Fluid Mechanics

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Lecture - 24
Non-Dimensional analysis - 2

So the last class right, you we started discussing Non-dimensionalization we said you know given a number of variables we do not want to deal with all of them. So, what is the best way to reduce it? Right. So, we took some simple equation and then we started non-dimensionalizing it and then we found that you know it could indeed be reduced. So, the expressions the simple expression that we had was that of a you know the trajectory of a falling particle.

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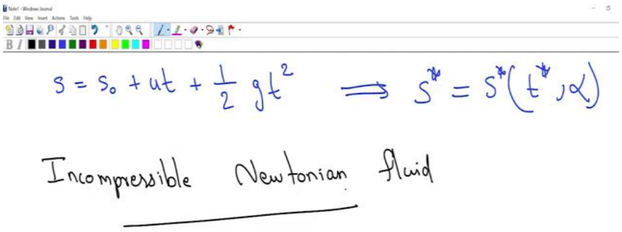

And, if that was actually given as s equal to s 0 plus u t plus half g t square depending upon what method you adopted to non-dimensionalize it, you could write it typically as a non-dimensional distance as a function of a non-dimensional time and some additional parameter, that is all what we had done right.

$$S = S_0 + ut + \frac{1}{2}gt^2$$

$$S^* = S^*(t^*, \alpha^*)$$

Then I said and in this particular case ok; in this particular case we knew the equation and therefore, we could go ahead, but typically that would not be the case ok. So, what is the general way of doing it? Before we discuss that let us look at non-dimensionalization of the governing equations that we already know ok.

So, we are going to look first incompressible Newtonian fluid and we will do the same thing that we did for the trajectory of the particle and see what non-dimensional parameters are going to come out ok. So, that is going to tell you how you know what is the most important parameter that determines the fluid flow and then we will look at when we do not know the equations how to proceed. So, we are looking at a very general system , we have you know some fluid which is flowing and we want to describe the dynamics. We will write down the governing equations which would include the continuity equation and the momentum conservation equation.

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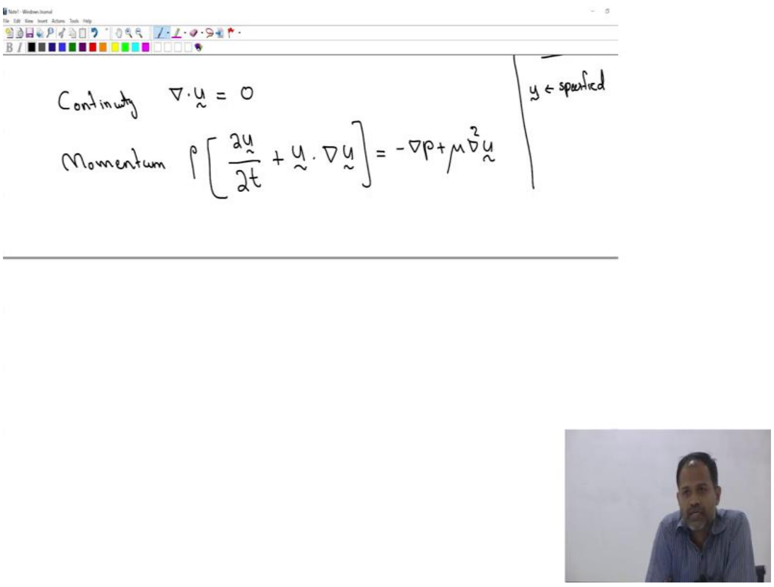

So, let us just write it down:

Continuity: $\nabla.\vec{u} = 0$

Momentum: $\rho\left[\frac{\partial u}{\partial t} + \vec{u}.\nabla\vec{u}\right] = -\nabla P + \mu\nabla^2\vec{u}$

And, you will additionally have boundary conditions which would be you know some velocities that you must either u is specified which must be 0 or something else depending upon the problem. So, this is typically what the problem will look like right.

So, now let us so, there are difference between the previous case that we considered and this is that in the previous case, it was just an algebraic equation, this is a partial differential equation that is the only difference ok, but otherwise there is no difference. And therefore, there is no difference in the way you want to non-dimensionalize it also to non-dimensionalize we need to select some scales. We do not know how many scales we should choose, but in this case I am going to tell you how many we will select ok. We are going to say that we will have a characteristic velocity we will have a characteristic length and we will have some characteristic pressure ok. So, that is what we are going to do.

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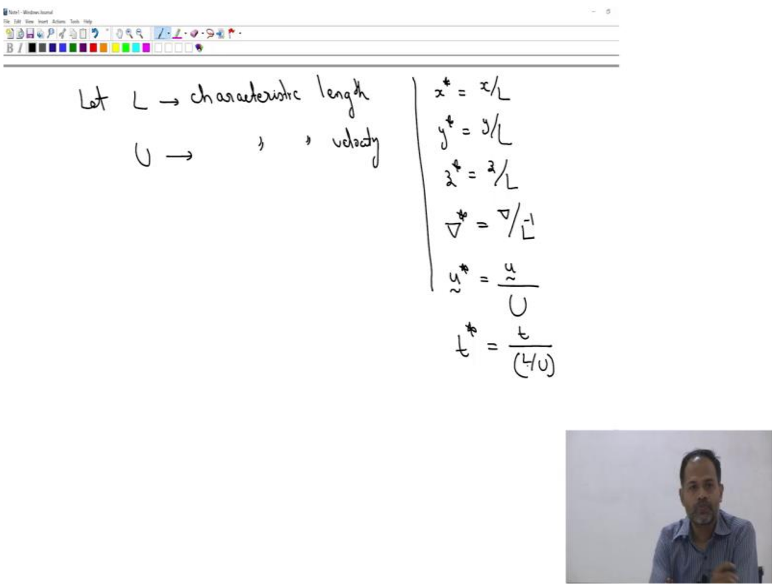

