Fluid Mechanics

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Lecture - 25
Non-dimensional analysis - 3 Buckingham Pi Theorem

So, we were looking at Buckingham Pi Theorem in the last class. Let us just go through it once again so, that you get it straight.

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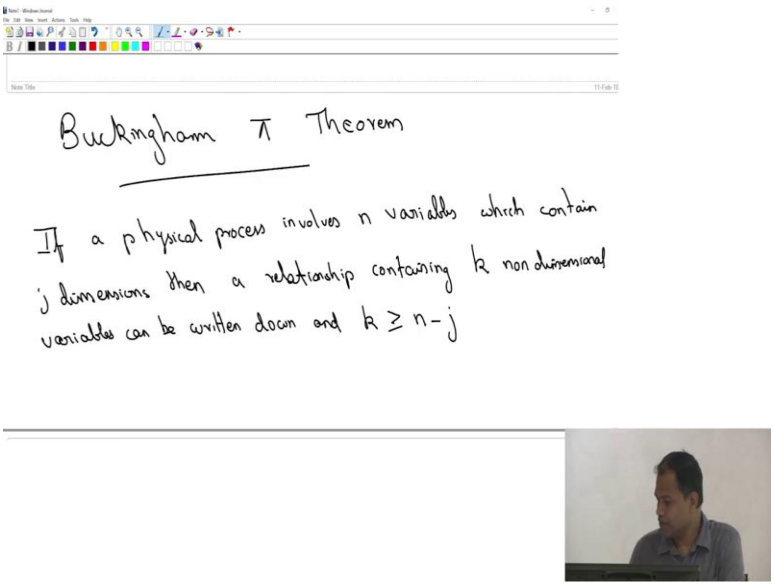

So, we really did not complete the theorem that day. So, let me just state the theorem. So, we actually what we did is we actually try to do an example. So, what it says is that if a physical process involves n variables which contain j dimensions, then relationship containing k non dimensional variables can be written down and k will be greater than or equal to n minus j ok.

So, suppose you have a physical process, which contains let us say n variables and it involves j dimensions then you can find a k which is n minus j or greater than n minus j and you will be able to find k non dimensional groups and you will be able to write down a relation between them.

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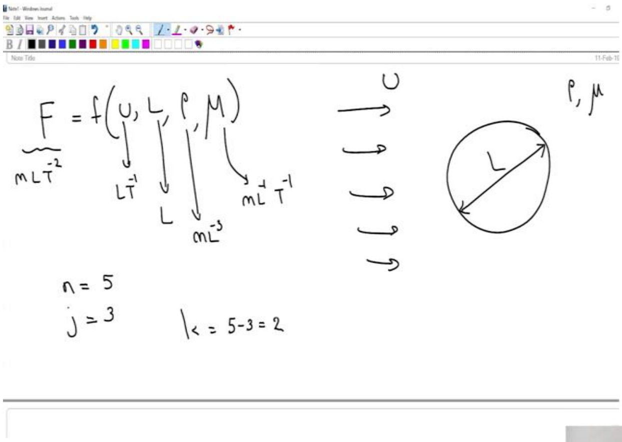

So, the problem that we had was some object with a characteristic dimension L and a fluid approaching it with a velocity U and has parameters sorry the properties density rho and viscosity mu and you were interested in calculating what is the force exerted by the object on the fluid. And you said that is actually going to be a function of you know U, rho and mu.

$$F = f(U, L, \rho, \mu)$$

And we said we have identified now, 5 different variables and we will write down the dimensions of each of them. So, So that means, we have identified n is equal to 5 j can be 3 because its mass length and time and therefore, we should find k which is equal to 5 minus 3 2 two non dimensional variables that we should be able to identify and then we said that we have to select the repeating variables first.

