**Fluid Mechanics**
**Prof. Sumesh P. Thampi**
**Department of Chemical Engineering**
**Indian Institute of Technology, Madras**

**Lecture -10**
**Introduction to cylindrical coordinate systems**

What we are going to do today is to do the same in cylindrical coordinates because finally, we want to do a flow through a pipe and that is a very important problem. But, we should know what are the governing equations in various coordinates systems because then we can try to attempt various geometries ok.

So, what we did in the last class was the simplest of the case where you know you deal with the Cartesian coordinate system and then you could therefore, solve for the flow profile in a 2D channel and you do know that you know the governing equations look a bit different when you are in other coordinate systems. So, let us just look at why it happens so before we actually attempt to solve the problem in a pipe.

So, what we are going to try doing today is to find out how do we find out what is the right governing equation in various coordinate systems. So, we will use cylindrical coordinate system as an example because that is going to be most useful for this course.

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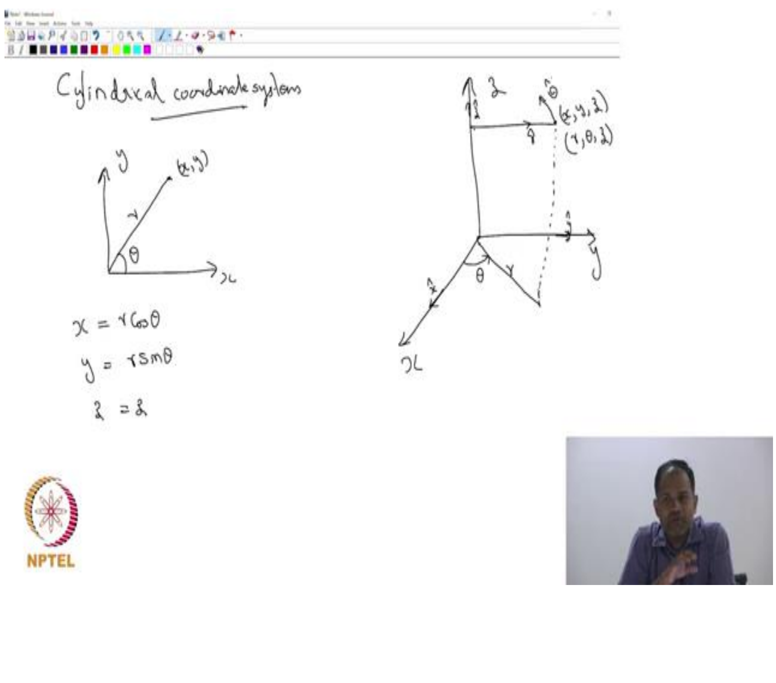

So, what we have is a Cartesian coordinate system. Let us start with that because that is the most familiar case ok. So, you have x, y, z a right handed coordinate system and any point that you look at, you will describe that point using a x, y, z right that is actually what we mean by Cartesian coordinates. Now, in cylindrical coordinates we want to replace it with an r θ z ok. How does one define r? The way to define r is you project this point onto the x y plane and this distance is what actually is known as r and how does one define θ.

Θ is essentially this angle ok. So, what so that means, that z so, things are changing so. So, really you have z in the Cartesian coordinate system remains as the same z in the cylindrical coordinate system instead of x and y you are going to use r and θ to describe any point. So, r is going to tell you what is the radial distance from z-axis. So, that is a z-axis it tells you how far radially it is from z-axis. Θ is going to tell you if you were to have an x-axis how much rotated it would be. So, therefore, r and θ will give you the same information as x and y give. That is clear?

So, if you look at just x y coordinates x. So, we can just since z is not changing we can actually look at all the analysis by looking at the x y plane. So, we will just draw x y plane that is our x that is our y. Any point that was actually described using x y we are going to change it into r θ coordinate system. So, that is going to be our r now and that is going to be our θ now. So, if this point was denoted using x y then what is the relation between x r and θ?

$$x = rcos\theta; y = rsin\theta; z = z$$

Now, the important thing is to realize what happens to the unit vectors in these coordinate systems. So, in Cartesian coordinates system, if I have a unit vector that I denote using x cap; here you know a unit direction y cap and this is z cap any vector that I talk about I can resolve it into components in x y z direction you all know that ok. Now what we really need is when we have a vector we also need to resolve it in r direction θ direction and z direction if you want to really make use of cylindrical coordinates system ok.