We will select let L be a characteristic length in the problem, let U be a characteristic velocity in the problem right. So that means:

$$L \rightarrow Characteristic\ length$$

$$U \rightarrow Characteristic\ velocity$$

$$x^* = \frac{x}{L}; y^* = \frac{y}{L}; z^* = \frac{z}{L}; \nabla^* = \frac{\nabla}{L^{-1}}; \overrightarrow{u^*} = \frac{\vec{u}}{U}; t^* = \frac{t}{L/U}$$

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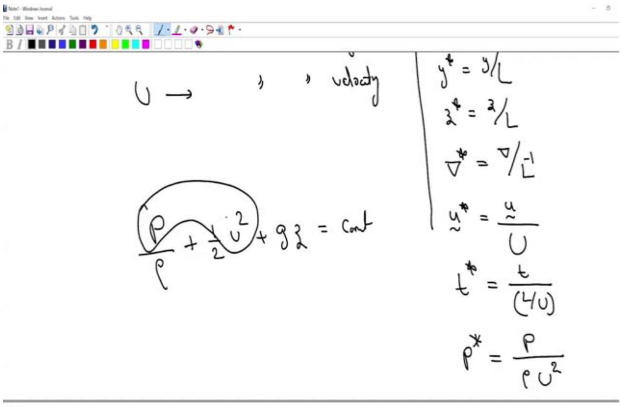

Pressure now there are various ways to do this and we have seen this and it. So, we discussed it in the last class that it really does not matter what exactly we choose right, we want exactly we select to non-dimensionalize. I am going to suggest that we will non-dimensionalize pressure by a quantity ρ U square ok. One way of thinking about it is that if you look at Bernoulli’s equation; Bernoulli’s equation is really p by ρ plus half U square plus g z right is a constant. So, you can see that p essentially would be like ρ U square or p could be like ρ g z; if actually hydrostatic pressure was more important ok.

So, by really selecting p is like ρ U square we are saying that in the problem it is likely that my pressure will be you know acting like my ρ U square that is the only recently have selected it ok. So, those are my non-dimensional variables that list so, we can go ahead. So, this is so, what I what you see on the right hand side is my non-dimensionalization. So now, we can go ahead and substitute and see what happens ok, how many reduced variables we are going to get.

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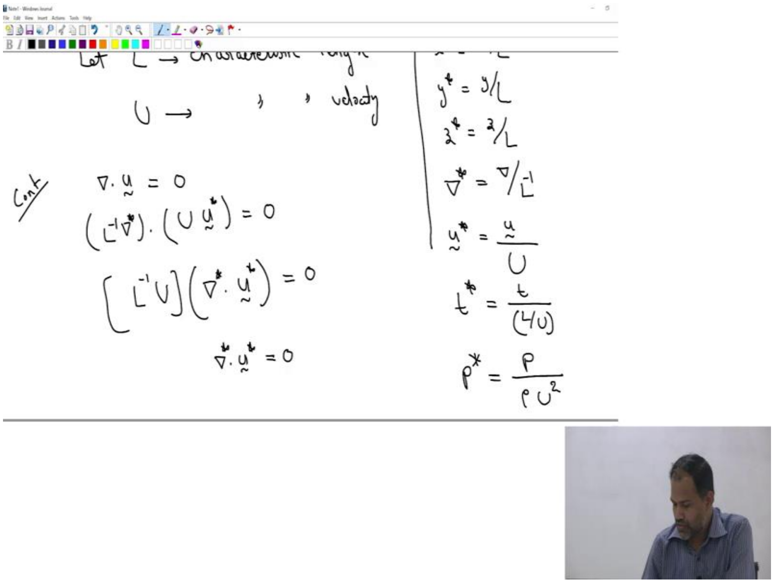

So, the first equation again is continuity equation:

$$\nabla . \vec{u} = 0$$

$$(L^{-1}\nabla^*)(U\overrightarrow{u^*}) = 0$$

$$[LU][\nabla^* . \overrightarrow{u^*}] = 0$$

$$[\nabla^* . \overrightarrow{u^*}] = 0$$

All the variables remain as such except that they have become now non-dimensional, there was no non-dimensional parameter that came up during the non-dimensionalization. Now, if you remember we had done something very similar when we derived the equation for a flow between a plates or tube flow one of them where we did the non-dimensionalization. Does anybody remember? We had a differential equation, we actually non-dimensionalized it with a characteristic length, a characteristic velocity and we ended up with something very similar. There again we did not have any non-dimensional parameter so, this is something very similar.

In other words the continuity equation basically you know whether its dimensional form, non-dimensional form whatever it is there is no parameter that is going to be prescribing what you know the mass conservation is. The mass conservation will be just given by that irrespective of what the flow parameters are, that is all. There is no particular parameter that you can identify which could actually change the way the solutions are going to be, just based on the mass conservation.