$$[F] = MLT^{-1}$$

$$[U] = LT^{-1}$$

$$[L] = L^1$$

$$[\rho] = ML^{-3}$$

$$[\mu] = ML^{-1}T^{-1}$$

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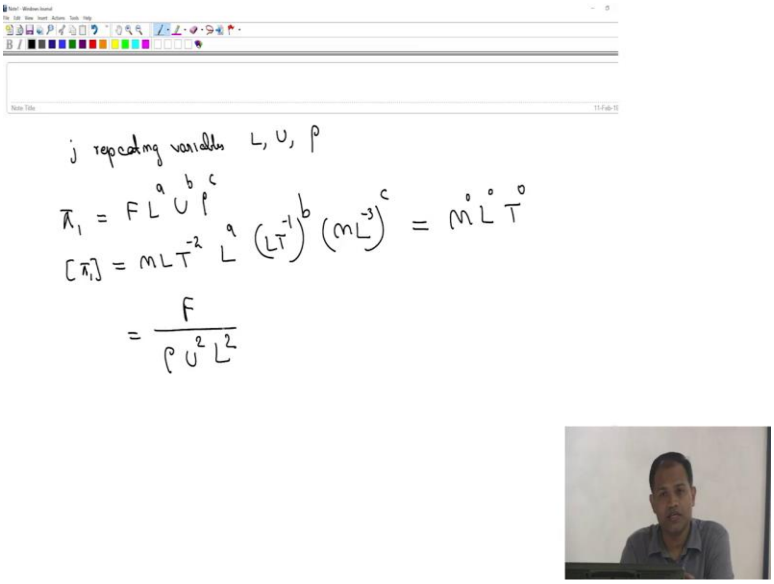

So, j repeating variables which we chose as length U and rho and therefore, we wrote down our first combination π_1 And we said that has to be equal to a non dimensional number; that means

$$(MLT^{-2})(L)^a(LT^{-1})^b(ML^{-3})^c = M^0L^0T^0$$

$$\pi_1 = \frac{F}{\rho U^2 L^2}$$

So, then we can talk about the next parameter.

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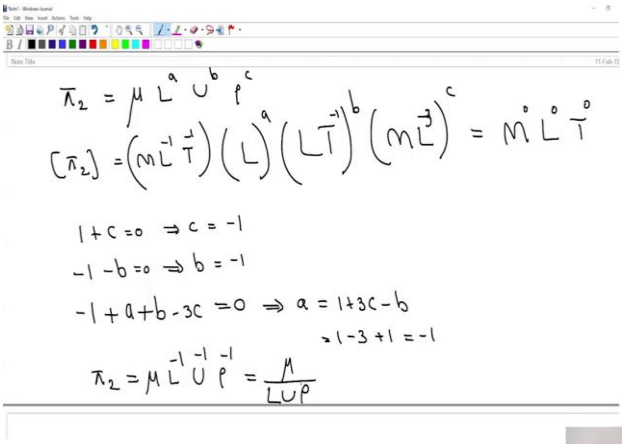

So, the second non-dimensional number pi 2 can be generated with μ;

$$\pi_2 = \mu L^a U^b \rho^c$$

$$[\pi_2] = [ML^{-1}T^{-1}][L]^a[LT^{-1}]^b[ML^{-3}]^c = M^0L^0T^0$$

Therefore,

$$1 + c = 0 \rightarrow c = -1$$

$$-1 - b = 0 \rightarrow b = -1$$

$$-1 + a + b - 3c = 0 \rightarrow a = -1$$

$$\pi_2 = \mu L^{-1}U^{-1}\rho^{-1} = \frac{\mu}{\rho UL}$$

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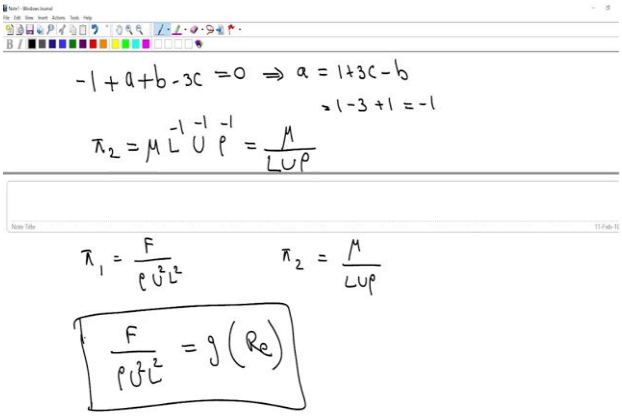

Buckingham pi theorem says that there exists a relationship between π_1 and π_2

$$\pi_1 = \frac{F}{\rho U^2 L^2}$$

$$\pi_2 = \frac{\mu}{\rho U L}$$

$$\frac{F}{\rho U^2 L^2} = g(Re)$$

So, this is one way of doing it there is another way of doing it which is relatively simpler and it is known as Ipsen’s method.

