Fluid and Particle Mechanics

Prof. Sumesh P. Thampi
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture – 12
Solutions of Navier Stokes in the cylindrical ordinate system - 1

So, what we are going to do today is pipe flow.

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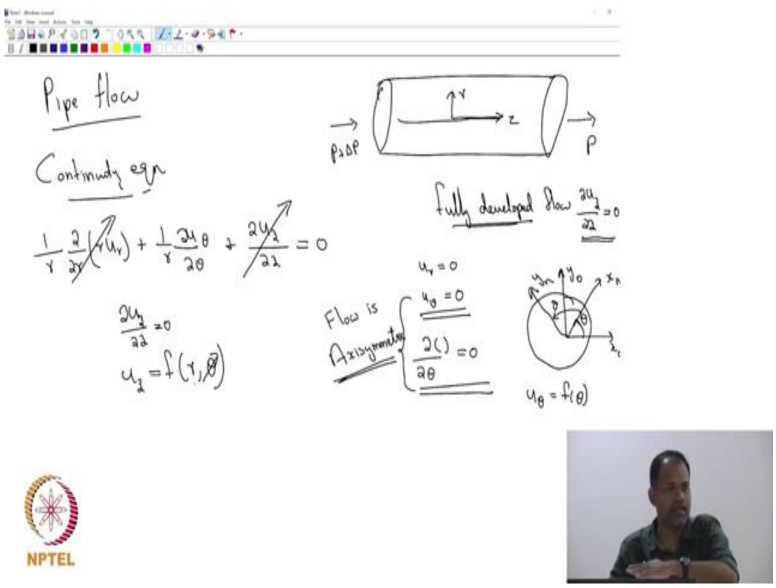

In terms of calculations the calculations are very simple straightforward you would have also probably seen it, but the concepts that get introduced during this calculation they are very important and that is because it is the same concept that will that we will keep talking again and again in various contexts throughout the course and probably throughout the you know your degree here ok. So, that is the importance of pipe flow.

So, the calculations essentially are very similar to what we did in the 2D channel ok. You basically want to find out how you can describe flow in a cylindrical pipe ok. So, when I say pipe flow it could be actually any cross section and when you go to a real system ok. For example, you go to an industry you need not always find a cylindrical pipe you might find a square pipe rectangular pipe and so on.

In fact, you could ask the question what is the pipe that I should choose when I have an; when I have an application. When I want to let say transport things from one point to another let us say crude oil or some chemical what should be the shape that you should be choosing? At the moment you do not know and we probably will not talk much about it, but we will see what various shapes would mean that will come at a later part, but this forms a basic you know example.

So, you have a pipe and there is a fluid that is going in and coming out and in order to attain that what you can do is you can apply a pressure gradient. You will apply a pressure p plus ∂ta p on this side and this side can be pressure and if as long as ∂ta p is greater than 0 fluid is to go in to go from one side to another and again we are going to look at fully developed flow.

So, as we said that day when the moment the fluid enters here there will be an entrance region where the flow is developing; that means, if I am going along this axis along the pipe axis I would see that at every point the flow is changing, but beyond a point flow does not change right and that is what we termed as fully developed flow. So, fully develop flow is that flow which is coming after the entrance length and we are basically dealing with fully develop flow because entrance length may not be so long while we might be talking about pipes which are like you know kilometers long then we do not really care what happens at the initial portion. So, let us as well neglect that initial portion and work only with the fully developed flow for the time being and let us use now which coordinate system would we choose for this problem? Why would we choose cylindrical coordinate system?

Because it is a cylinder. So, let us say now I want to solve flow in that Cartesian ok. So, what decides what you know how should we choose the coordinate system? We will see that I will answer the question once we go along. So, let us choose the cylindrical coordinate system. We will say that that is our z axis that is our r and then θ is going to be you know perpendicular to the plane and around the z axis.

We can, so, what is the first thing to do you will write down the governing equations which is continuity and momentum equation. We will start with the continuity equation and we will make few more assumptions. So, we derived continuity equation yesterday what was that? I hope you will tell. Are you going to tell?

