Fluid Mechanics
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Lecture – 04
Equation of continuity- Differential approach

So, to analyze fluid flow ok, we could take two different approaches one is an integral approach and one is a Differential approach.

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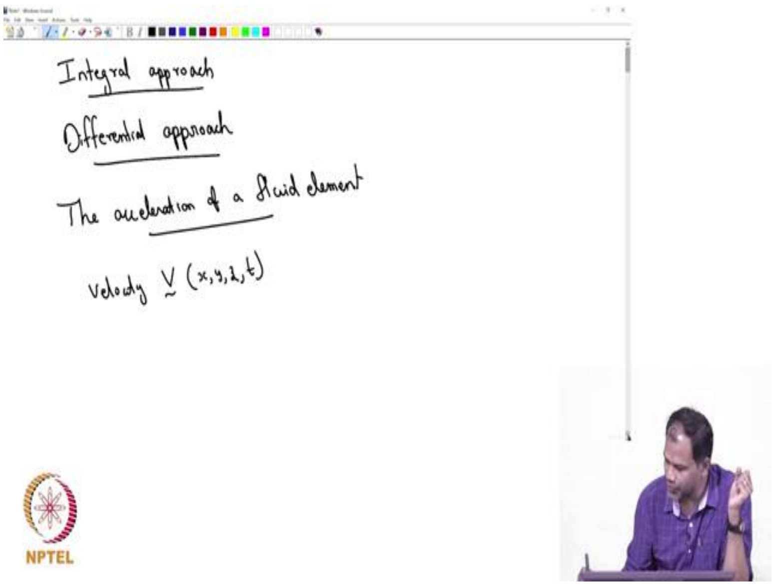

And we are going to actually follow the differential approach in this course because it is going to tell you how the fluid flow is at various points in domain. And it is going to give you actually the details of how does the flow look like and it is important to know what it is ok. So, we are going to look at now the differential approach of solving fluid mechanics problems. So, the first thing that I want to define is the acceleration of a fluid element.

So, let us say you have some arbitrary flow and you can define then a velocity field. So, say let us say the velocity is some capital V, I am putting that squiggle to show that it is actually a vector, because velocity is a vector it has a magnitude and a direction. But the velocity will not be uniform, the velocity will be a function of space it will change with time. So, in order to represent that I can say that it is a function of x, y, z,t so at every point in space; it is going to be different:

$$Velocity = \vec{V}(x, y, z, t)$$

So, that is the velocity field that I have gotten in some domain, but remember we is said to be a function of x, y, z and time, so; that means, that every point the velocity is changing and it also going to change with time and my intention is to now calculate what is the acceleration of the fluid element. Now, you could calculate acceleration in two different ways.

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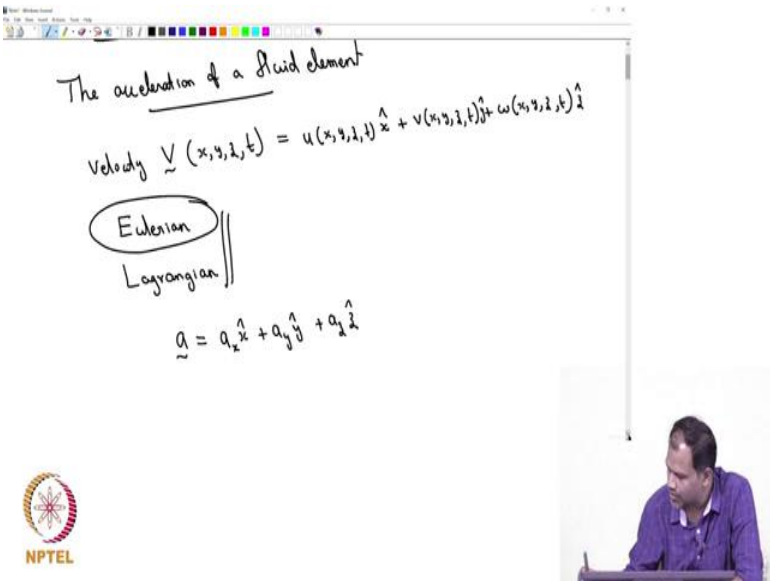

But let us see first of all velocity ok. So, velocity; so since it is a vector, it has an x component, it has a y component and an z component. So, let me write it explicitly:

$$\vec{V}(x, y, z, t) = u(x, y, z, t)\hat{x} + v(x, y, z, t)\hat{y} + w(x, y, z, t)\hat{z}$$

So, this is basically nothing, but the velocity written in a vector form with each component u v and w are the components of velocity in each directions. And our intention now is to calculate what is the acceleration associated with this velocity field? There will be an acceleration because velocity is changing ok; it is changing in space and time; so you will definitely have an acceleration.