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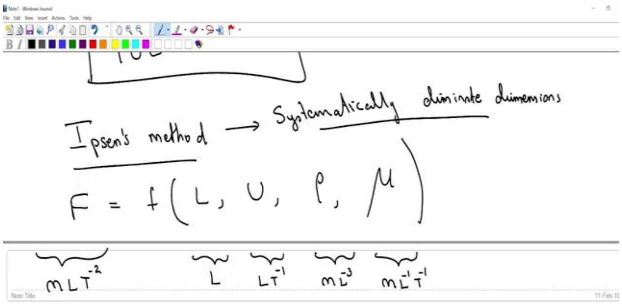

So, in this case so, we will look at the same problem and try to derive the same relationship, but in a slightly different procedure. We will write we want to get force as a function of what all things L U rho mu that is our intention at the moment. The first thing that you do is again write down the dimensions of each of them:

$$F = f(U, L, \rho, \mu)$$

$$[F] = MLT^{-1}$$

$$[U] = LT^{-1}$$

$$[L] = L^{1}$$

$$[\rho] = ML^{-3}$$

$$[\mu] = ML^{-1}T^{-1}$$

So, in this method what you have to do is you have to systematically eliminate dimensions ok. So, what you should do Ipsen's method says that systematically eliminate dimensions that is all it says which means this eliminate we have got mass length and time let us say we decide to eliminate mass first.

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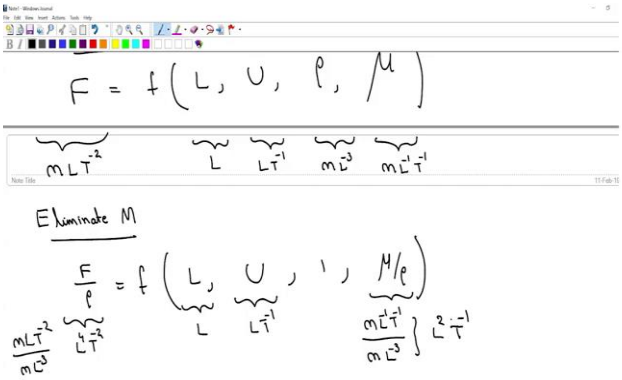

Eliminating Mass:

$$\frac{F}{\rho} = f\left(L, U, 1, \frac{\mu}{\rho}\right)$$

$$[L^4T^{-2}] = f([L], [LT^{-1}], [1], [L^2T^{-1}])$$

So, now, this expression if you look at this expression has only length and time unit I can choose to eliminate length or time ok. So, let us say we do time first.

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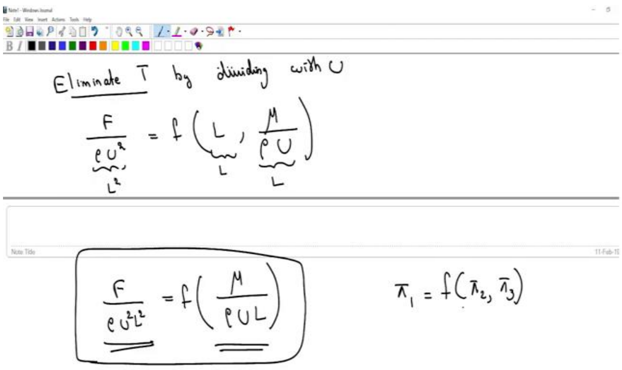

Eliminating time:

$$\frac{F}{\rho U^2} = f\left(L, \frac{\mu}{\rho U}\right)$$

$$[L^2] = f([L],[L])$$

So, now the entire expression only has length in it, I can now eliminate L:

$$\frac{F}{\rho U^2 L^2} = f\left(\frac{\mu}{\rho U L}\right)$$

So, the good thing is that here you do not need to solve you know the algebraic equations you do not need to really identify you know how many; you know you do not need to figure out what are the repeating variables and so on it sort of comes out naturally yeah. Huh. I did not understand can you explain again. Yes. No. So, j is 3. So, you will get two non dimensional variables n minus j non dimensional variables.