So, now the momentum conservation so, we have several parts we will take one by one. So, we have an unsteady term then we have a non-linear term, then we have a pressure term and then we have a viscous term; let us non-dimensionalize each of them and substitute.

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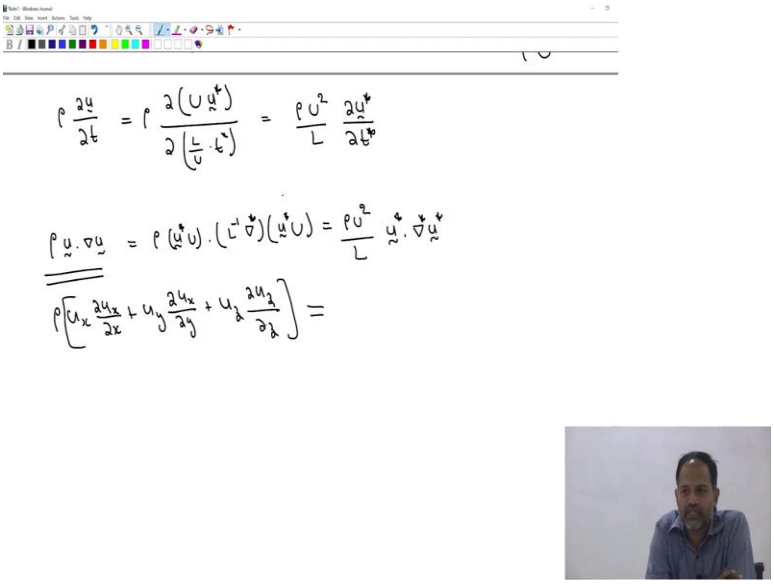

$$\rho \frac{\partial \vec{u}}{\partial t} = \rho \frac{\partial (U\overrightarrow{u^*})}{\partial \left(\frac{L}{U}\ t^*\right)} = \frac{\rho U^2}{L} \frac{\partial \overrightarrow{u^*}}{\partial t^*}$$

$$\rho \vec{u}.\nabla \vec{u} = \rho\left(\overrightarrow{u^*}U\right).\left(L^{-1}\nabla^*\right)\left(\overrightarrow{u^*}U\right) = \rho \frac{U^2}{L} \overrightarrow{u^*}.\nabla^* \overrightarrow{u^*}$$

$$-\nabla P = \left(\nabla^* L^{-1} P^* \rho U^2\right) = \rho \frac{U^2}{L}\left(-\nabla^* P^*\right)$$

$$\mu \nabla^2 \vec{u} = \mu L^{-2} \nabla^{*2}\left(u^* U\right) = \frac{\mu U}{L^2} \nabla^{*2} \overrightarrow{u^*}$$

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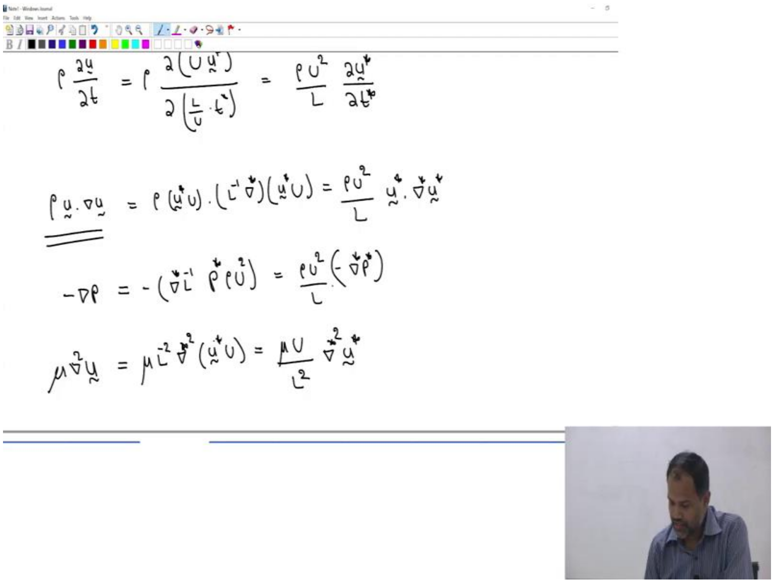

Let us go back and substitute in the equation now. We get:

$$\frac{\partial u^*}{\partial t^*} + \overrightarrow{u^*}.\nabla^*\overrightarrow{u^*} = -\nabla^* P^* + \frac{1}{Re}\ \nabla^{*2}\overrightarrow{u^*}$$