The question is how are you going to calculate the acceleration? You can do the calculation in two different ways either you can talk about an Eulerian approach or you can talk about a Lagrangian approach ok. So, the Eulerian approach is that if you are looking at a quantity and you are fixed ok. So, the frame of reference is fixed, you are looking at a particular point and you are looking at the changes; that is what you will have as an Eulerian field. As opposed to the Lagrangian field in which you will be actually following a fluid particle.

So, let me give you an example; let us say you know you, you have a you are your monitoring traffic ok. And let us say that you have got you are looking at various vehicles which are moving and you are trying to find out what is the velocity with which different things are moving ok. And you reach some junction; let us say some you know signal not signal actually at some junction, you will stay there and you will look at what is the velocity with which each vehicle is going ok.

So, you are looking at a particular point which is actually that junction and you are looking at the speed of the vehicles. So, that is what is meant by Eulerian approach because you are looking at a particular point and then trying to find out what is the velocity field ok. On the other hand, let us say you are looking at a particular vehicle let us say you show you picked one car and then you are trying to follow that car and trying to find out what is the velocity of that car? Then it becomes a Lagrangian approach that is because you are picking up a element a fluid element or in this case it is a car and then you are actually following the car. So, then it becomes a Lagrangian approach ok.

Now, when I have written V as a function of x, y, z and t; I am actually defining my Eulerian field because I am looking at a particular point and then I am trying to define what is the velocity field ok. So, this is the difference between Eulerian and Lagrangian approach and we would be actually using the Eulerian description because we have a fixed domain and we are interested in telling how things change at a particular point and in time. And now I want to define acceleration associated with this velocity field acceleration ok.

So, acceleration is going to be:

$$\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$$

Now, let us say you were throwing a stone and you are trying to follow the stone which basically becomes the Lagrangian approach. In which case its very easy to calculate the acceleration because you will find out how much does the velocity change in time when you are looking at the stone. So, that is how you will calculate the acceleration.

Now here you have to do the same thing you want to look at a fluid element and you want to find out how much does the velocity change when you are looking at a fluid element. The only complication is that the fluid element that you are looking at time t would have reached somewhere else at time t plus ∂ta t. And you are not interested in finding out what is the velocity or the acceleration at that point you were originally interested in finding out the acceleration of a fluid element at a particular point.

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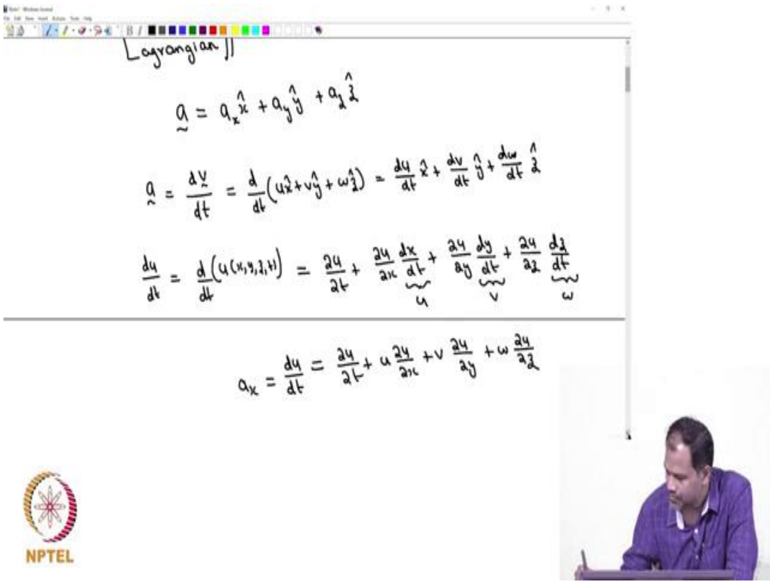