Oh! then you will get three non dimensional variables since see this we have. So, for example, here you have F by rho U square L square is only a function of another non dimensional variable right. So, similarly if you have 3 of them you will find pi 1 is a function of pi 2 comma pi 3 you do not know what f is, but this is the best you will be able to write.

So, originally if your problem contains six non dimensional variables; let us say let us take the same situation.

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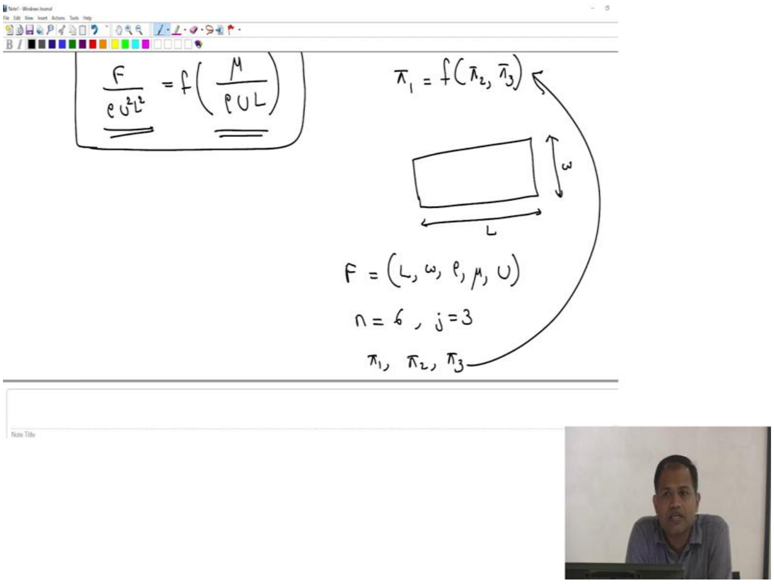

Let us say so, you only asked last time probably that if the shape was different. Let us say that it has a length L and let us say a width w let us say the third dimension is also w, then our F is going to be a function of L comma w comma rho mu U.

So, here n is going to be 6, but in this case again j is going to be 3 ok. So, you can expect to find 3 non dimensional variables. Let us see you will find out them as pi 1, pi 2, pi 3; then you can say that this is what the relationship going to be and so on. What would be the extra non dimension parameter that will come in this case can anybody guess? It'll be just w by L everything will remaining same right you only have an extra w k. So, w by L is the extra parameter that is going to come there which is basically the aspect ratio of the body which is a non dimensional parameter. And that is so, simple in this case because we have added just something which is a dimension of length and something physical ok. So, that is fine.

So, we have now learned how again a given problem if you are familiar with the number of variables or the number of variables that could be relevant, then you can write down a simple relationship instead of investigating the dependence of each of the variables ok. So, that so now, if you look at the flow what did we do? Initially we said if we knew the relation how to non dimensionalize the variable. So, we did it using an algebraic relation, we also did how to deal with the differential equations. Then now what we have done is we did not know anything we only knew that these are possibly the variables that are affecting and then we are we have written down a relation.

So, one of the crucial things that you have seen in this step is that you can write down the dimensions of any physical variable as a power law expression.