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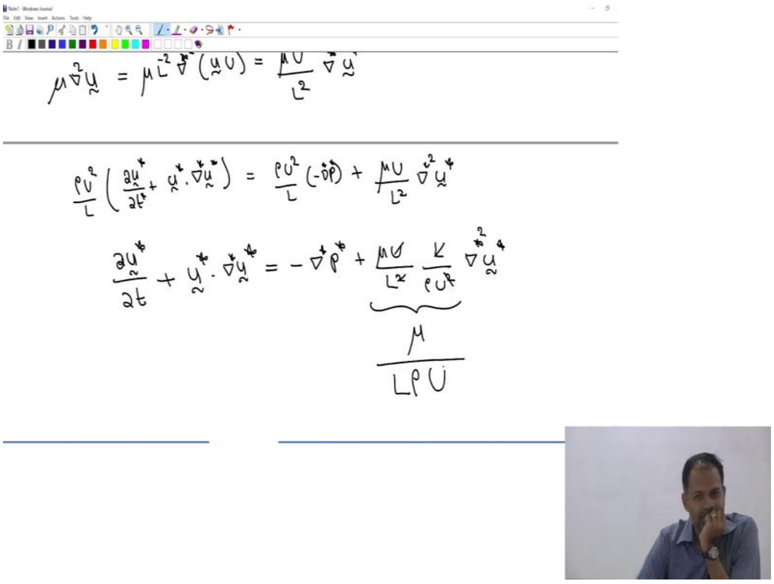

So, what non-dimensional number did we end up with? Reynolds number ok.

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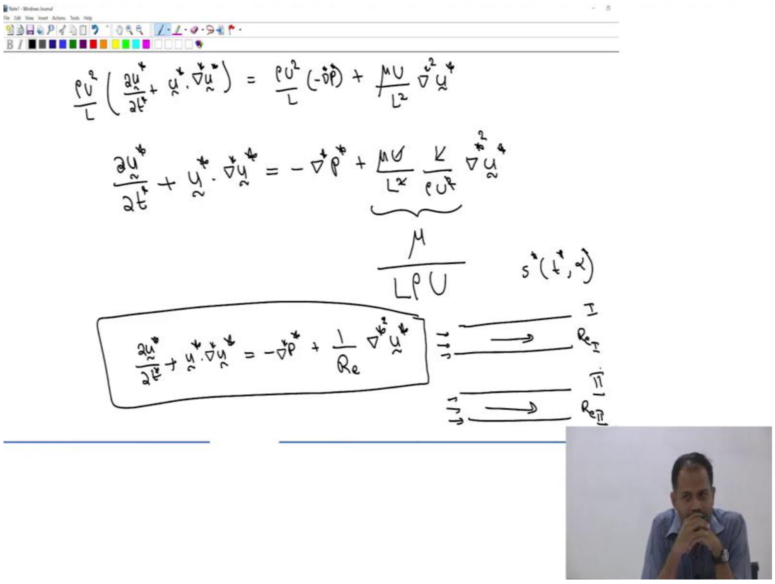

That is the non-dimensional form of the momentum conservation equation ok.

You can see that there are now non-dimensional variables which are u * t * and p * and there is only one parameter and that parameter is nothing, but the Reynolds number ok. Depending upon what is the Reynolds number you can have equation, if form of the equation will remain the same, but the answer will come out to be a function of Reynolds number ok; that is why Reynolds number actually plays a very important role.

You know that for example, Reynolds number is very small you get laminar flow when it is very large you get turbulent flow and so on ok. So, you are actually ending up with different different solutions based on one non-dimensional parameter. It does not matter what is the size of the pipe you are using, what is the fluid that you are using. It only matters what the Reynolds number is, that is what the equation or the non-dimensionalization of the equation that just said clear. Should I repeat? I am just saying that we have you know really non-dimensionalized our governing equation, when we non-dimensionalize the continuity equation ok, it did not give you any non-dimensional parameter.

We have non-dimensionalized the momentum conservation equation and we have ended up with just one non-dimensional parameter and that parameter is Reynolds number. Remember when we had our original problem ok, we had s * as a function of t * comma alpha right. So, that trajectory of the particle we said alpha determines what is the relation between s * and t *, where s * was some non-dimensional distance, t * was some non-dimensional time. And, based on the value of alpha alone we could say what where the trajectory is going to be. So, alpha is the only non-dimensional parameter that determines the solution.

Similarly, here this is a partial differential equation, the solution is determined only by R e ok. When you solve it for example, will come out to be a function of R e along with time and space and the value of R e is that something going to be different right. You can take one pipe let us say case I and another pipe case II. In each case you can select some Reynolds number I, this is another Reynolds number II ok. And, you can let us say let a fluid go in and out and let us say you will select you know different fluids.

So, if you by the way you had selected different fluids you are going to get two different Reynolds numbers. The flow is going to be different and it will be different simply based on the Reynolds number nothing else, that is all we are trying to say and that is all what the equation has said ok. Again any doubts? Yeah.

Yes. So, the so, in this particular case I have kept it same, but otherwise you could, you could take a bigger pipe you could take a smaller pipe. So, that is why see we are talking about for example, t * or length when you take out the solution in terms of an x *, it does not matter what the actual dimension is.

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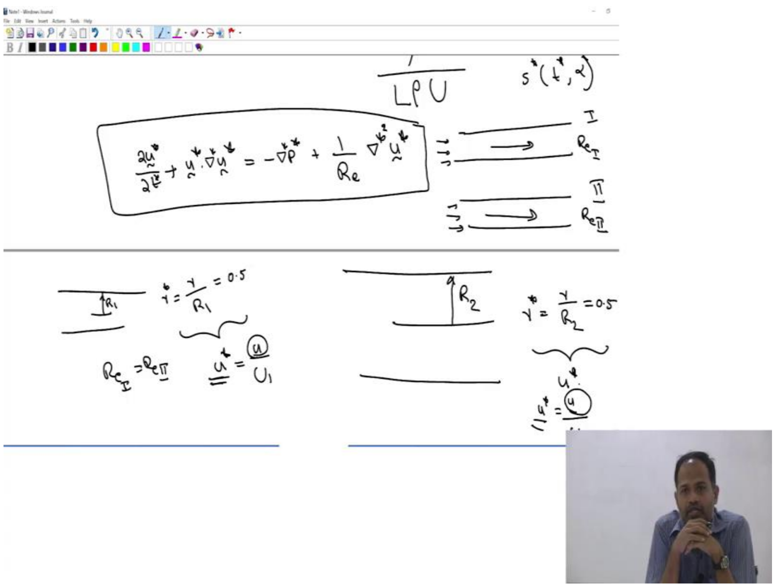