So, we can define let us say acceleration as the total change in velocity ok; which I am writing it as dv by dt where v is my velocity field; I am going to differentiate my velocity with respect to time and going to calculate what is the acceleration.

$$\vec{a} = \frac{d\vec{V}}{dt}$$

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}(u\hat{x} + v\hat{y} + w\hat{z}) = \frac{du}{dt}\hat{x} + \frac{dv}{dt}\hat{y} + \frac{dw}{dt}\hat{z}$$

So, therefore, if I want to take a total derivative; I should be really using the chain rule

$$\frac{du}{dt} = \frac{d}{dt}\big(u(x, y, z, t)\big) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

I have just applied the chain rule because u is a function of x, y, z and t. Now,

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

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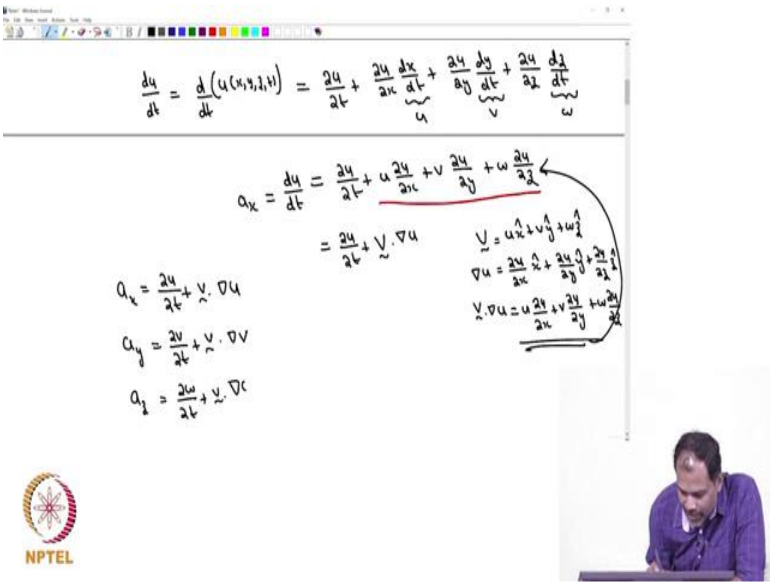

So, that is a component of acceleration in the x direction. This can be simplified further:

$$a_x = \frac{\partial u}{\partial t} + \vec{V}.\nabla u$$

$$Where, \vec{V} = u\hat{x} + v\hat{y} + w\hat{z}$$

$$\nabla u = \frac{\partial u}{\partial x}\hat{x} + \frac{\partial u}{\partial y}\hat{y} + \frac{\partial u}{\partial z}\hat{z}$$

And what I am doing is; I am taking a dot product between this velocity vector and the gradient vector.

$$\vec{V}.\nabla u = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

Now,

$$a_x = \frac{\partial u}{\partial t} + \vec{V}.\nabla u$$

$$a_y = \frac{\partial v}{\partial t} + \vec{V}.\nabla v$$

$$a_z = \frac{\partial w}{\partial t} + \vec{V}.\nabla w$$

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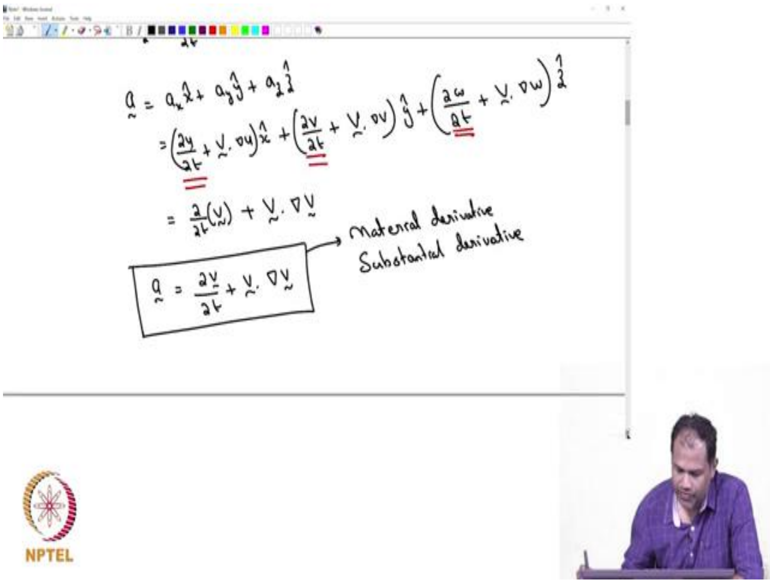

