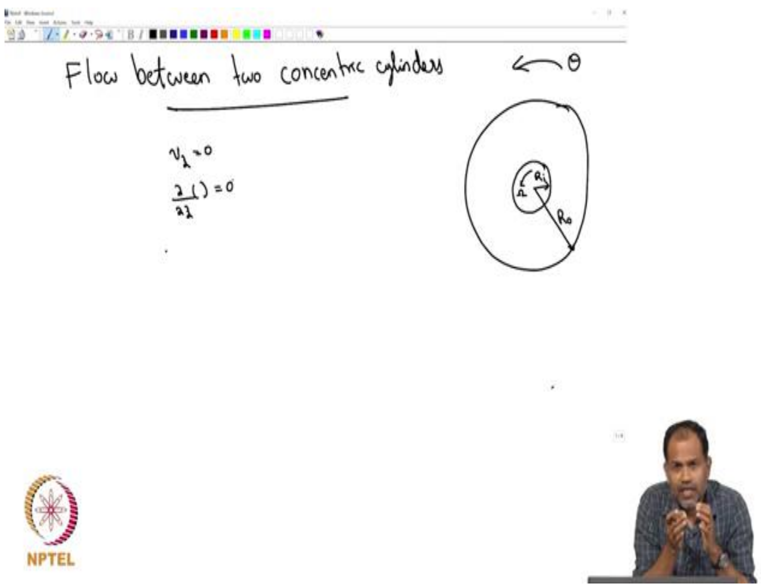
**Fluid and particle Mechanics**
**Prof. Sumesh P.Thampi**
**Department of Chemical Engineering**
**Indian Institute of Technology, Madras**

**Lecture - 17**
**Taylor couette flow**

So, we have seen examples of how we can simplify the Navier Stokes equations namely the mass and momentum balances in a Cartesian coordinate system. We have also seen why these equations would be different for other coordinate system. For example, in a cylindrical coordinate system and the spherical coordinate system and so on and we learned that the main difference is because the unit vectors that you have for non-Cartesian coordinate systems is location specific and therefore, they give rise to non-trivial you know terms in the equation and therefore the equations would indeed look different.

So, we can see an example of a cylindrical coordinate system and see how do we simplify the equations and that flow is a very nice flow because it gives you a lot of other insights into the system. So, this is flow between 2 concentric cylinders ok.

(Refer Slide Time: 01:33)

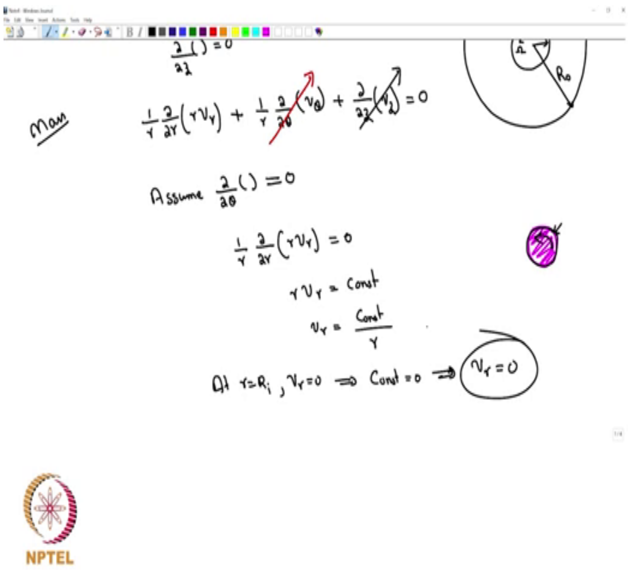



So, you have 2 cylinders 1 inner cylinder and 1 outer cylinder, I am giving the top view. So, let us say that is here inner cylinder and that is here outer cylinder and let us say R i is the radius of the inner cylinder and R o be the radius of the outer cylinder and the fluid is between the inner cylinder and the outer cylinder and let us say that this inner cylinder is rotating. It is rotating with an angular velocity omega and so the. So, you have a fluid now between 2 cylinders; the inner cylinder is rotating because of which the fluid outside will actually start moving, but because the outer cylinder is not rotating the velocity of the fluid exactly on the outer surface or outer cylinder should be 0 and our idea is to find out how does the velocity change from the inner to the outer cylinder ok.

