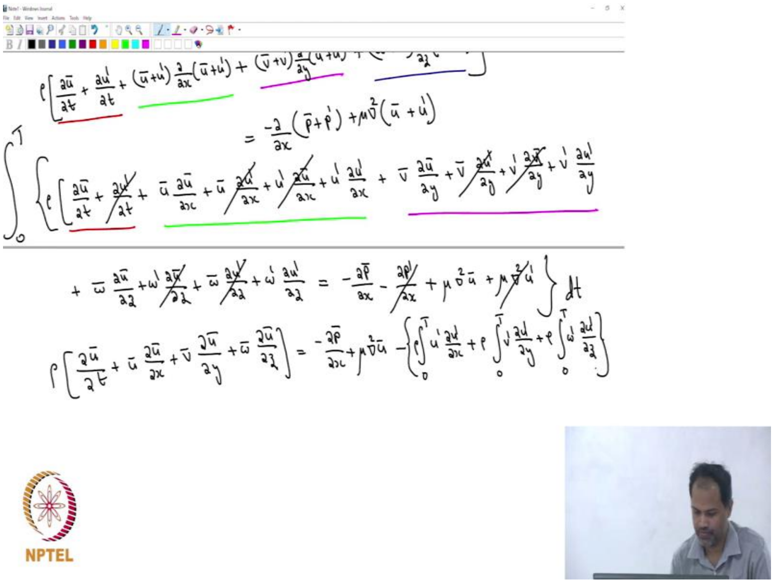

$$\rho\left[\frac{\partial(\bar{u}+u')}{\partial t}+(\bar{u}+u')\frac{\partial(\bar{u}+u')}{\partial x}+(\bar{v}+v')\frac{\partial(\bar{u}+u')}{\partial y}+(\bar{w}+w')\frac{\partial(\bar{u}+u')}{\partial z}\right]$$
$$=-\frac{\partial(\bar{p}+p')}{\partial x}+\mu\nabla^2(\bar{u}+u')$$

$$\rho\left[\left(\frac{\partial\bar{u}}{\partial t}+\frac{\partial u'}{\partial t}\right)+\left(\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{u}\frac{\partial u'}{\partial x}+u'\frac{\partial\bar{u}}{\partial x}\;u'\frac{\partial u'}{\partial x}\right)+\left(\bar{v}\frac{\partial\bar{u}}{\partial y}+\bar{v}\frac{\partial u'}{\partial y}+v'\frac{\partial\bar{u}}{\partial y}+v'\frac{\partial u'}{\partial y}\right)\right.$$
$$\left.+\left(\bar{w}\frac{\partial\bar{u}}{\partial z}+\bar{w}\frac{\partial u'}{\partial z}+w'\frac{\partial\bar{u}}{\partial z}+w'\frac{\partial u'}{\partial z}\right)\right]=-\frac{\partial\bar{p}}{\partial x}-\frac{\partial p'}{\partial x}+\mu\nabla^2\bar{u}+\mu\nabla^2u'$$

Is this simple expansion ok? What is the next step we should do? Integrate ok. Integrate this expression from 0 to T and throw away all unnecessary terms that we have. So, we this entire expression integrated from 0 to T dt. So, that is what we are doing. So, all the terms that have just u prime just one fluctuating variable will go away when we do this integration right.

But there if there are any products that are going to stay there this is the way that happened for the continuity equation right. For continuity equation we had seen that we could interchange the order of differentiation and integration. And therefore, all the terms that were there as just primes left you know it was it was gone, but then if there was a u prime square or a u prime v prime all of them are going to stay back.

