Fluid Mechanics
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Lecture – 05
Equation of linear momentum 1

So, the last class we looked at the differential equation that describes mass conservation and that is because we wanted to express mechanics of fluids, using a differential approach and mass conservation is a fact that is associated with any system that we are going to deal with and therefore, we expressed that fact as a partial differential equation. And when we did that we basically had two variables in that system one was density and the other was the velocity field.

We also simplified the relation for a constant density and then derived the equation of an incompressible fluid as well. So, we are going to continue and. So, we are trying to talk about dynamics of fluids and as far as dynamics of fluids are concerned, these things are moving they are in motion and therefore, the way they start moving is because you are applying some external force onto the system. And so and the kind of fluid flow that is generated is a consequence of how much is the force that you are applying.

So, what you are really looking for is some kind of a relation between the velocities that are generated and connect that to the force that you have applied. Now for simple systems like for example, if you are throwing a stone you know how to write down the velocity of the stone at every point, you will actually write down the Newton’s second law which is nothing, but force is equal to mass into acceleration. And if you know the force you can calculate what is the acceleration and therefore, you will know what is the velocity at any instant of time.

So, we need something very similar a corresponding expression, but for fluid flow and also we need to do it in an Eulerian frame. Because we are going to look at velocity at every point in space and in time and we want to talk about forces acting at a particular point and see what is the kind of velocities that are generated. So, that is what we were going to look at ok. So, we are going to derive now equation of linear momentum conservation.

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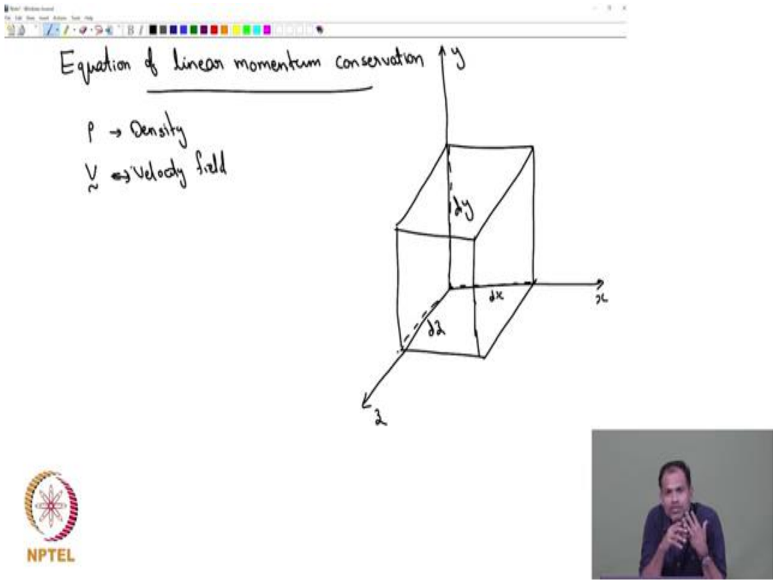

So, we are going to talk about how what is the law that is going to come about, then we are going to talk about conservation of linear momentum. And we will proceed exactly the way that we did yesterday, we will first write the will first consider a control volume let us consider a control volume ok. So, that is a control volume and let us say that we are going to fix it at the origin of our coordinate system. So, that is the coordinate system. So, choose x y and z ok.

So, this length is dx that length is dy and this length is dz. So, these are the three lengths and what we are going to say is that, the fluid actually has a density ρ. So, ρ is the density of the fluid and ρ is the density of the fluid and let us say V is the velocity of the fluid so it’s a velocity field ok. So, which varies at x y z and time and what we need to do now is that we will find out, how much is the momentum that is going into the system and how much is the momentum that is going out of the system and whatever is left out will give rise to the accumulation. So, just like we proceeded yesterday, we will actually talk about momentum in and out through each face.