Or combining all of them; I can write the acceleration vector:

$$\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z} = \left(\frac{\partial u}{\partial t} + \vec{V}.\nabla u\right)\hat{x} + \left(\frac{\partial v}{\partial t} + \vec{V}.\nabla v\right)\hat{y} + \left(\frac{\partial w}{\partial t} + \vec{V}.\nabla w\right)\hat{z}$$

So, if I combine this that and that I get that is equal to ∂ by ∂ t of this is the x component of velocity, y component of velocity, z component of velocity; that is just velocity itself plus the other three terms combine to give you:

$$\vec{a} = \frac{\partial \vec{V}}{\partial t} + \vec{V}.\nabla\vec{V}$$

So, that is the way you have to calculate acceleration in an Eulerian frame of reference ok. So, really this kind of derivative is called material derivative or substantial derivative or even sometimes total derivative ok.

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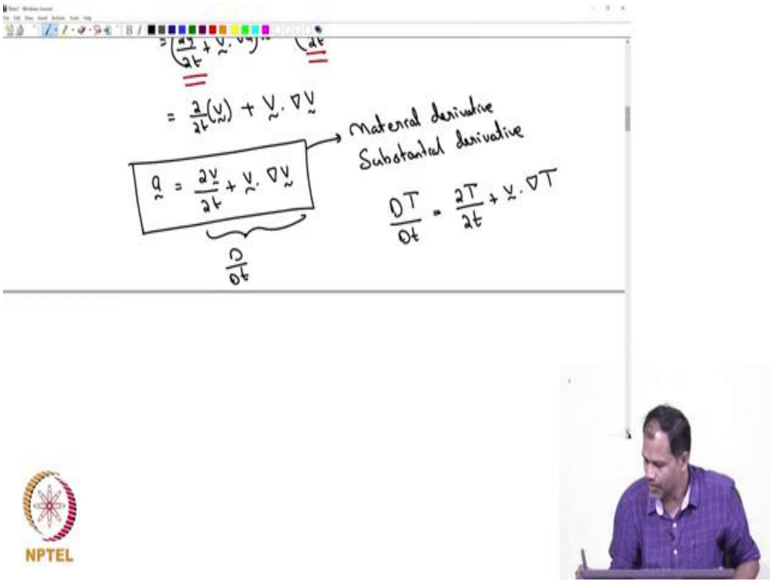

So, this basically gives you how what is the; this expression gives you the acceleration, but you could find out the material derivative of any quantity of interest. For example, you can calculate the material derivative for temperature, pressure. So, if you want to calculate it for temperature that would be; so, this is typically denoted with a simple D/Dt as:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{V}.\nabla T$$

So, that tells you what is the material derivative of temperature; what is the physical meaning of material derivative? It is just that if you are following a fluid particle; what is the change that you are going to see? So, if you are going to calculate the material derivative of temperature, if you are following a fluid particle; what is the change in temperature that you are going to see; so, that is what material derivative represents. So, in this box expression if you are following a fluid particle; what is the change in the velocity that you are going to see and that is nothing, but the acceleration. So, that is the way you have to define acceleration.