So, the appropriate coordinate system that we want to consider here would be cylindrical coordinate system. Let us assume that the cylinders that you are looking at are really long ok; so, much long in the z direction. So, the z direction is going to be perpendicular to the plane that we are considering and the cylinders are so long that we can neglect all variations in the z directions. So, we will and also the velocity in the z direction. So, because this inner cylinder is rotating you would expect that the fluid flow is in the θ direction really.

So, if I am taking the cylindrical coordinate system I have an R which is in the direction of the radius, then I will have a θ direction which is in the plane and then there is an z direction and that is perpendicular to the plane you will assume that V z is equal to 0 and ∂ by ∂ z are also equal to 0 because the cylinder is infinitely long perpendicular to the surface there is nothing that is going to change in that direction. If that is the case then let us see how do the equations simplify ok.

(Refer Slide Time: 04:18)




So, the first thing that we need to write down is the mass conservation equation.

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)+\frac{\partial}{\partial z}(v_z)=0$$

And as I said a you can take it from any standard textbook because the equations would look a little different from the one which we write in Cartesian coordinate system. So, again you have r θ and z direction. So, there are 3 terms one depends upon v r the second depends upon v θ and the third depends upon v z.

Now, we need to simplify this equation, we know that we are assuming that the flow is purely azimuthal; that means, it is only in the direction of θ it is and the pipe is infinitely long.

Now, also we are going to say that there is nothing that is going to change in the θ direction because θ direction if you look at that is the flow is just continuous there is no θ that is different. So, θ equal to 0 or θ equal to pi by 4 it is just same. So, you would not expect anything that is changing in the θ direction and therefore, derivatives of any quantity in the θ is also going to be 0.

$$\frac{\partial}{\partial z}(\quad)=0$$

$$\frac{\partial}{\partial \theta}(\quad)=0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+0+0=0$$

On integration:

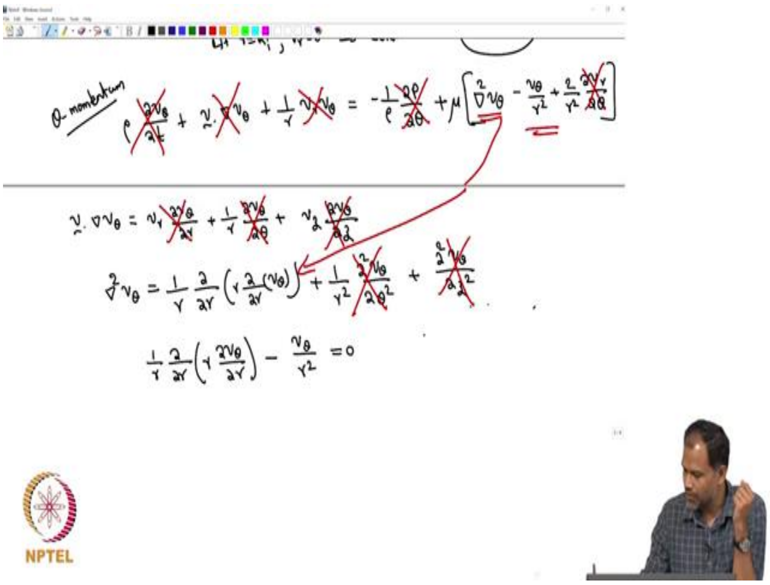
$$v_r=\frac{constant}{r}$$

Now, in order to evaluate this constant we need to calculate what we need to have some physical insight. So, what should be let us say the value of v r on the surface of the inner cylinder. Let us imagine that is your inner cylinder and this inner cylinder is rotating you know in a counterclockwise fashion with some angular velocity. What do you expect the value of v r on the surface of the cylinder? You cannot have fluid that is going into the cylinder because the cylinder is a solid cylinder. So, v r on the surface of the cylinder has to be 0. So,

$$At\ r = R_i, v_r = 0 \rightarrow Constant = 0$$

So, now we have said we have found that v r is 0 everywhere in the fluid domain v z is equal to 0 because we have assumed an infinitely long pipe and there is no flow along the length of the pipe and there is only fluid flow is in the θ direction and therefore, we need to only worry about the θ component of the fluid velocity. We can write down only the θ momentum equation, but before that it is worth looking at the radial component of velocity or r momentum equation as well.