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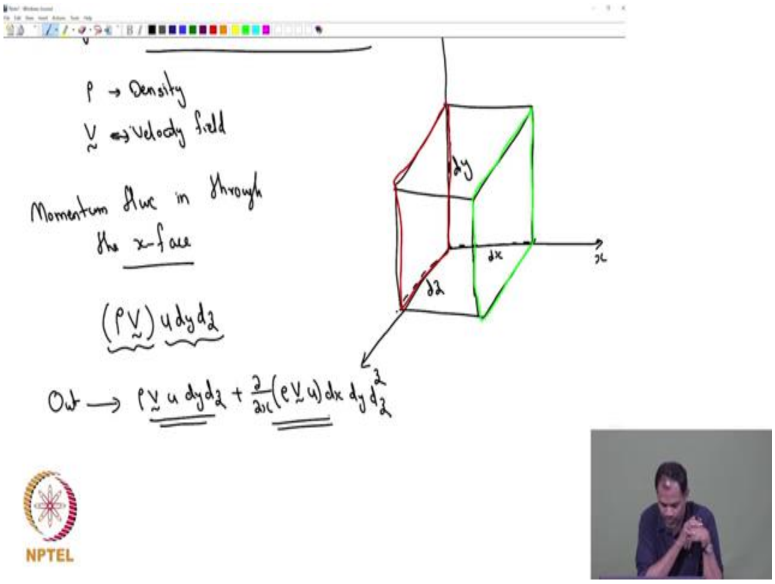

So, let us mark let us say the first face that we are going to consider is what I have marked with red color ok. So, that is a face that is perpendicular to the x direction and its area is dy dz. So, if you, so, what is the momentum that is associated with the fluid? So, we need to get let us say momentum flux; momentum flux in through the x face. So, that is going to be equal to ρ times V. So, that is going to be the momentum density because its mass times velocity is your momentum.

So, this is density is mass per unit volume. So, mass per unit volume times velocity is going to give you the momentum density and multiplied by u times dy d z. So, u times dy dz is going to be the flow rate. So, velocity times area is the flow rate. So, this is the flow rate times the momentum is the total momentum that is getting in through the face dy by dz and what will be the momentum that is out through the other face? Let us say the face that I mark it as green again we can use the Taylor series expansion:

$$momentum_{in} = (\rho\vec{V})u\, dy\, dz$$

$$momentum_{out} = (\rho\vec{V})u\, dy\, dz + \frac{\partial}{\partial x}(\rho\vec{V}u)dx\, dy\, dz$$

So, similarly we can do it for each face. So, let us do that.

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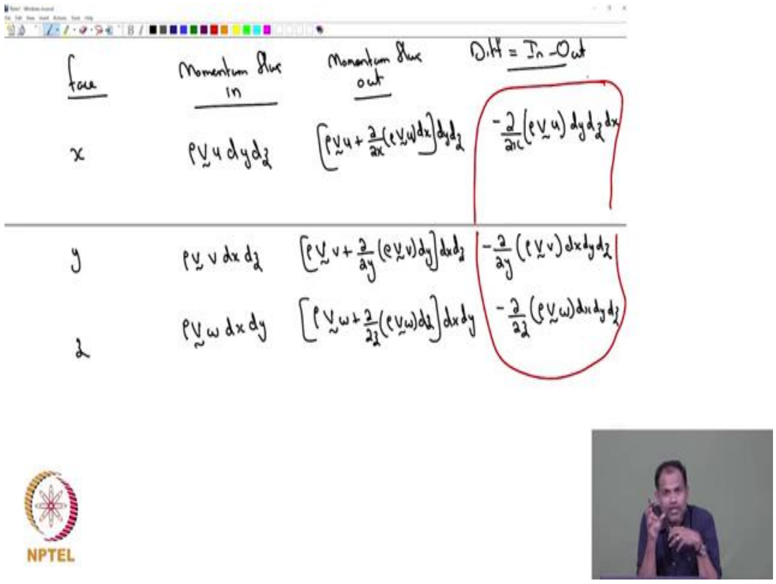

So, let us say face momentum flux in momentum flux out and the difference which is in minus out. Now, we can do the same thing for the y face. So, let us look at the y face now. So, the y face is the momentum that is entering from the bottom and then the momentum that is going above from the top. So, similar calculation.