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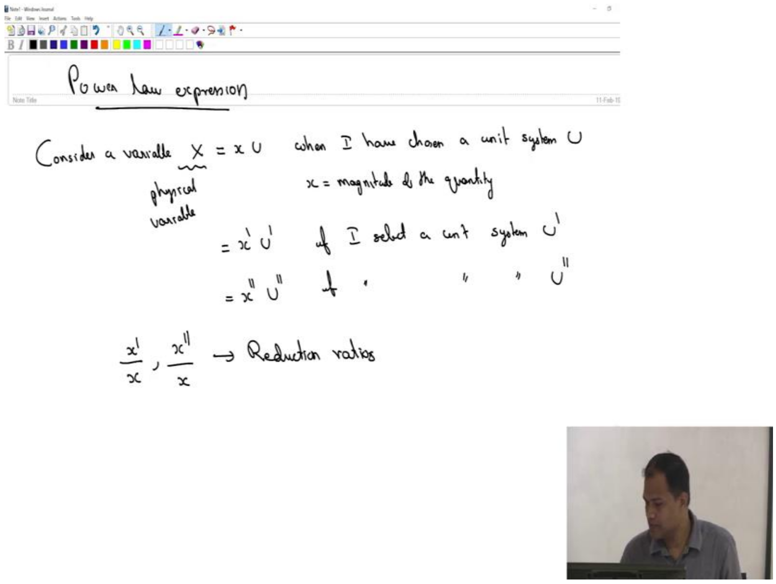

So, I mentioned it a few classes before that we have written down the variable any variables the dimensions as a power law expression ok. So, is this always going to be the case? Can we always write the dimensions as a power law expression? And we are going to prove that yes it is possible and therefore, all the things that we have said so far we will continue to be hold for any problem that you pick up is the objective clear?

Consider a variable let us say X which I will write as x U when I have chosen unit system U. So, what is that I have chosen? A unit system U in which x is some variable and this small x is therefore, the magnitude of that quantity; x is some physical variable, but x some let us like f is equal to 10 Newton force is equal to 10 Newton that is all what I have written, but in a general form or I could write this as x prime U prime if I select a unit system U prime or x double prime U double prime if I select a unit system U double prime.

So, what I am saying maybe I will take my SI system or CGS system or MKS system I could always write it right. So, for example if its length I can write it as 1 meter 100 centimeter anything that I like. So, that is what I have written now and we know that all this should be related right all these will be related by some factors. For example, x prime by x will be some number right x prime is the magnitude in a unit system x double x is a magnitude in some other unit system and they will be related by some number.

$$X = xU$$

$$X = x'U'$$

$$X = x''U''$$

So, similarly x double prime by x for example. These are called reduction ratios. Reduction ratios are nothing, but the numbers that you need to multiply when you are moving from 1 system of units to another system of units.

$$\frac{x'}{x}, \frac{x''}{x} \rightarrow reduction\ ratios$$

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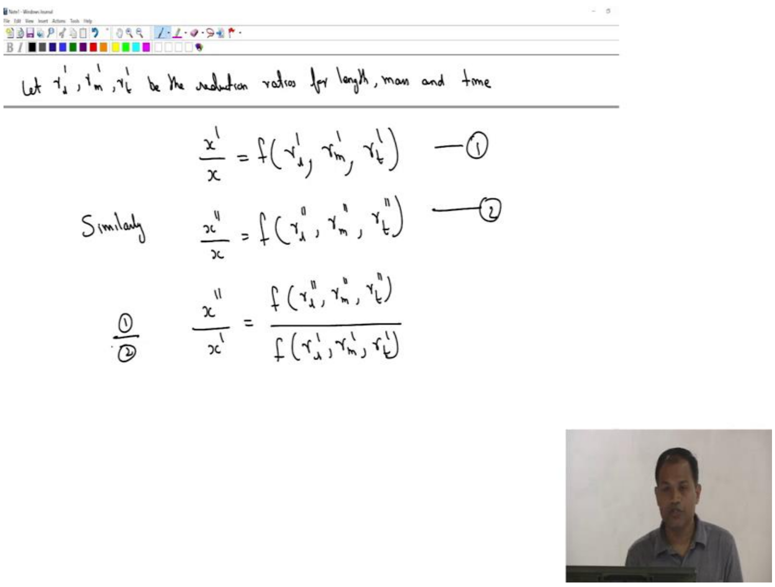

$$r_l', r_m', r_t' \rightarrow reduction\ ratios\ for\ length, mass\ and\ time$$

So, when I define my x I did not say what it is its some variable some physical variable this vector and you know r l prime is for example, the reduction ratio for the length k. When I am changing from oh unit system U prime to unit system U what is the what is the number that I should be multiplying with that is r l prime. If it is I am talking about length or if it is mass, let it let us call it r m prime if it is time let us call it r t prime ok.