(Refer Slide Time: 08:11)




Because that is going to give you some insight into the problem; so, we can look at θ momentum equation that is what is going to give you information about the about the flow profile.

So, let us start by doing that, we start by writing down the θ momentum equation

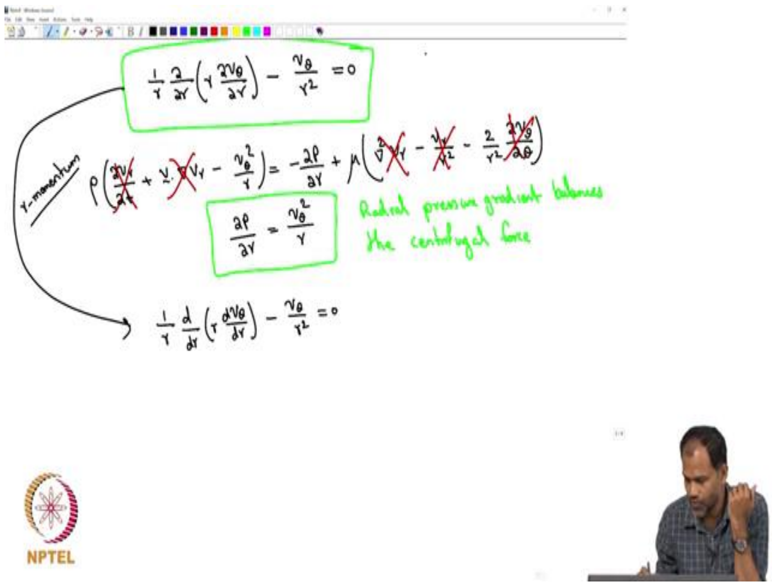
$$\theta - momentum\ equation: \rho \frac{\partial v_\theta}{\partial t} + \vec{v}.\nabla v_\theta + \frac{1}{r}\ v_r v_\theta$$
$$= -\frac{1}{\rho}\frac{\partial P}{\partial \theta} + \mu\left[\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]$$

$$\vec{v}.\nabla v_\theta = v_r\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} = 0$$

$$\nabla^2 v_\theta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} = 0$$

(Refer Slide Time: 12:01)




So, that is what the θ momentum equation reduced to. We can look at what r momentum equation reduced to:

$$\rho\left(\frac{\partial v_r}{\partial t} + \vec{v}.\nabla v_r - \frac{v_\theta}{r}\right) = -\frac{\partial P}{\partial r} + \mu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)$$

$$\frac{\partial P}{\partial r} = \frac{v_\theta^2}{r}$$

Now, if you look at this equation carefully the right hand side that is V square by r. So, that is not force or it is the for centrifugal force density the centrifugal force acting per unit mass ok. So, that is equal to the pressure gradient. So, in other words whenever the fluid elements are rotating it experiences a centrifugal force and that force is actually balanced by a pressure gradient in the r direction. So, something is rotating in the θ direction it experiences a centrifugal force and that force is exactly balanced by the pressure gradient. So, the radial pressure gradient balance the centrifugal force, radial pressure gradient balances the centrifugal force.