FACE	**MOMENTUM FLUX IN**	**MOMENTUM FLUX OUT**	**DIFFERENCE**
x	$(\rho\vec{V})u\ dy\ dz$	$(\rho\vec{V})u\ dy\ dz + \frac{\partial}{\partial x}(\rho\vec{V}u)dx\ dy\ dz$	$-\frac{\partial}{\partial x}(\rho\vec{V}u)dx\ dy\ dz$
y	$(\rho\vec{V})v\ dx\ dz$	$(\rho\vec{V})v\ dx\ dz + \frac{\partial}{\partial y}(\rho\vec{V}v)dx\ dy\ dz$	$-\frac{\partial}{\partial y}(\rho\vec{V}v)dx\ dy\ dz$
z	$(\rho\vec{V})w\ dy\ dx$	$(\rho\vec{V})w\ dy\ dx + \frac{\partial}{\partial z}(\rho\vec{V}w)dx\ dy\ dz$	$-\frac{\partial}{\partial z}(\rho\vec{V}w)dx\ dy\ dz$

So, this is the quantity that is interesting to us, because that tells you what is the momentum in minus momentum out. And that momentum in minus momentum out should be the should be equal to the rate of change of momentum in that particular volume. So, if there is in is not equal to out; that means, the momentum of the fluid in that particular volume is going to be changed and we know how to do that.

The rate of change of momentum.

$$Rate\ of\ change\ of\ momentum = \frac{\partial}{\partial t}(\rho\vec{V})dx\ dy\ dz$$

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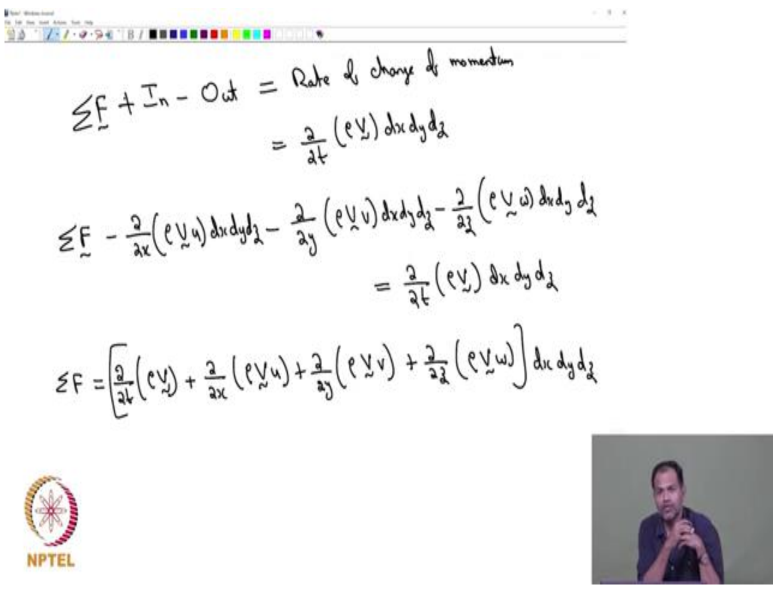

So, rate of change of momentum must be then equal to the parts that I have marked in that red box, but then if you think about it a little more seriously, what you know is that the momentum or the rate of change of momentum is not just because there is a difference in the momentum IN minus momentum OUT, you can get a rate of change of momentum if you apply a force also right.

So, for example, if something is moving with a certain momentum and if you apply a force, you are actually going to change its momentum. So, the rate of change of momentum is not just equal to IN minus OUT, it actually has to be also equal to the total forces acting on that fluid element ok. So, therefore, that right equation is some more forces plus momentum in minus momentum out has to be equal to the total rate of change of momentum.

So, therefore, we find that our equation should really be sum of forces that is acting plus all the quantities, that we have found out as in this red curve red box. So,

$$\sum F - \frac{\partial}{\partial x}\left(\rho\vec{V}u\right)dx\,dy\,dz - \frac{\partial}{\partial y}\left(\rho\vec{V}v\right)dx\,dy\,dz - \frac{\partial}{\partial z}\left(\rho\vec{V}w\right)dx\,dy\,dz$$
$$= \frac{\partial}{\partial t}\left(\rho\vec{V}\right)dx\,dy\,dz$$