So, therefore, we need to define unit vectors in r θ z coordinate system which is going to be simple because this is your r and then how what you defining a r is essentially the distance from the z-axis to any point ok. So, r is really that ok. So, r cap if I define, r cap is just going to be a vector ok; a unit vector which tells you know what is the direction which your radius is going. So, it could be so, this is right now in this plane y is a plane, r cap would be like this if I talk about an x z plane, r cap will come out of the plane and so on you understand that. So, if you have an z-axis r cap could be in these directions just the way you have radial directions going away ok.

And θ cap is basically the direction in which θ is changing. So, θ cap would be something of that sort ok; a unit vector that is defined in the θ cap direction. So, you imagine my hand is the z-axis ok, r cap would change like that θ cap is going to tell you how things are changing in this direction. So, this can be so, if I am looking at this point θ cap could be a direction if I am to talking about a point here θ cap will be this direction and so on ok.

So, r cap is the unit vector that is coming out of z θ cap is going to tell you how things are going to change this way and z cap of course, will remain as usual along the z-axis. So, that are those are the unit vectors in cylindrical coordinate system ok. Now, we are going to do is we are going to connect this unit vectors in cylindrical coordinates with unit vectors in Cartesian coordinates ok. That is going to be useful this going to give us useful relations.

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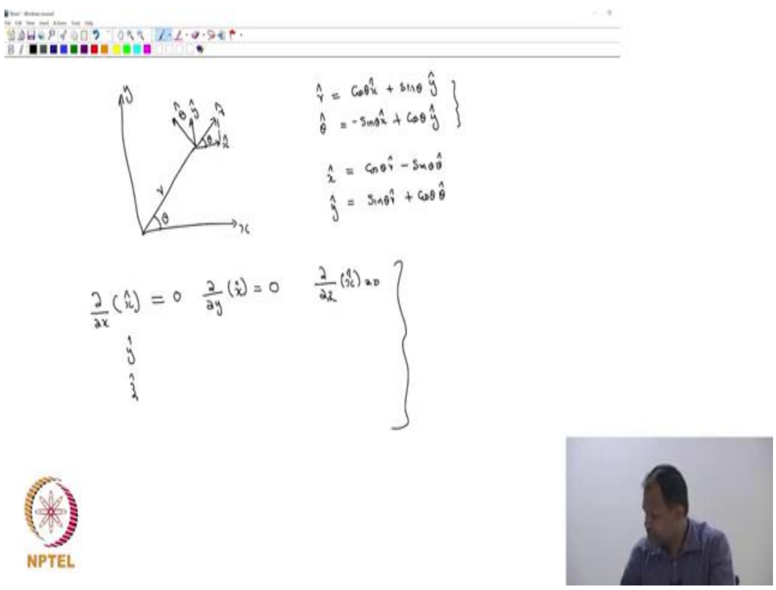


So, again we can a just stick to the x y plane; so, x y the point at a distance r at an angle θ. So, by definition that is my unit vector r cap, that is going to be my unit vector θ cap, this is how x cap would have looked like that is how y cap would have a looked like and this angle is going to be θ all of you agree. So, then how can I write down r cap, r cap is a unit vector. So, if I draw finish the triangle this length is going to be just cos θ.

This length is going to be just sin θ. So, if I do cos θ x cap plus sin θ y cap that is just going to give me r cap right; so,

$$\hat{z} = \hat{z}$$

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

So, then now write down what is x cap and what is y cap? You can write down using the same thing or you can solve these equation whichever way you like. You can multiply the first equation with cos θ, multiply the second equation with sin θ add up and you will get x cap and the other way ok.

$$\hat{x} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

Ok and y cap.

$$\hat{y} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$$

Yeah. So, that is basically the relation the inverse relation. So, now, we know given Cartesian coordinate system how to convert it into cylindrical coordinate unit vectors and the other way ok. Now, so these are just some relations that we need the most important difference that make Cartesian coordinate system different from all other coordinate systems is this. What is ∂ by ∂ x of x cap that is when I am moving along the x direction how does my unit vector x change? Does it change? Is just does not change by magnitude by definition and it does not change the direction so it is 0.