Let us say if you are talking about a; you are talking about a small pipe and then you are talking about a large pipe. Let us say in the small pipe the diameter was R 1, in the large pipe let us say that sorry the radius is R 2. In the small thing we would have non-dementialated the non-dimensional R * would be R divided by R 1, here R * would be R divided by R 2. So, this R * would vary between 0 and 1, here the R * would be vary between again 0 and 1 ok. So, in the scale of that 0 and 1 whatever velocity that you get ok, will remain same; here you mean on the one on the left hand side and the one on the right hand side.

So, let us say you are looking at a point R * is equal to 0.5, here also you are looking at a point R * is equal to 0.5. Your equations would give you a u * here; here it will give you a u *. This u * the u * on the left hand side and the u * on the right hand side is going to be same. But, the absolute magnitudes will be different because in this case you would have non-dimensionalize u * with some characteristic velocity which you would have defined it as small u divided by some capital U 1.

In the right hand side you would have defined your u * as some velocity divided by sorry you do not need that divided by u 2. That means, this u that you are going to get in the bigger pipe and this u that you are going to get in the smaller pipe, the numbers would be different ok, but this u * would be same in both the cases.

No oh sorry I so, I am talking about here as a Reynolds number being different and therefore, Reynolds number is the parameter. If you want to get this Reynolds number in problem I should be same as a Reynolds number in problem II, then only you get this match. And, that is the beauty of talking about it, we do not care about what are the actual dimensions or the fluids ok. In fact, this is a very important principle that is used when you have to make you know models in a lab instead of testing out in a field, which we will discuss next week and this is a principle that we would use ok.

So, what we have done so, far two cases one was the trajectory of the particle, second is the governing equation and these two cases are when we knew the form of the equation and therefore, we have gone ahead and done the non-dimensionalization. Now, in most of the cases we would not know the equation because, if we knew the equation we could have solved it in some way ok. So, we would not know the equation, we would probably know what are the relevant variables and how would we simplify that situation that is what we are going to look at ok. And, there are two methods for doing it, the first method is what you call the Buckingham pi theorem. And then there is an Ipsen's method which we will see if we have time today.

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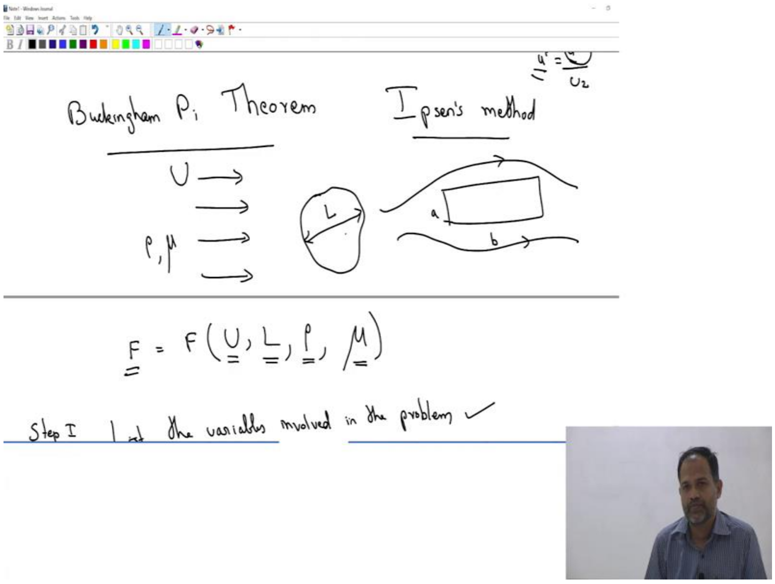

So, the situation that we are going to consider is the situation that we actually *ted talking about when we *ted non-dimensional analysis which is flow past a body ok. We have some object you know of some characteristic size L and then there is a fluid that is going to go past it ok. And, one of the questions that we post was what is the force that this object would exert on the fluid because we know that this we are keeping the object stationary; that means, the fluid is going to slow down.

So that means, the you are go the object is exerting some force on the fluid flow and you want to know what is that force, we even hypothesized a form for that right. So, let us say that if the fluid is coming with a velocity U and the fluid has a density ρ and viscosity mu.