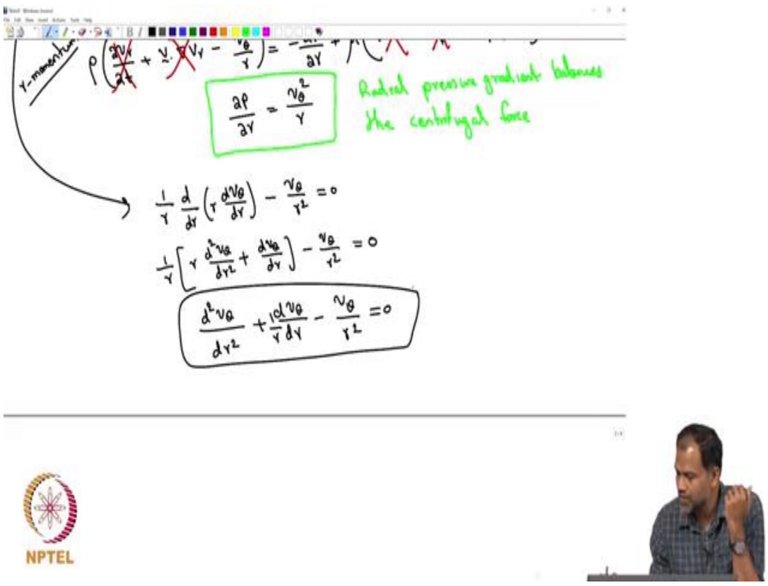
So, that is the information that you get the r momentum equation. The θ momentum equation of course, gave you a differential equation which we need to solve and find out what is we solving it now.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) = \frac{v_\theta}{r^2}$$

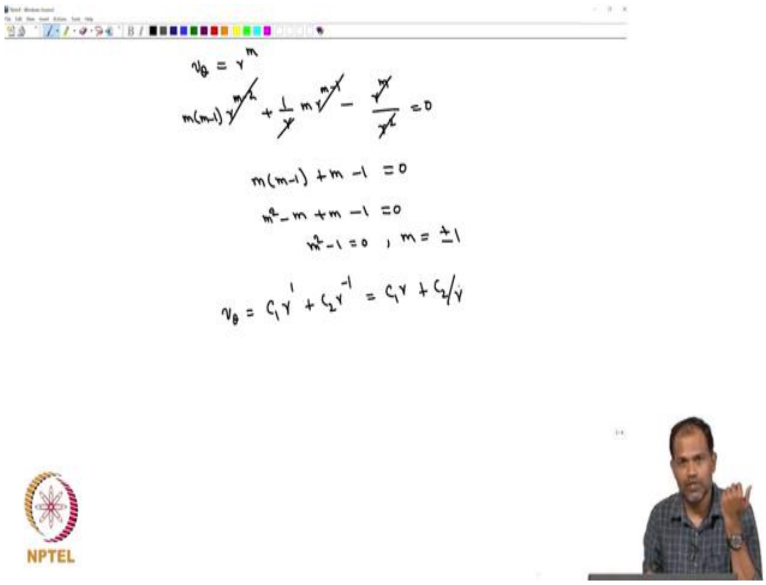
$$\frac{d^2 v_\theta}{dr^2} + \frac{1}{r}\frac{dv_\theta}{dr} - \frac{v_\theta}{r^2} = 0$$

So, that is the differential equation that we need to solve to calculate what is v θ. Now, these kind of differential equations ok. So, you know d square y by d x square plus 1 by x d y plus y by x square is equal to 0.

(Refer Slide Time: 15:05)



(Refer Slide Time: 16:13)




So, these equations are called Euler Cauchy equations and they admit a solution of a power law ok. For example, here the solutions that these equations will be something like r to the power of m. So, that is the kind of solution that we will have. So, we can actually try to find out what that would be. So,

$$If\ ,\qquad v_\theta = r^m \rightarrow Substituting\ in\ the\ differential\ equation$$