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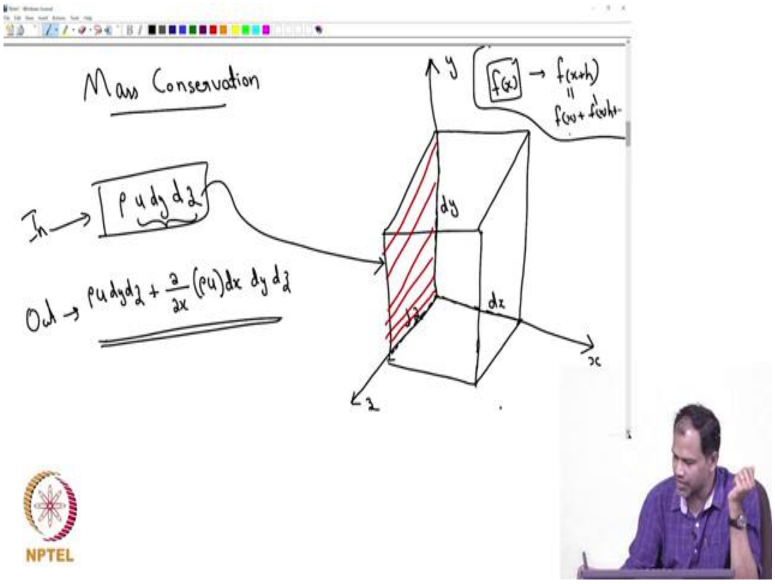

Now having found; acceleration, the next thing that you are going to you do; so you are going to look at a fundamental law that we will keep using ok. And that is nothing, but mass conservation; it is a very simple fact ok. Unless you know there is some reaction or there is some source you know or unless; you know there is a sink the mass of the fluid; mass of the fluid quantity that you are going to look at, it always remains fixed; it is not going to change.

So, you basically want to express that fact as a mathematical equation and that is what we are going to do now. So, what we will do; again remember we are actually doing a differential approach. So, our intention is to write down this fact of mass conservation as a differential equation. And the way you would do its will start by considering a fluid element; so let us say we will take a fluid element ok. So, that is a fluid element which is located at the origin of a coordinate system; so I will draw the coordinate system.

So, we will select this as x axis, that as y axis and this as the z axis ok. So, each of this small elements; so to the small elements; let us say this side is dx, this is dy and this is d z. And let us say there is some fluid that is flowing and you are looking at a small domain. So, this is not a fluid element; this is a small element that you are considering in your entire domain and there is a fluid that is must be flowing in the domain.

And then let us say that fluid is going to; so we will consider each face first let us say the face that I am going to shade ok. So, that is the face that we are considering in the first, which is actually face that is perpendicular to x direction ok; which is in located in the y z plane. And what I am trying to calculate is; what is the total amount of fluid that is going to go through that that face ok?

So, if I look at the face x and I want to calculate what is the total fluid that is entering; so that is going to be. So, we what I want to write is the total mass that is entering and the total mass is going to be equal to density times the volumetric flow rate; so density times the volumetric flow rate. So, the volumetric flow rate is going to be ρ times u times d y, d z. So, this d u times d; so u is the velocity that is perpendicular to the plate velocity times area dy dz is a small area; u dy dz gives you the flow rate multiplied by density; gives you the mass flow rate:

$$m_{in} = \rho u \; dy \; dz$$

So, this is the mass that is going to be entering from this side. Now I want to again write down what is the mass that is going to leave. Now in the absence of any other information the best way to do it is to write down it has a Taylor expansion. So, if ρ u times dy by dz is entering the left face, then let us say on the you move a distance dx the one which is exit in is going to be:

$$m_{out} = \rho u \; dy \; dz + \frac{\partial}{\partial x}(\rho u) dx \; dy \; dz$$

So, this is nothing, but the Taylor series expansion. So, it is basically that fact is that what we have used to write down what is the; what is the amount that is going out. So, let us actually try to make a table now based on this figure.

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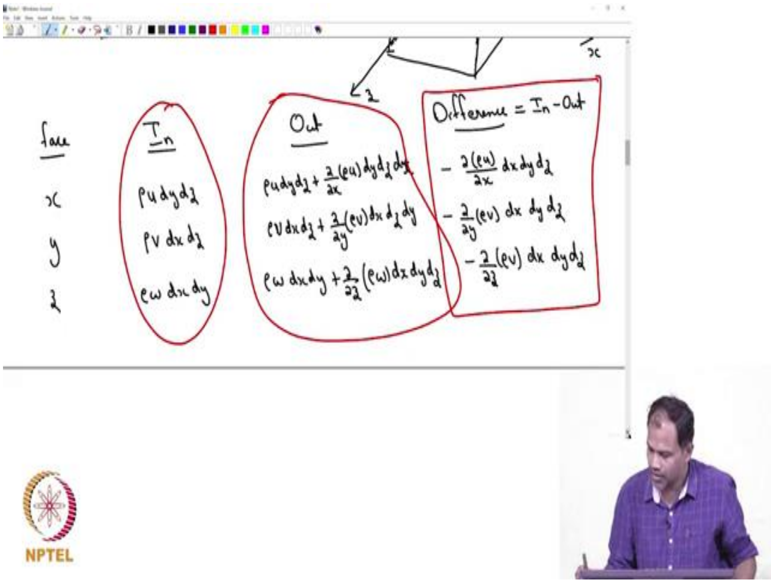