What is ∂ by ∂ y of x cap? If I am moving in y direction does anything happen to my unit vector x cap; no ∂ by ∂ z of x cap again 0 and so is true with whether I use y cap or z cap. In other words none of the unit vectors there are no gradients whatever differences I am going to take it is all going to be 0 ok.

$$\frac{\partial\hat{x}}{\partial x} = 0;\ \frac{\partial\hat{x}}{\partial y} = 0;\ \frac{\partial\hat{x}}{\partial z} = 0$$

That is not true in any of the curvilinear coordinate system.

So, why I am saying that let us say this is my z-axis. Let us say this pen represents by r cap ok. If I am changing in this direction my r cap direction is changing right, then I am going like this I am going in θ direction. When I am going in θ direction; so, if I have a point here this is the direction which I have defined my r cap if I go to a different point my r cap has changed the direction even though it is magnitude has not changed ok.

So; that means, that if I am see if I am looking at the derivative of unit vectors that need not to be 0 the way we have written it down for the Cartesian coordinates system. So, we need to actually find out how unit vectors change when I go in different directions and that is important because we talk about all differential equations, we talk about spatial gradients, we talk about divergences, curves and gradients and so on. So, the change in unit vectors therefore, become important that is clear. So, let us calculate that it is very easy.

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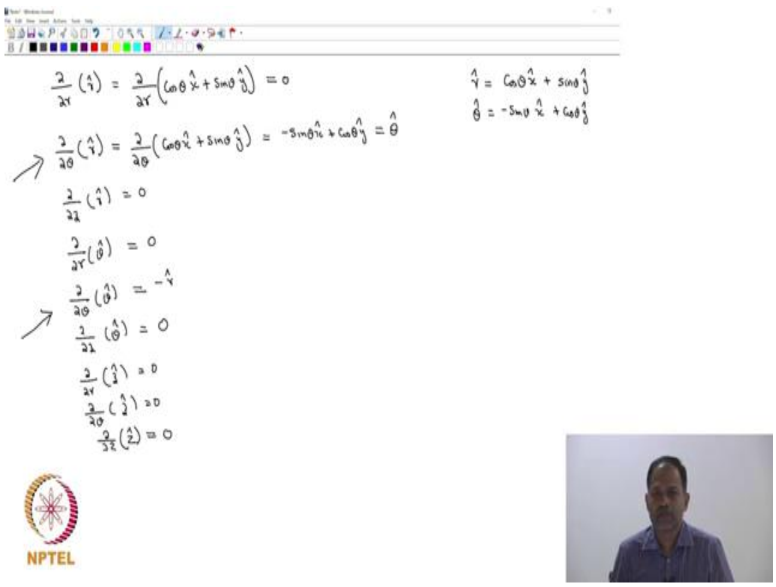


So,

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

Now θ is not a function of r by definition θ and r are 2 independent coordinates, x cap and y cap are constant unit vectors. So, this is basically a constant as far as r variation is concerned and therefore, that is equal to 0 ∂ by ∂ r of r cap equal to 0 and that has to be the case because if I define an r cap like this if I go in this direction the unit vector is not going to change ok. So,

$$\frac{\partial}{\partial r}(\hat{r}) = 0$$

Now, let us do ∂ by ∂ θ of r cap. What is ∂ by ∂ θ of r cap?

$$\frac{\partial}{\partial \theta}(\hat{r}) = \frac{\partial}{\partial \theta}(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}) = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} = \hat{\theta}$$

Also,

$$\frac{\partial}{\partial z}(\hat{r}) = 0$$

$$\frac{\partial}{\partial r}(\hat{\theta}) = 0$$

$$\frac{\partial}{\partial r}(\hat{z}) = 0$$

$$\frac{\partial}{\partial \theta}(\hat{z}) = 0$$

$$\frac{\partial}{\partial z}(\hat{z}) = 0$$

So, this is basically how the unit vectors are changing in the cylindrical coordinate system ok. So, there are non non-trivial 2 non-trivial quantities; one is ∂ by ∂ θ of r cap another is ∂ by ∂ θ of θ cap ok. And, that can actually bring in some additional terms whenever you talk about special gradients.