$$m(m-1)r^{m-2} + \frac{1}{r}\,mr^{m-1} - \frac{r^m}{r^2} = 0$$

$$m(m-1) + m - 1 = 0$$

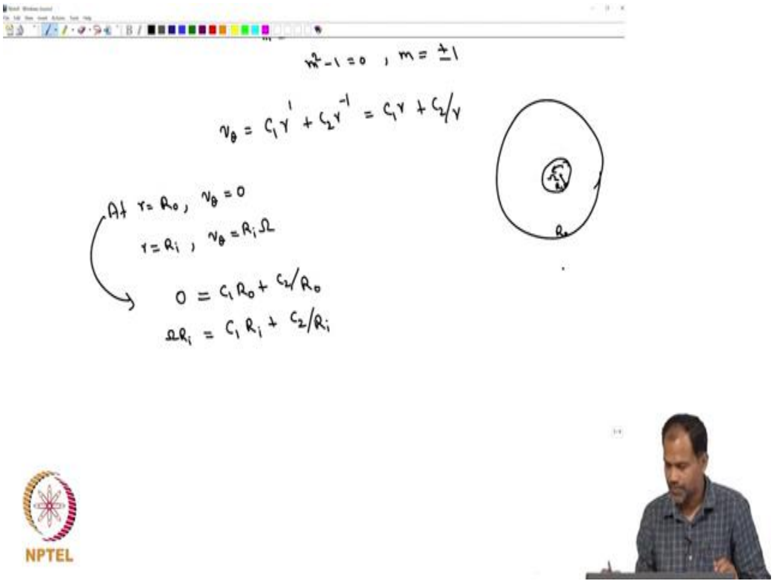
$$(m-1)(m+1) = 0$$

$$m = \pm 1$$

That means, my solution is going to be

$$v_\theta = C_1 r + \frac{C_2}{r}$$

(Refer Slide Time: 18:09)

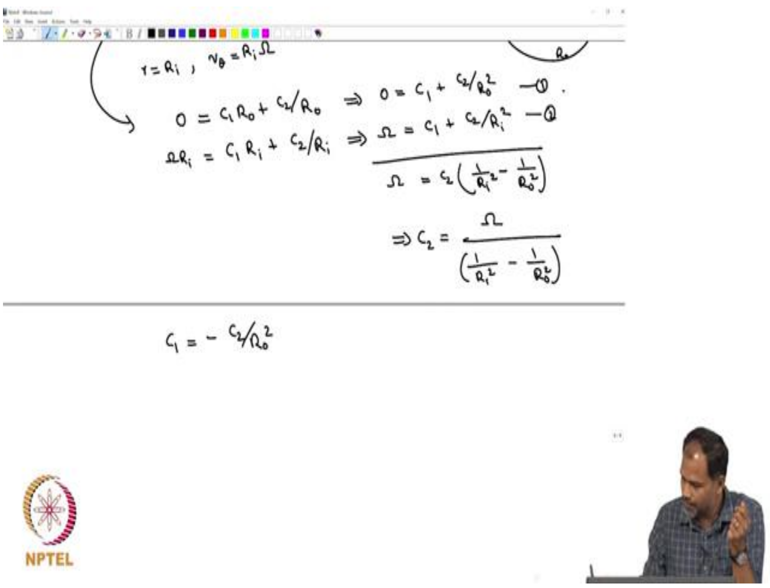



We have an inner cylinder that is rotating with an angular velocity omega. We have an outer cylinder that is static. So, and the inner cylinder has a radius R i the outer cylinder has a radius R o. So, we can say that

$$At, r = R_o \; ; \; v_\theta = 0$$

$$r = R_i \; ; \; v_\theta = R_i \Omega$$

(Refer Slide Time: 19:28)




So, we can substitute and find that

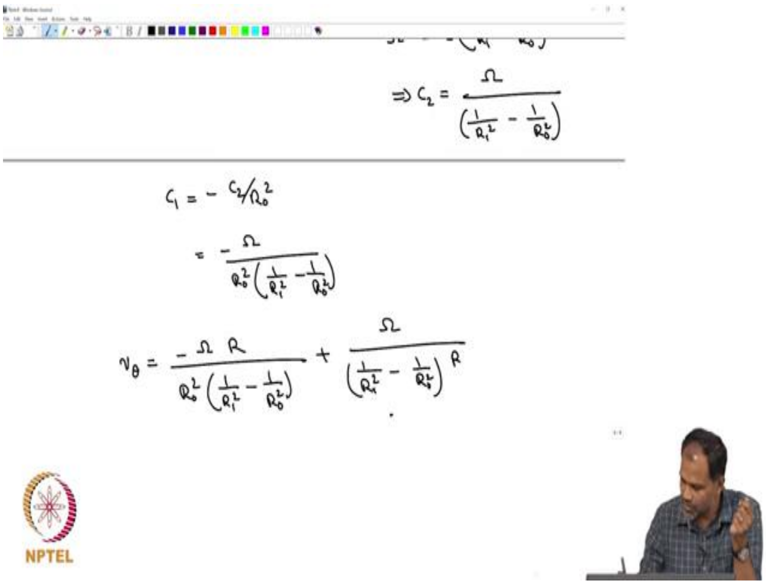
$$0 = C_1 R_o + \frac{C_2}{R_o} \rightarrow C_1 + \frac{C_2}{R_o^2} = 0$$