FACE	IN	OUT	DIFFERENCE
x	$\rho u\, dy\, dz$	$\rho u\, dy\, dz + \frac{\partial}{\partial x}(\rho u) dx\, dy\, dz$	$-\frac{\partial}{\partial x}(\rho u) dx\, dy\, dz$
y	$\rho v\, dx\, dz$	$\rho v\, dx\, dz + \frac{\partial}{\partial y}(\rho v) dx\, dy\, dz$	$-\frac{\partial}{\partial y}(\rho v) dx\, dy\, dz$
z	$\rho w\, dx\, dy$	$\rho w\, dx\, dy + \frac{\partial}{\partial z}(\rho w) dx\, dy\, dz$	$-\frac{\partial}{\partial z}(\rho w) dx\, dy\, dz$

Now, if you have gotten what is the difference; you can add up this difference and what can you say about this difference? So, see certain amount of fluid entered the volume and then certain amount left, whatever be the difference that you have that difference what should happen to that difference? That difference will constitute to change in the mass in that particular element right.

So, in minus out should give rise to the accumulation. So, if mass is accumulating in that fluid element; that means, it is going to essentially change the density of that fluid in that volume.

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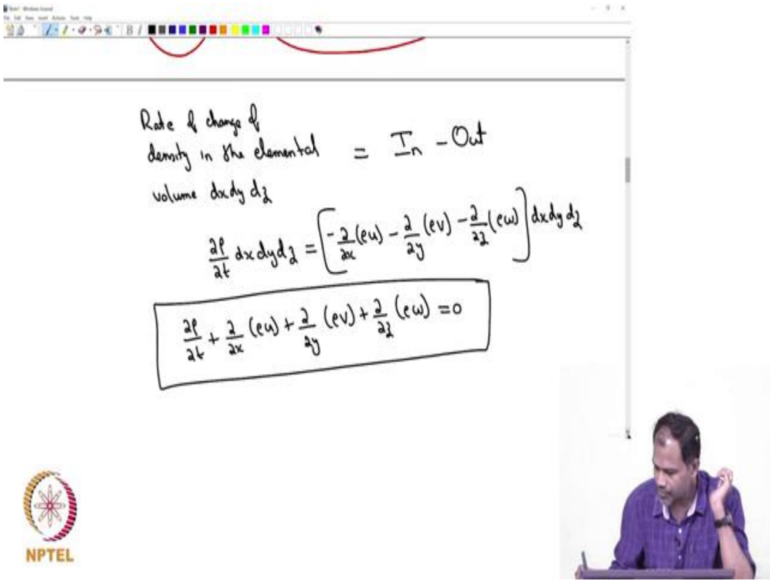

So, we can say that if we write down the rate of change of density in the elemental volume dx, dy, dz; that should be equal to In minus Out; because if In is exactly equal to Out, there is no accumulation, density cannot change, there should not be anything else, but if there is a change that should be reflected as a change in density.

So, then we can calculate the rate of change of density:

$$Rate\ of\ change\ of\ density\ in\ the\ elemental\ volume\ dx\ dy\ dz = \dot{m_{in}} - \dot{m_{out}}$$

$$\frac{\partial \rho}{\partial t} dx\ dy\ dz = \left[-\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w)\right] dx\ dy\ dz$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

That equation is the equation that represents the fact that mass is conserved ok. So, if there is a change in density that will be seen as a difference in the flow rate between two faces or if there is a difference in the flow rate there should be seen as a change in density ok; so this is the equation that describes the mass conservation.