Similarly, so, within I know that I can write x prime by x is going to be sum function of r l prime r m prime r t prime ok. So, the reduction ratio for length reduction ratio of mass reduction ratio of time each of them if I combined in some form I should be able to get the reduction ratio for any other quantity that I am dealing with remember I am only worried about 3 dimensions now 3 particular mass length and time only. As you expand the problem you could have other things coming into the picture.

Similarly I could write

$$\frac{x'}{x} = f(r_l', r_m', r_t')$$

$$\frac{x''}{x} = f(r_l'', r_m'', r_t'')$$

$$\frac{x''}{x'} = \frac{f(r_l'', r_m'', r_t'')}{f(r_l', r_m', r_t')}$$

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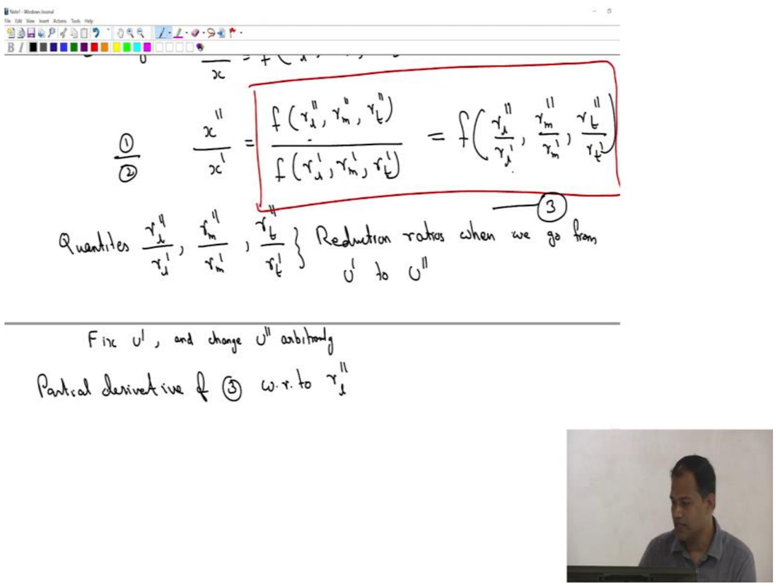

Now,

$$\frac{x''}{x'} = \frac{f(r_l'', r_m'', r_t'')}{f(r_l', r_m', r_t')} = f\left(\frac{r_l''}{r_l'}, \frac{r_m''}{r_m'}, \frac{r_t''}{r_t'}\right)$$

So, what have I written really? See if you look at the right hand side I take the reduction ratio for length mass and time separately and combine them or if I if you look at the expression in the in the middle this part I calculate the quantity in 2 different units and then take the ratio then I am going to get the same thing. For example, if I am talking about force I calculate. So, this part let us say the double prime is I have calculated it in some units let us say mass into acceleration I have done let us say I have done it as kilograms times meter per second square or this quantity the one in the denominator I will do it in CGS unit let us say I will do grams per grams in 2 centimeter per second square if I do the force ratio or I do the mass ratio and acceleration ratio and multiply them the answers are going to be same.

Essentially this part is just that fact that whether I do the forces and take the ratio or if I take the each of the components mass, length and time separately and then take the ratio and do combine appropriately I am going to get the same expression your same ratio same number that is what we have any doubts? What we will do is we will fix U double prime and change U prime arbitrarily. So, so far I talked about fixing an SI system and saying that one of the system I am going to fix the other 1 I am going to consider to be a variable so, that I can do differentiation and integrations and so, on.

So, I will say that in this quantity in this box expression, I am going to keep my double primed variables as fixed quantities and my primed variables as really variables so, that I can do differentiation. In fact, what I want to do is I want to take partial derivative of let us call this expression 3 with respect to let us say r l prime ok. So, I am keeping my primed variables as variable really. So, if I differentiate it. So, they will do the other way I will keep the U prime I will keep U prime as the variable as the fixed quantity and U double prime as arbitrary so, that I can differentiate the primed variable. So, you could do it either way you will see that its symbols another.