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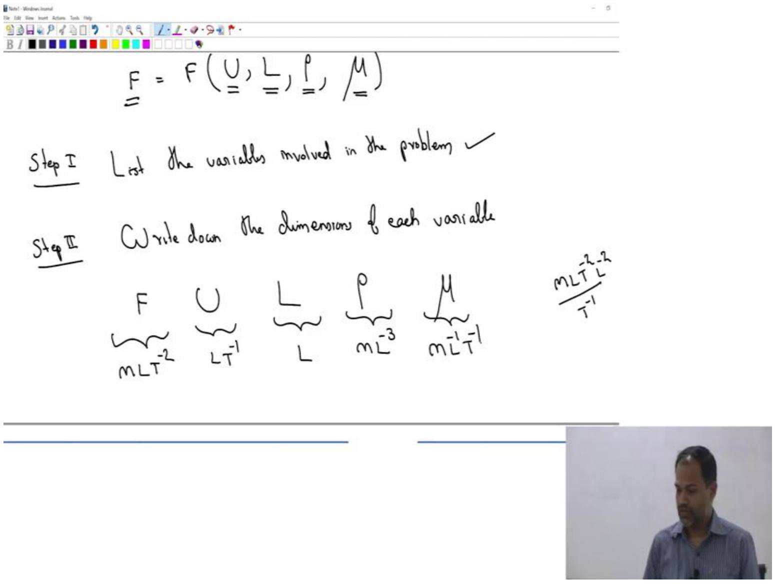

Then we said the force will be a function of the you know the velocity with which the fluid is coming, the size of the object, the density of the fluid, the viscosity of the fluid and so on right.

$$F = f(U, L, \rho, \mu)$$

And, this is the example that we chose where we said we have like we have force 1 2 3 4 5 different variables and if you want to do an experiment we will do huge number of combinations. And therefore, then it becomes you know a large data analysis problem rather than making sense out of it ok. So, we want we said it can be further simplified ok. In this case now we do not know what is the force that is going to be exerted, but we can still calculate some form of the force ok.

And, Buckingham theorem tells you how to do that, we will do the proof on Monday we will just follow the procedure first and see what the procedure is and then we will prove that what we have done is right. So, right so, the steps involved step I is list the variables involved in the problem. Now, this is a very crucial step because you are *ting from nowhere ok, you do not know what are the variables involved. But, you could do let us say some preliminary experiments, vary something and see that things are different which will give you some idea about what might be the variables involved ok.

And, when you *t working on a problem let us say you are trying to analyze a reactor you will have some idea maybe the size is important, some of the fluid properties are important the you know things like that. So, it is a its the selecting the variables it is actually a skill and there is no real way of doing it. So, in this particular problem I said we know that you know the fluid the speed with which the fluid is coming, it can be an important parameter. The size of the object, it does look like you know it can be an important parameter. I did not say for example, the what the thermal conductivity of the fluid is important because, one would not expect that. I did not notice a the surface tension is important because, there is no really surface involved ok.

So, it is some so, you should not select too many parameters because, then it is not going to give you any simplification. The thing that you should really look for us what could be the relevant parameters, you could keep adding it to this later and see whether you know things are different or not. So, in this particular problem we have taken now five different variables right; force, length, velocity, density and viscosity of the fluid. These are the five things that we have and our objective is to know derive an equation which will be a non-dimensional equation. So, first thing is done list the variables involved in the problem, step II write down the dimensions of each variable.

So, we have a force, we have velocity, we have a length, we have a density, we have a viscosity unit. So,

$$[F] = MLT^{-1}$$

$$[U] = LT^{-1}$$

$$[L] = L^1$$

$$[\rho] = ML^{-3}$$

$$[\mu] = ML^{-1}T^{-1}$$

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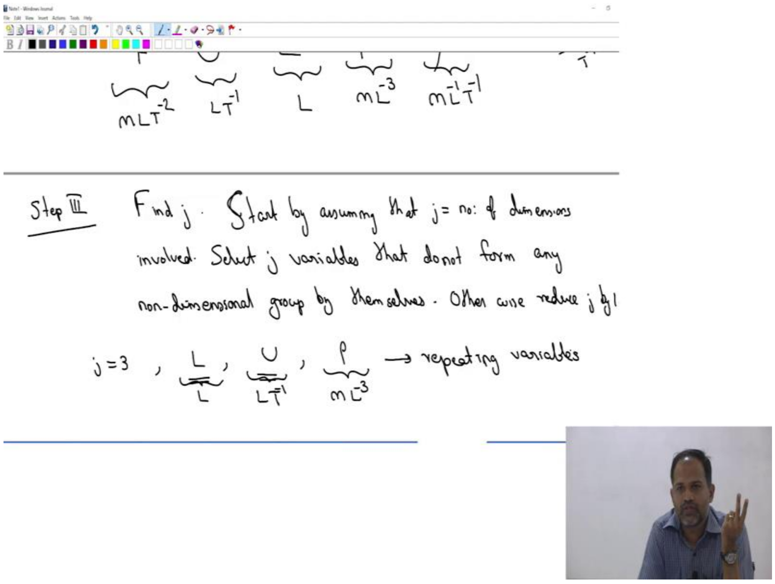

So, this is the next step. So, what we need to do now is we have got five variables, we have got the dimensions of them. Now, we have to select a number from it which I am going to call j and I will select j to be the number of dimensions involved. So, in the problem we have length, mass and time three dimensions involved. So, I will select j to be 3 to *t with ok, I will select j equal to 3 and then I will select j equal to 3 and; that means, I have to select now three variables from five variables that I know ok.