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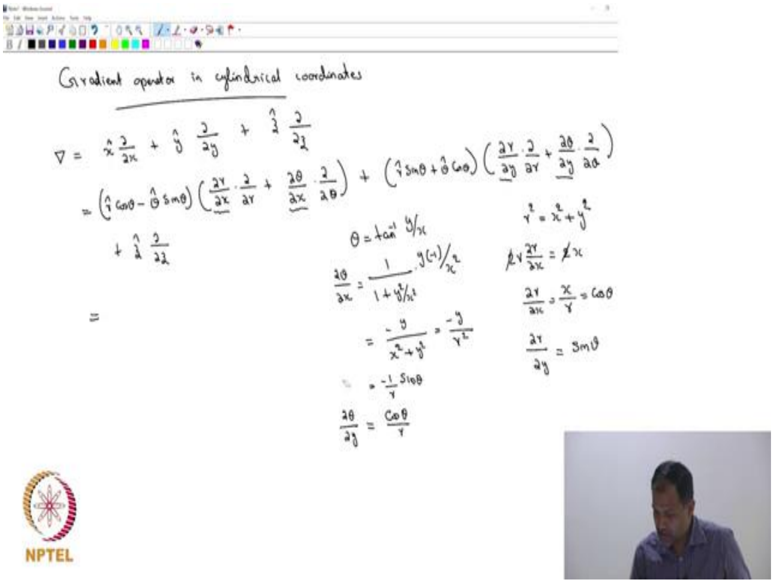


So, we are going to derive now gradient operator in cylindrical coordinates. In Cartesian coordinates we have,

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Substituting expressions we derived earlier:

$$\nabla = \left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\right) + \left(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}\right)\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}\right) + \hat{z}\frac{\partial}{\partial z}$$

We can write r as:

$$r^2 = x^2 + y^2$$

Now,

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$$

Now we need ∂ θ by ∂ x. So,

$$\theta = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{y}{r^2} = -\frac{1}{r}\sin\theta$$

Also,

$$\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$$

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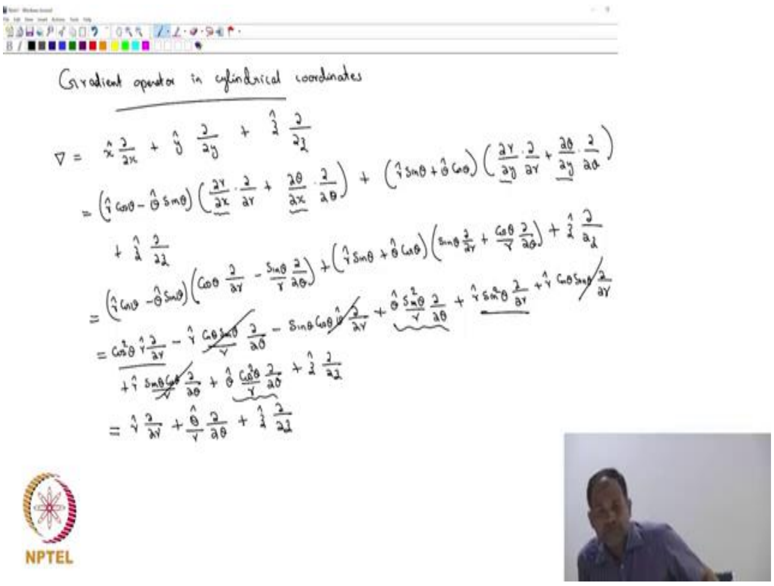


So, now we can substitute and we can simplify:

$$\nabla = \left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) + \left(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}\right)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) + \hat{z}\frac{\partial}{\partial z}$$

$$\nabla = \cos^2\theta\,\hat{r}\frac{\partial}{\partial r} - \frac{\hat{r}\cos\theta}{r}\frac{\partial}{\partial\theta} - \sin\theta\cos\theta\,\hat{\theta}\frac{\partial}{\partial r} + \frac{\hat{\theta}\sin^2\theta}{r}\frac{\partial}{\partial\theta} + \hat{r}\sin^2\theta\frac{\partial}{\partial r} + \hat{r}\cos\theta\sin\theta\frac{\partial}{\partial r} + \frac{\hat{r}\sin\theta\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\theta}\cos^2\theta}{r}\frac{\partial}{\partial\theta} + \hat{z}\frac{\partial}{\partial z}$$

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \hat{z}\frac{\partial}{\partial z}$$