$$\frac{\frac{\partial f}{\partial r_l''}(r_l'', r_m'', r_t'')}{f(r_l', r_m', r_t')} = \frac{\frac{\partial f}{\partial r_l''}\left(\frac{r_l''}{r_l'}, \frac{r_m''}{r_m'}, \frac{r_t''}{r_t'}\right)}{r_l'}$$

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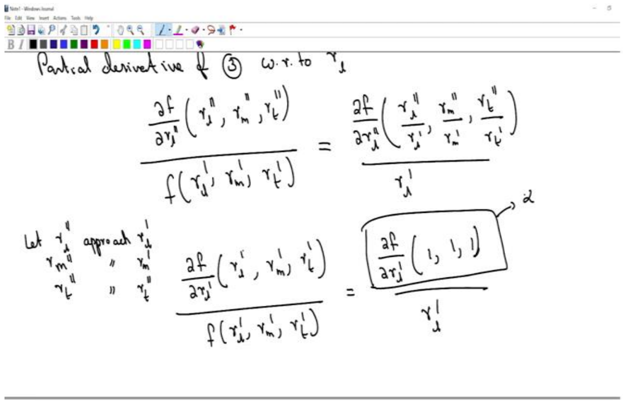

Now, having done that, let r l double prime approach r l prime ok. $r_l'' \rightarrow r_l'$

$$\frac{\frac{\partial f}{\partial r_l'}(r_l', r_m', r_t')}{f(r_l', r_m', r_t')} = \frac{\frac{\partial f}{\partial r_l''}(1,1,1)}{r_l'}$$

I will just call that alpha for the convenience also this is really a function of primed variables I will just drop the primes.

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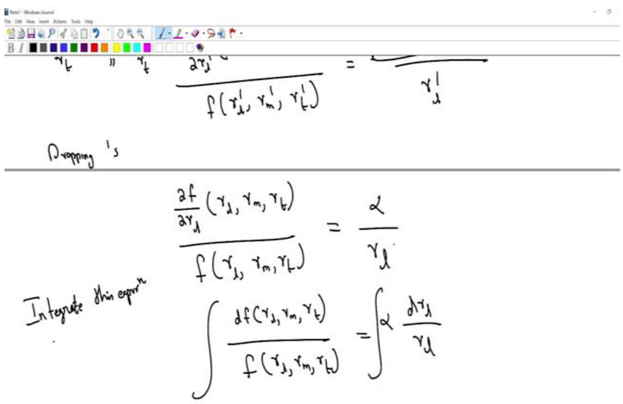

$$\frac{\frac{\partial f}{\partial r_l}(r_l, r_m, r_t)}{f(r_l, r_m, r_t)} = \frac{\alpha}{r_l}$$

Now, I am going to integrate this quantity

$$\int \frac{df(r_l, r_m, r_t)}{f(r_l, r_m, r_t)} = \int \frac{\alpha}{r_l} dr_l$$

$$\ln f(r_l, r_m, r_t) = \alpha \ln r_l + \ln g(r_m, r_t)$$

$$f(r_l, r_m, r_t) = r_l^{\alpha} g(r_m, r_t)$$

$$f(r_l, r_m, r_t) = r_l^{\alpha} r_m^{\beta}, r_t^{\gamma}$$

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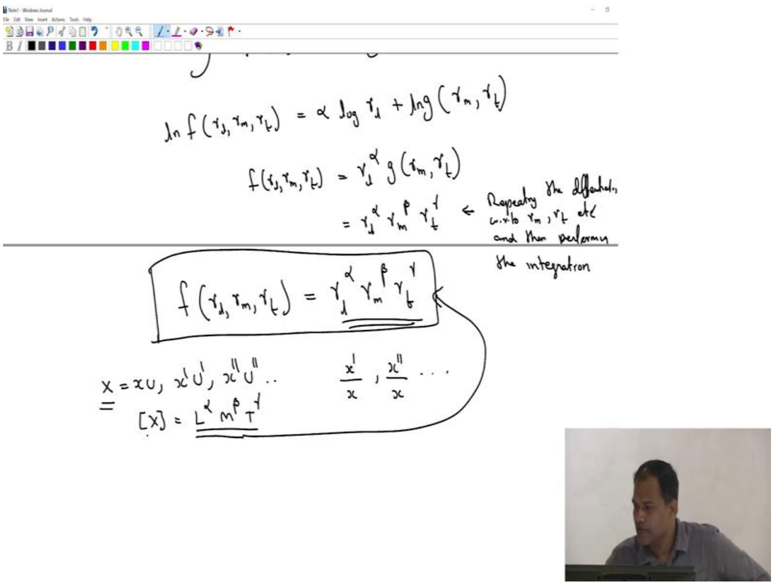