So, the three variables that I will select, I could select any three variable. So, right now I am going to select length, velocity and density as the three variables. Now, when I select them what I need to ensure is that they do not form a non-dimensional number by themselves. So, let us see if I select L U ρ so, L has the dimension of length, U has the dimensions of L T inverse, and ρ has the dimensions of M L to the power minus 3. Can you write ρ, L and U in any combination that you like so, that it can become non-dimensional? No, right that is because one thing is that ρ has an M in it, there is there are no other quantities which has got mass in it. So, whatever way you write that M is not going to go away.

Similarly, U has a T in it so, that T is anyway not going to go away. So, you can ensure, so now, you are assured that L U and ρ just cannot form a non-dimensional number ok. So, these are then this is these are the variables which we call are the repeating variables, which will be part of each non-dimensional group that you are going to find ok. You will find actually some non-dimensional groups, each of them will have L U and ρ as part of the non-dimensional group along with something else and that along with something else is something we need to find.

For example: if you are not able to find you know see we have selected 3 and we are not able to find a you know variables which are not forming non-dimensional group then we should reduce it. We should reduce the value of j by 1 which basically says instead of looking for three variables select two variables as the repeating variables and proceed. Is that clear? Ok. So, we have got L U ρ yeah.

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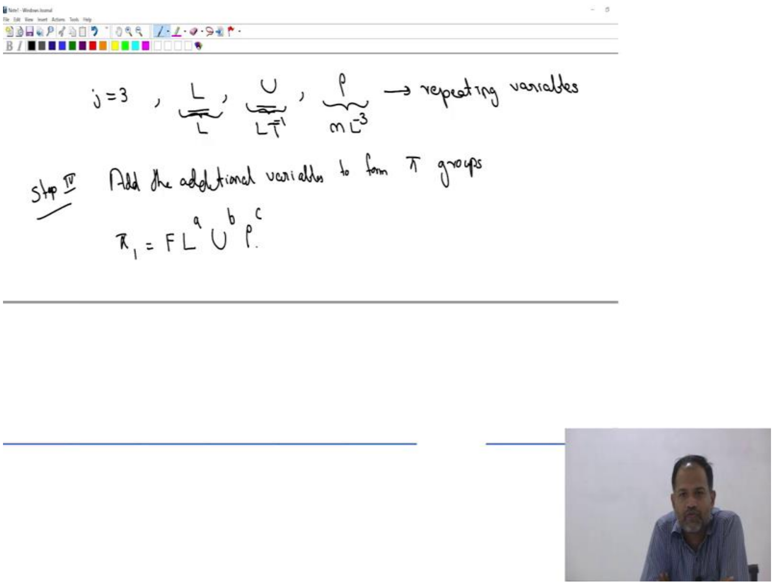

Now, it is easy step IV add the additional variables to form pi groups. So, we have L U and ρ, the variables that we have not considered yet are force and viscosity; so, we can form two pi groups one with F and one with mu. So, our first pi group which I will call pi 1 is going to be just F times L to the power a U to the power b ρ to the power c. So, L, U and ρ are my repeating variables, I raise each of them with a certain power, I multiply it with some force ok. And, that is what I am going to call my first non-dimensional group to be.

$$\pi_1 = FL^{a1}U^{b1}\rho^{c1}$$

I do not know what a b c are, but it is likely that there might be an a b c which might exist which can give you a non-dimensional group ok. Ask no ok, I will take a pause and I will take questions ok. Let me repeat what we have done so, far maybe then that will help.

What is our problem right now? We have a object, we have a fluid that is flowing past it and we want to know what is the force that this object is going to exert on the fluid ok. You are clear that the fluid is going to slow down right because, you have kept an object there so, the fluid is going to go slow down. Fluid was actually coming with a velocity U and it is going to go down to some other velocity which we do not know, but our interest right now is to find out what is this force ok.

Now, this is a very important problem as you will see many applications of this. So, that is the that is the objective that we have. Now we, but we do not know anything; one possibility was that we solve the governing equations. So, which were complicated equations and you knew I mean you have actually seen that only in certain situations, you can solve it ok. This particular problem is a very difficult problem ok, you do not know exactly at the moment how to solve it. So, you are actually trying to do an alternate approach, that alternate approach is what we are going to we are trying to learn right now ok.

So, it involves certain number of steps and we are just going to follow those steps and see what we are going to end up with ok. And, that result is what is one calls as Buckingham pi theorem ok, basically it is a theorem that tells you that how many non-dimensional variables you can identify and there exists a relation between those non-dimensional variables. So, I am telling you that we will do the proof on Monday we will just follow so, that we are familiar with the you know the idea of dimension. So, that you know the proof becomes simpler. So, the problem now is to calculate the force we are yeah. Yeah.

It will depend on the shape of the body.

So, how would you include the shape? What is that?

Angle does not have dimension so, I do not care. What else would you include? Ok. So, you are right so, for example, if it was a rectangle then we could have chosen two different lengths right or if it was a * I could have chosen multiple lengths ok. So, right now I am assuming that there is only one particular length that is relevant for my problem. If you wanted to actually do a rectangle you could define an a, you could define a b and you can say it is a function of a and b. So, if you like you can assume right now that it is a sphere of diameter L.

How do you know b does not have any effect? See fluid is going to go like that right so, the regions of b it is coming in contact a lot. There you are applying the no slip boundary condition, the fluid is indeed going to go down right, whether it is a smaller one or whether it is a longer one the fluid will slow down up to different extent.