$$\Omega R_i = C_1 R_i + \frac{C_2}{R_i} \rightarrow \Omega = C_1 + \frac{C_2}{R_i^2}$$

$$Eliminating\ C_1, we\ have\ C_2 = \frac{\Omega}{\left(\frac{1}{R_i^2} - \frac{1}{R_o^2}\right)}$$

$$C_1 = -\frac{C_2}{R_o^2} = -\frac{\Omega}{R_o^2\left(\frac{1}{R_i^2} - \frac{1}{R_o^2}\right)}$$

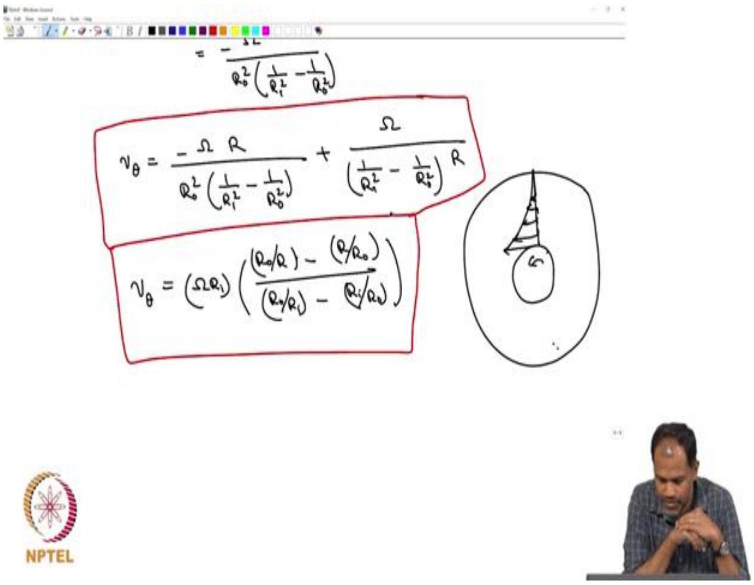
(Refer Slide Time: 20:28)




And therefore, you are ready to get your expression

$$v_\theta = -\frac{\Omega R}{R_o^2\left(\frac{1}{R_i^2} - \frac{1}{R_o^2}\right)} + \frac{\Omega}{\left(\frac{1}{R_i^2} - \frac{1}{R_o^2}\right) R}$$

(Refer Slide Time: 21:29)



So, if you want you can rearrange that expression you can do it yourself and nicer way to represent it would be:

$$v_\theta = (\Omega R_i)\left(\frac{\left(\frac{R_o}{R}\right)-\left(\frac{R}{R_o}\right)}{\left(\frac{R_o}{R_i}\right)-\left(\frac{R_i}{R_o}\right)}\right)$$

Just rearranging that equation if you like. So, that is the expression for the velocity profile between 2 concentric cylinders. In fact, if you plot it if that is the inner cylinder and that was the outer cylinder and if the inner cylinder is rotating you will see that the velocity profile would. So, the velocity at that point would be something of that sort and then it should become 0 and it would decay like that nice to plot it and see.

But that is what the velocity profile is going to be. We just box that is the expression you need to remember anything it is just that is the information that you get when you solve the cylindrical problem. It also tells you how you should deal with the equations in cylindrical coordinate system, how some of the non-trivial terms you know contribute and the pressure centrifugal force balance is the other thing that nicely came out of this analysis.