You can do it and you will find that you are going there is another thing that is going to change L was just an arbitrary choice what have you really done? We found that f of r l r m r t some function of reduction ratios is just going to be a power law expression that is what we found. Reduction ratios are the ones which you will use when you are changing from one system of units to another system of units. We started by saying that we are going to look at a quantity x which we will write it as either x U or x prime U prime or x double prime U double prime and so, on and we said we are looking at quantities like x prime by x x double prime by x and so, on ok.

Now, if you want your if the reduction ratios are going to be power law expressions, then the only way x can be X will X has to be having a form L to the power of alpha M to the power of beta T to the power of gamma if x has the form of L to the power of alpha M to the power of beta T to the power of gamma then only I can have my reduction ratio having you know my reduction the reduction form will basically have this power law expression right if my x had any other form not just this power law form anything else then my reduction ratio cannot be like this because the reduction ratio is by definition when I am going from one system of units to another system of units. So, any X that I choose if the dimensions do not have this form then I cannot have this form of reduction ratio.

So, now we have found that the only way you know in an arbitrary case, the reduction ratios will always have a power law expression; that means, dimensions will always be power law expressions for any physical variable that we choose and you have seen that right any system any unit that you write will always be something to the power of something its never in any other form that is what we have proven now that that has to be the case always doubts yeah. Sure here.

Oh i. So, this is. So, this was my equation right dou f by dou r l prime. So, the only variable that I have is r l prime. So, I just drop that r primes or you could just continue writing everything and you will end up with an expression saying that f of r l prime r m prime r t prime is r l prime to the power alpha. So, there is no physical this thing there yeah. Which one. Ha r l prime should I go up no its general huh. Yes. Can you be bit loud?

No it cannot no because by definition it contains only length, it is like 100 it could be. So, when you are going from cgs to si r l prime is just 100 like that it does not contain by definition mass or time mass and time are fundamentally different dimensions. So, therefore, r l prime cannot be a function of rm or rt; however, x prime could be I mean we wrote it as x prime by x which may which for example, could be a reduction ratio for force which means it will contain both length mass and time in it.

So, what we have done is either we write it as a combination or we write it separately that is what we did and that is what exactly that box the expression was telling this red box the expression that on the left hand on the right hand side I do it as a function of I will change all the lengths I will change all the masses I will change all the times and appropriately combine it or on the left hand side I am saying that I will take the force in 1 units I will take the force and other units and I will take the ratio of it I will come I will get the same number go down.

So,. So, far we. So, till we got this expression we kept these two things as two different units r l prime and rl double prime or rather U prime and U double prime has two different units now I am saying that one system is up and when we kept one of them fixed and other changing. So, that we could do a differentiation and we are saying that there was the one which we are changing is actually approaching the 1 which we have kept fixed ok. So, therefore, each of these ratios r l double prime by rl prime is essentially just one number I mean its basically same quantity because U double prime and U prime are the same system of units now ok.

So, there is no reduction ratio really that is why I just wrote it as 1 1 or simply you say that I am going to say my rl double prime is equal to rl prime and so, on for all primed quantities and that is why its a constant and you can see that that constant is was comes out to be as a factor which says that you know the quantity is that you know that comes as powers is essentially the derivative you know with respect to that unit system that is what that power means ok.

So, you go back you revise all these things the proof is not very important the proof is there because I thought its a nice thing to do you can just learn it and more important is to learn how to do the problems we will do one more proof and that is we are going to prove the Buckingham theorem. So, we I just said that if you have n variables and j dimensions, then we will have n minus j non dimensional parameters right. So, we will again do a similar proof like that and then we will wind up this session this topic ok. I will see you tomorrow.