Which surface area?

Of the object. So, you could so, that you are saying that a into b is your non-dimensional parameter, sorry it is a dimensional parameter that you are worried about; I do not know at the moment whether a into b is the right parameter or is it something else. So, I am going to keep my a and b separate, I am not even caring about right now in the problem I am just saying L is the only parameter that I will worry ok. So, I will continue. So, this is so, we have written down a expression not really an expression just a functional form, where we have said that F is dependent on the other four variables in the problem which we have identified as velocity, length, density and viscosity ok.

So, we have got five variables and we know the dimensions of each of them. We have identified the dimensions of each of them, that was step number II. Simply writing down what is it in terms of mass, length and time that is a simple thing to do right. Now, I am saying that really how to the next step really is to you know tell how many non-dimensional groups are there ok, that is why we are coming up with this variable j ok.

So, let me state the no its ok. So, we are now out of these five we have to select so, it is like selecting the scaling parameters, when we did the trajectory right we selected two scaling parameters ok. So, here we are actually going to select three of them and these three are L, U and ρ at the moment. We have taken L U ρ and I am saying that we can form two groups which we will call pi groups containing L U and ρ pi 1 and pi 2.

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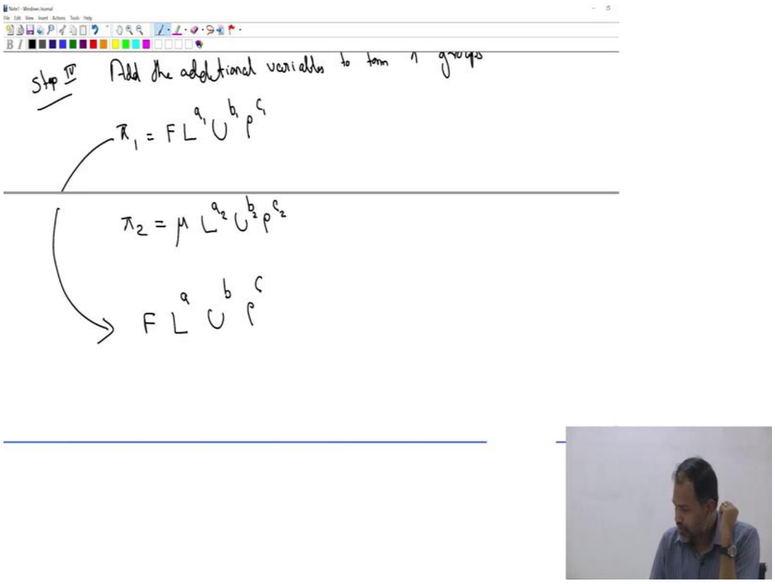

So, L to the power a U to the power b ρ to the power c so, these are the two pi groups that I have that I am writing down. In the first pi group I have select put all the three variables, in the second pi group also I have put all the three variables ok; L U ρ L U and ρ, in the first one I have added my force, in the second one I have added viscosity.

$$\pi_2 = \mu L^{a2} U^{b2} \rho^{c2}$$

So, now you can see that I have covered all the five variables I am not saying so, this we let us call it a 1 b 1 c 1 and let us call this a 2 b 2 and c 2. I do not know whether what are the values of a 1 b 1 and c 1 ok, but they have all of them have got some dimensions. So, I may be able to find out some a 1 b 1 c 1 such that says that it forms a non-dimensional number. So, let us try to find out a 1 b 1 c 1 ok. So, how would we do that? If we talk about pi 1 that is F times L times U times ρ.

$$(MLT^{-2})(L)^a(LT^{-1})^b(ML^{-3})^c = M^0L^0T^0$$

$$M \rightarrow c = -1$$

$$T \rightarrow b = -2$$

$$L \rightarrow a = -2$$

$$\pi_1 = \frac{F}{\rho U^2 L^2}$$

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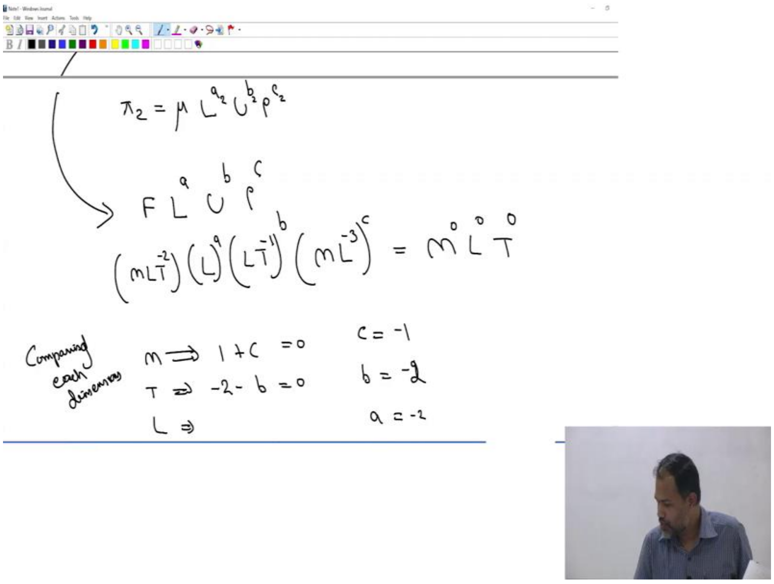

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