No. I am going to write it down in cylindrical coordinate system.

$$continuity\ equation: \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

This one what you see on the screen. So, that is the continuity equation. So, you have u r u θ u z. So, these are the three components of velocity as we described yesterday. So, we have fluid which is going from one side of the pipe to the other ok. So, the component of velocity that we are interested in is u z ok. So, we really want to know what is u z and if you remember what we did in the 2D channel what was the component of velocity that we had. We had a component which is parallel to the channel axis we did not have a component that is perpendicular to it. So, here also it is unlikely that we will have a u r ok; u r is likely to be 0 and we will right now assume that u r is indeed 0 you can question that assumption, but at a later point.

So, let us assume that u r is equal to 0 we will also assume that u θ is equal to 0 ok.

$$u_r = 0; u_\theta = 0$$

Now assuming u θ equal to 0 is much more sensible because what is u θ? U θ is essentially. So, let us look at a cross section of this pipe. Let us say that is a cross section ok. So, flow is coming from behind the screen through the screen to outside and let us say you know you have an x, you have a y and that is what you define as θ right. So, then I could say that my u θ which will come out to be some function of θ if I say I could also draw my coordinate system in that fashion right that is also an acceptable coordinate system in which I would define my θ that way right.

So, the θ that I would have defined in my original coordinate system and the θ that I will define in my new coordinate system would have been different ok. So, it is unlikely that therefore, there can be u θ dependence because you would not be able to find out a mathematical expression that will do this, that any coordinate that you choose whether you know you choose this x o, y o, x n, y n whatever coordinate system you choose there will not be any mathematical function that is going to tell you that you know you are going to get the same number ok.

So, it is unlikely that u θ will be coming out to it will come out as a non-zero quantity. That is one reason to a priori assume that I can actually start by saying that u θ is 0. Is that clear ok? So, even if it is not it is I mean let us just assume that u θ is equal to 0. We will also assume that ∂ by ∂ θ of anything is equal to 0.

$$\frac{\partial}{\partial\theta}(Any\ variable) = 0$$

What is meant by that? As I go in the θ direction. So, as I am going in this direction again I am assuming that there is nothing that is changing ok.

So, that is, so, imagine, so, you exactly have a circular pipe ok. So, why should any point be different than any other point that is the idea. So, therefore, you are going to assume that ∂ by ∂ θ of anything is also going to be 0. So, these two assumptions are known as axisymmetric. So, we are essentially assuming that flow is axisymmetric. So, flow is axisymmetric. So, basically meaning that around the axis around the z axis flow is symmetric that is the assumption that we have gone in or we have taken and we are only looking for the u z component. The u z component is typically called the axial velocity that is velocity along the axis.

You can, so, you cannot assume both of them to be 0 as a consequence of a fully developed flow. You can take one of them another can be shown that fully developed flow is consistent with this approximation. So, let me just show you. So, let us say if you take u r is equal to 0 u θ equal to 0 the continuity equation says that is gone, that is gone and only this is left which is whatever fully developed flow is ok.

So, it is consist. So, only thing I am saying is that fully developed flow and these two assumptions are consistent ok. Now on the other hand if you wanted to start by saying that I have fully developed flow. So, ∂ u z by ∂ z is equal to 0 then suppose you started with continuity equation and you could have said ∂ u z by ∂ z is equal to 0. So, this I am going to put to 0 then you have to either assume u r equal to 0 or u θ equal to 0 to get the other is equal to 0.

So, if I say u r is equal to 0, I will pull push that out then I have ∂ u θ by ∂ θ which will come out to be a constant ok, but it will be 0 at the wall and therefore, it will be 0 everywhere.

So, by definition this is our definition of fully developed flow. If u z is changing with z then it is not called fully develop flow ok. For example, the flow that is coming from here to here things are changing in z direction. So, you can say the flow is actually speeding up or slowing down that is speeding up is in space as well it could be in time also, but it is. So, it is accelerating in space. So, wherever it is accelerating or decelerating you will not call it a fully developed. It is a developing flow against a fully developed flow ok, alright.