So, therefore, I can throw away all the necessary terms that we have written down now which would include that that this and that. So, the moment I do the integration all of them we will go away and therefore, what I have left out is rho times dou u bar by dou t plus u bar dou u bar by dou x plus v bar dou u bar by dou y plus w bar dou u bar by dou z is equal to minus dou p bar by dou x plus mu del square u bar minus integral 0 to T u prime dou u prime by dou x multiplied by rho plus rho integral 0 to T v prime dou u prime by dou y plus rho integral 0 to T w prime dou u prime by dou z that is what we end up with.

$$\rho\left[\left(\frac{\partial\bar{u}}{\partial t}\right)+\left(\bar{u}\frac{\partial\bar{u}}{\partial x}+\int_{0}^{T}u'\frac{\partial u'}{\partial x}\,dt\right)+\left(\bar{v}\frac{\partial\bar{u}}{\partial y}+\int_{0}^{T}v'\frac{\partial u'}{\partial y}\,dt\right)+\left(\bar{w}\frac{\partial\bar{u}}{\partial z}+\int_{0}^{T}w'\frac{\partial u'}{\partial z}\,dt\right)\right]=-\frac{\partial\bar{p}}{\partial x}+\mu\nabla^{2}\bar{u}$$

$$\rho\left[\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\bar{w}\frac{\partial\bar{u}}{\partial z}\right]=-\frac{\partial\bar{p}}{\partial x}+\mu\nabla^{2}\bar{u}-\rho\left\{\int_{0}^{T}u'\frac{\partial u'}{\partial x}\,dt+\int_{0}^{T}v'\frac{\partial u'}{\partial y}\,dt+\int_{0}^{T}w'\frac{\partial u'}{\partial z}\,dt\right\}$$

So, some of the terms stayed back right all the terms which had two primed variables which came which were the products of the fluctuations they will not go away during the integration they will stay ok. So, what you are seeing is that the left hand side is its early like what we had for our you know the exact solution all the terms that had these double primes, I have moved to right hand side for convenience and written down and you see that if you compare with the actual the full expression that we had this is what we used to have but now this has come out to be extra which are basically some integrals which we of course, do not know.

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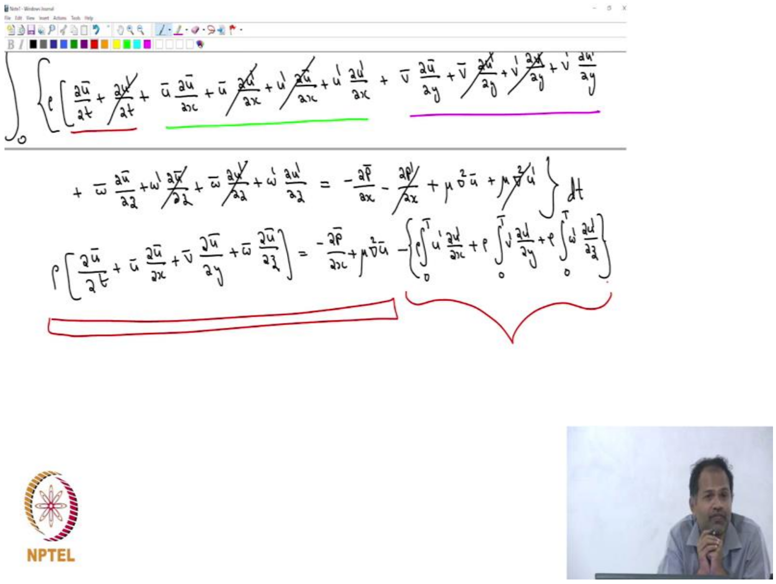

Or another way of saying it is that, the mean velocity does not satisfy the x momentum equation ok. Then if you want to write down the x momentum equation for the mean velocity along with the terms that you originally had, there are going to be additional contributions those additional contributions are in terms of fluctuating velocities or rather products of fluctuating velocities. But those are the products of fluctuating velocities that we are really looking for we do not know what it is. So, we actually try to get that down get remove that, but we could not remove it they will not go away they will remain there and that is basically the main issue with turbulent flow.

So, the whole problem comes out is can we get rid of them or can we use some other fact to remove that and therefore, able to solve the equation is that clear? So, this will happen for any you know we y momentum is z momentum all the equations that is clear ok. So, basically you are now convinced that its not a simple problem to solve ok. So, therefore, we actually need to do some alternate ways of dealing with these equations which we will see in the next class any questions?

Yeah.

there is a dts.