So, that is the expression that now we have gotten when we have said that the sum of the forces plus momentum in minus momentum out is equal to rate of change of momentum and now we need to simplify this expression. So,

$$\sum F = \left[\frac{\partial}{\partial t}\left(\rho\vec{V}\right) + \frac{\partial}{\partial x}\left(\rho\vec{V}u\right) + \frac{\partial}{\partial y}\left(\rho\vec{V}v\right) + \frac{\partial}{\partial z}\left(\rho\vec{V}w\right)\right] dx\,dy\,dz$$

$$\sum F = \left[\rho\frac{\partial\vec{V}}{\partial t} + \vec{V}\frac{\partial\rho}{\partial t} + \rho u\frac{\partial\vec{V}}{\partial x} + \vec{V}\frac{\partial(\rho u)}{\partial x} + \rho v\frac{\partial\vec{V}}{\partial y} + \vec{V}\frac{\partial(\rho v)}{\partial y} + \rho w\frac{\partial\vec{V}}{\partial z}\right.$$
$$\left. + \vec{V}\frac{\partial(\rho w)}{\partial z}\right] dx\,dy\,dz$$

$$\sum F = \left[\left[\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\vec{V} + \rho\frac{\partial\vec{V}}{\partial t} + \rho u\frac{\partial\vec{V}}{\partial x} + \rho v\frac{\partial\vec{V}}{\partial y}\right.$$
$$\left. + \rho w\frac{\partial\vec{V}}{\partial z}\right] dx\,dy\,dz$$

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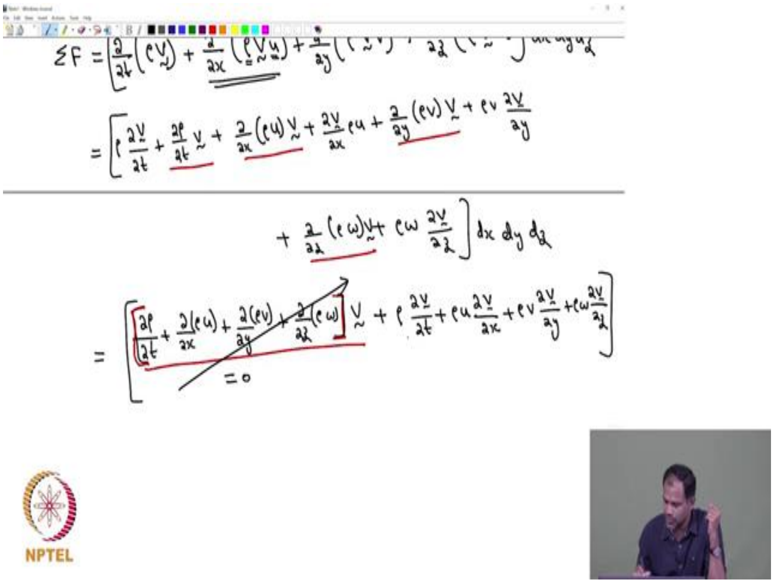

So, these are the terms that are in underlined as red in the above expression and other terms. Now what is this thing term in this square bracket that I have put that is nothing, but continuity equations. If you look at your notes it's essentially equal to 0.

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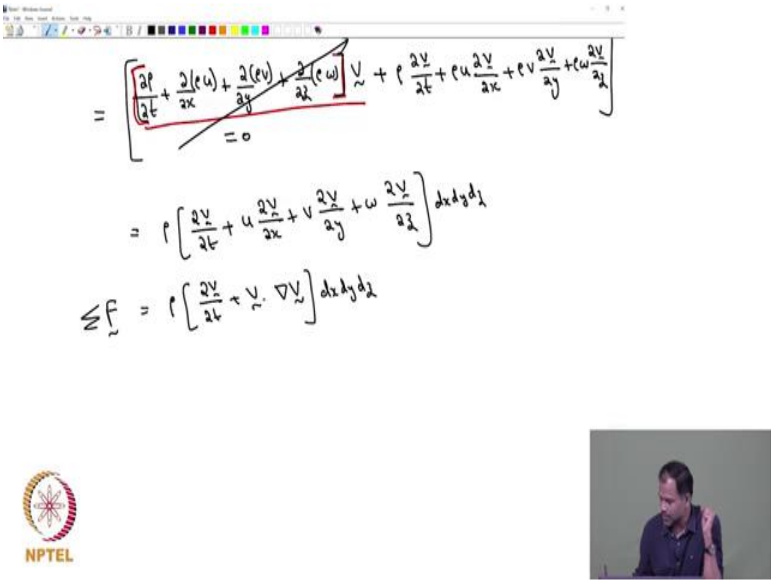

So,

$$\sum F = 0 + \rho\left[\frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}\right]dx\,dy\,dz = \rho\left[\frac{\partial\vec{V}}{\partial t} + \nabla.\vec{V}\right]dx\,dy\,dz$$

So, that is where we have ended up with. So, now, we need to talk about the forces acting on the fluid element.