So, those are the assumptions. Now we can just go ahead write down our you know force balances throw out all the unnecessary terms and then see what is left out. Any ∂bts otherwise on this? So, this one thing. So, we have ∂ u z by ∂ z is equal to 0; that means, u z is a function of r and θ, but because of the assumption of axis symmetry by definition things are not dependent on θ.

$$\frac{\partial u_z}{\partial z} = 0; u_z = f(r,\theta) = f(r)[axisymmetric\ assumption]$$

So, we do not have we have just u z as a function of r. Just the way in the other problem we had a velocity nor parallel to the channel is just a function of you know perpendicular direction. So, similarly here velocity is along the pipe axis and it will vary only in the radial direction and that is what we are gonna derive.

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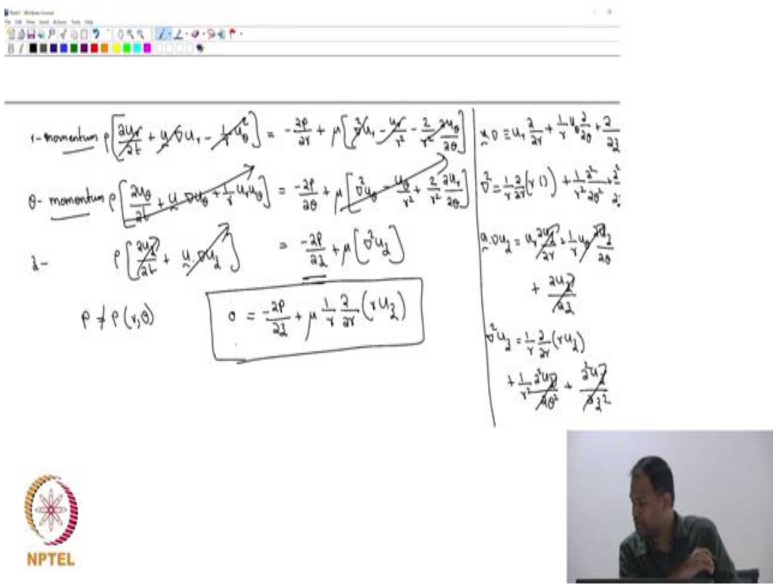

So, let me write down the equations r momentum equation.

$$r - momentum: \rho\left[\frac{\partial u_r}{\partial t} + \vec{u}.\nabla u_r - \frac{1}{r}u_\theta^2\right] = -\frac{\partial P}{\partial r} + \mu\left[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]$$

So, now, we should not be surprising to you why we get all these extra terms right. So, this right hand side is actually $\nabla^2 u$ right it is second derivative of u. So, we already saw what divergence did just second derivative. So, there are many more differentiations. All those unit vectors are going to contribute to many things. That is why you get all these extra terms here and some of the extra terms there, alright ok.

$$\theta - momentum: \rho\left[\frac{\partial u_\theta}{\partial t} + \vec{u}.\nabla u_\theta - \frac{1}{r}u_r u_\theta\right] = -\frac{\partial P}{\partial \theta} + \mu\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2} - \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right]$$

$$z - momentum: \rho\left[\frac{\partial u_z}{\partial t} + \vec{u}.\nabla u_z\right] = -\frac{\partial P}{\partial z} + \mu[\nabla^2 u_z]$$

$$\vec{u}.\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

From first two equations, we see that

$$P \neq P(r, \theta)$$

$$P = P(z)$$

That is going to be nothing, but the pressure force that you have applied ok. Pressure is high on one side and pressure is low on the other sides just exactly that. So, p is only a function of a z.

On simplification, the final differential equation that we obtain is:

$$-\frac{\partial P}{\partial z} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = 0$$

What kind of a this is a partial differential equation or an ordinary differential equation that is the question? So, we said p is not a function of r and θ. So, I can write this ∂ p by ∂ z as really d p by d z and therefore, u z is only a function of r. So, that is again an ordinary derivative ok. So, it is no more a partial differential equation it is an ordinary differential equation and you know how to solve it.