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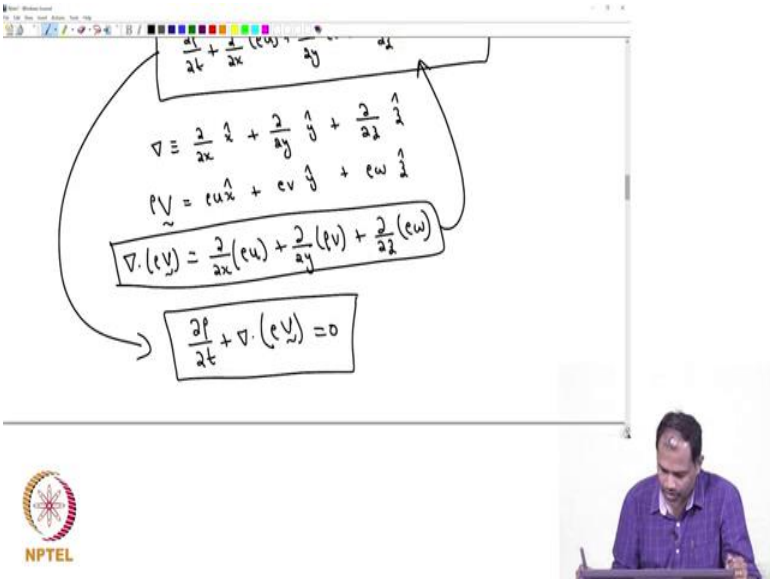

We can write this equation in a much more convenient form:

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

$$\rho\vec{V} = \rho u\hat{x} + \rho v\hat{y} + \rho w\hat{z}$$

$$\nabla.\rho\vec{V} = \frac{\partial}{\partial x}(\rho u)\hat{x} + \frac{\partial}{\partial y}(\rho v)\hat{y} + \frac{\partial}{\partial z}(\rho w)\hat{z}$$

$$\frac{\partial \rho}{\partial t} + \nabla.\left(\rho\vec{V}\right) = 0 \; [Mass\ conservation\ equation]$$

Now at this point I will talk about an incompressible fluid.

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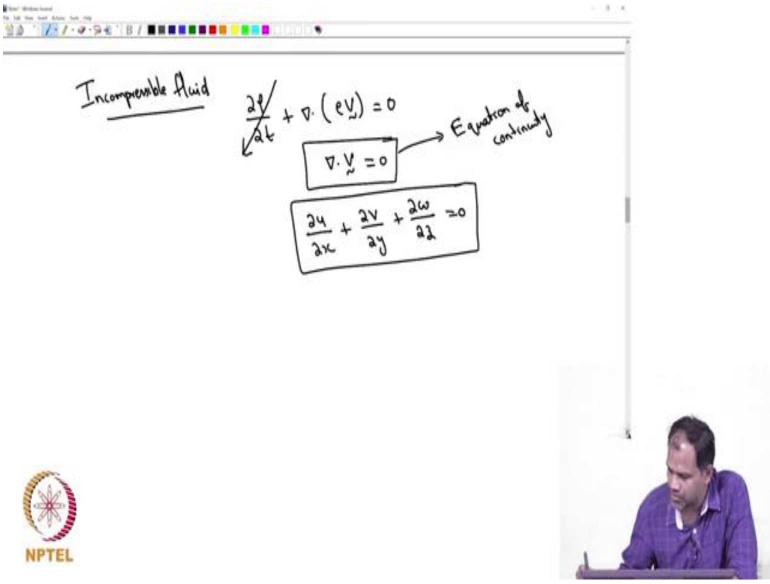

So, incompressible fluid ok; let us assume that ρ is a constant density of the fluid that you are considering is a constant. And if it is a constant, then it is; it is not changing with time, it is not changing with space. So, if you have $\frac{\partial\rho}{\partial t} + \nabla.\left(\rho\vec{V}\right) = 0$ it is not changing with time and you can take ρ out because it is not changing with space.

$$\frac{\partial\rho}{\partial t} + \nabla.\left(\rho\vec{V}\right) = 0$$

$$\nabla.\vec{V} = 0\ (Incompressible\ fluid)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

So, this is the mass conservation equation and its very simple to write ok. So, because its divergence of V; this is typically also called equation of continuity.