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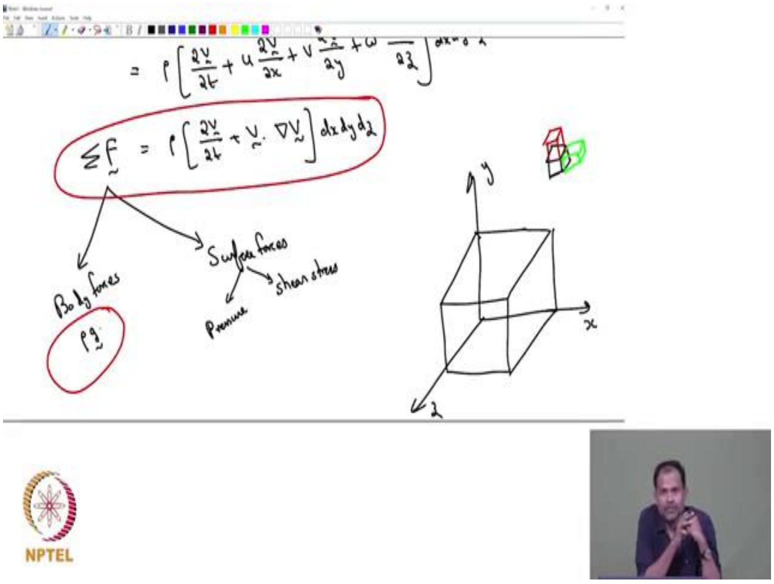

The forces can be actually two types; one is called body forces and the other is called surface forces. So, when I say body forces it’s an external force that acts on the entire the on or the force that depends on the volume of the fluid element that you are looking at and you know one such force for example, gravity ok. Gravity depends upon what is the amount of the fluid that you are looking at and here its obvious that if I take dx dy dz to the left hand side, I really need what is the force density or the force acting per unit volume and the force acting per unit volume is nothing, but ρ times g ok.

So, that I, so, the really the forces m times g, but if you are talking about the force per unit volume its ρ times g ok. So, ρ g is the force that is going to come up as a body force. Now surface forces are the force that acts on the surfaces of the fluid element. So, if you have let us say let us look at the same fluid element, will make another diagram ok. So, that is the fluid element and we had it has x y, I will just system x y z ok.

Now on any face of it let us say this fluid element is part of the fluid this fluid element is surrounded by other fluid elements. So, the other fluid elements basically will be exerting some forces on this fluid element as well and that force is exerted on the surface of this fluid element. So, what I am trying to say is that let us say if that is one of the fluid element above that draw again ok.

So, that is one; then above it there is going to be another fluid element, then on the side there is going to be another fluid element and so on. So, the red fluid is going to exert some force on the black fluid, the black fluid is going to exert some force on the green fluid. So, this is all forces exerted on the surface not on the not its that force exerted is essentially proportional to the surface area because that force is getting transferred across the area ok. So, those are the kind of forces that you call as surface forces and there are two surface forces one is something that you already know and that is the pressure the second is the shear stress.

So, if you remember the definition of the Newton's law of viscosity, we defined it as stress is equal to a constant times strain and that constant is what we called as viscosity. So, stress is again defined as force per unit area. So, the pressure forces shear stresses are examples of surface forces ok. Because you can see that that force is defined per unit area unlike the body force which is defined per unit volume, because it depends upon the mass of the object that is under consideration. So, since we now understand what are the body forces and what are the surface forces.

So, as far as body force is concerned, we know what is the expression that we need to use in the equation that we have. So, this is the form of the equation at the moment, this is the form of the equation at the moment ok. So, the left hand side we have simply written a test total forces, but we need to be more explicit about it. So, ρ g is the body force contribution that is going to go into the sum of f and then we need to find out what are the what is the form of surface forces that should go in the expression for capital F.