So, often equation of continuity and mass conservation ok; they refer to the same fact and this particular form where we write divergence of V is equal to 0 only refers to the incompressible fluids.

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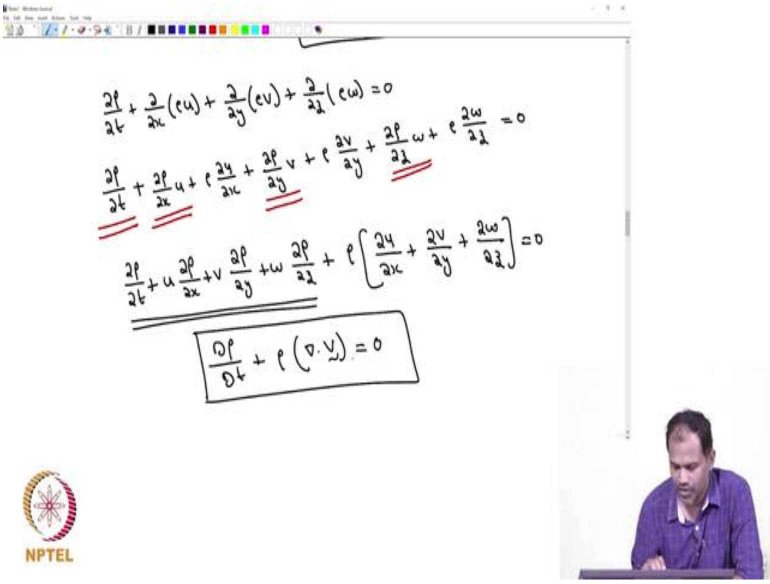

So, that is the equation for mass conservation; I am going to apply product rule to each of the terms.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0$$

$$\frac{D\rho}{Dt} + \rho\left(\nabla.\vec{V}\right) = 0$$

So, this is another form in which you can write the continuity equation ok. So, we wrote continuity equation in two forms.

Also,

$$If\,\frac{D\rho}{Dt} = 0 \rightarrow \nabla.\vec{V} = 0$$

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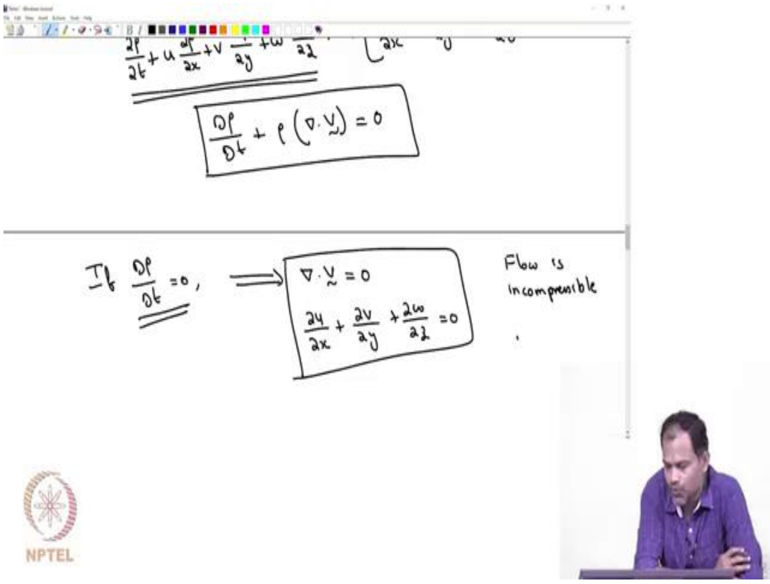

Now remember the density might be changing locally, but if you are following a particular fluid element and if its density does not change; then is D ρ by Dt equal to 0, in which case also you get divergence of V equal to 0:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

So, in this case you will say that the flow is incompressible; flow is incompressible as opposed to the fluid being incompressible. So, earlier we said that the fluid is incompressible and we threw away all the terms containing derivatives of ρ and then we found that you basically get divergence of V is equal to 0.

But you will also get divergence of V equal to 0; if you find that following a fluid element does not change its density ok; in which case again you will have divergence of V is equal to 0. So, most often the cases that we will consider would be either the flow would be incompressible or we will take the fluid to be incompressible. And in either case we will have divergence of V equal to 0 and that is the equation that we will use.