CH5230: System Identification

Spectral Representation 1

We've been reviewing the concepts of random processes and we have learned quite a bit about the random processes and time domain. And one of the things that we have learned is that the Auto covariance function really is a very important tool in the analysis of random signals because it tells us whether there is predictability and the signatures of the ACF as well as the PACF, tell us what kind of models we can build and so on. Now, as you have seen the auto covariance function is a measure that's based out of time domain but if you recall it analysis that we have done for deterministic processes, we have learned how to analyze deterministic processes in frequency domain, right? And one of the things that we have learned in frequency domain is the notion of a frequency response function. So the question that we want to have that we have here is, given that, there is a random process that can be represented as a white noise passing through a filter. Can I make sense for example, out of this frequency response function? Does it have the same meaning as the one that we have in a deterministic world. So in the deterministic world. We had this relation with us where, we wrote y[k] equals G q inverse u[k] that's basically y star. And then, we said that G of e to the minus j omega or G of E to the J omega is the frequency response function. It tells us how the system acts like a filter, what kind of frequencies are filtered and so on. Can I say, can I give the same interpretation to H of E to the j omega. Does it make sense? Because, if you recall the theoretical definition of G is that, it relates, y of y star of omega to U of omega. Where y star of omega and U of omega are the discrete time Fourier time transforms. Provided, they exist. Now, can I write a similar relation here for random signals is what we shall ask and we should answer as well. But before I go and show you the developments that lead to the answer, let me give you the answer itself. They are answer is no. That means, I cannot write a relation like this for the case of random signals simply because the Fourier transform of, or the discrete time Fourier transform of random signals do not exist. Why is that? Sorry? Giving you, it's a stationary process.

But y[k] will not be essential in convergent.

Convergent in what sense?

In, when k tends to infinity. [3:34 inaudible] when you can't get the full length answer.

No, no. You have to tell me the correct technical word. Convergent, there is a prefix attached to it. No. When does a discrete time Fourier transform exist of a sequence?

How much ever we [3:51 inaudible] into a [3:51 inaudible].

Really? You want to correct that? So when does the discrete time Fourier exist? We have written many times. That is for impulse response for any sequence. You just have to extend that result any sequence. For any sequence, when does the discrete time Fourier transform exist? When that sequences absolutely convergent. What is the problem? I'm not asking with respect to impulse response. You are forgetting a very important absolutely convergent. Now absolute convergence implies that the sequence has to decay. We know by definition, random signals do not decay with time. They exist forever. And therefore the Fourier transform doesn't exist. I mean, there is a weaker requirement for a DTFT to exist, which is that its energy should be finite. That is the weaker requirement. The strict requirement is the signal should be absolutely convergent, weaker requirement is that the energy should be finite. Now, we know, random signals, if you think of the energy, do you think we have finite energy? Do we know what, and how energy is defined as? Right. Well, in some sense, maybe you can think of it as some square, absolute values, we'll define energy shortly. They do not have finite energy either. So in other words, the Fourier transform of random signals does not exist. Therefore, I cannot attach such an interpretation in the random world. All right. But if you recall, I think I've given this relation, there is another relation that we wrote here. Did I give you this relation? Even if I don't, you can just quickly derive this right? How do I derive that? How do I derive this relation from here? Sorry?

Multiply that with y star of omega. Well, I mean, this y star may confuse you because star is also used in complex numbers to indicate conjugate. Okay? So momentarily will drop the star here. So how do you direct the solution from here? Basic complex number theory you should be comfortable. I see that from the quiz that your complex number found fundamentals basics are really shaky. You should know the concept of a conjugate, right? So think, how do I, how do you derive the relation? What is the difficult? So, if I don't ask you personally, you won't answer. Anyone is free to answer. Eight magnitude both sides and square, that's it or multiply both sides to the respective conjugates. You shouldn't be looking at me, you should be actually telling the answer. So just multiply with the conjugates. If this equation holds for y, it also holds for its conjugate. And multiply both sides you will get this equation. Now, it turns out that although this does not hold, we will show that a relation like this holds. In other words, although we may not be able to define y of omega, like that is we cannot define V of omega, here. We may be able to define something like this. Something like this. Very close to that, you get the point. I cannot define V of omega simply because of Fourier transform doesn't exist. But I may be to define mod v of omega square. Now, that may sound very strange, because how do you get mod v of have omega square, if I can, you cannot define v of omega, correct? And that is the interesting part with random signals that I do not have to know v of omega to compute mod v of omega square. And that is actually a very interesting and important, in fact, concert a milestone result in the analysis of random signals. And that's where, that's a result, central result that will review in today's lecture, but to understand that, we will have to briefly go through a journey in the deterministic world again and examine certain notions such as energy and power, and so on. Because, this quantity, eventually, slightly modified version of this for v slightly modified, not exactly. This we'll realize that it is nothing but the so-called power spectral density. But we'll talk about. I mean you have to understand what is energy? What is power? What is meant by density? What is meant by power spectral density and so on? Because if you don't understand those terms, you're going to have a tough time understanding the frequency domain description of random processes. So let's begin with some basic definition. So we here we have the notion of energy, right? Defined for continuous time and discrete time signals. They're very standard definitions they follow from physics. And we assume that this integrals exist for the continuous time and discrete time case.

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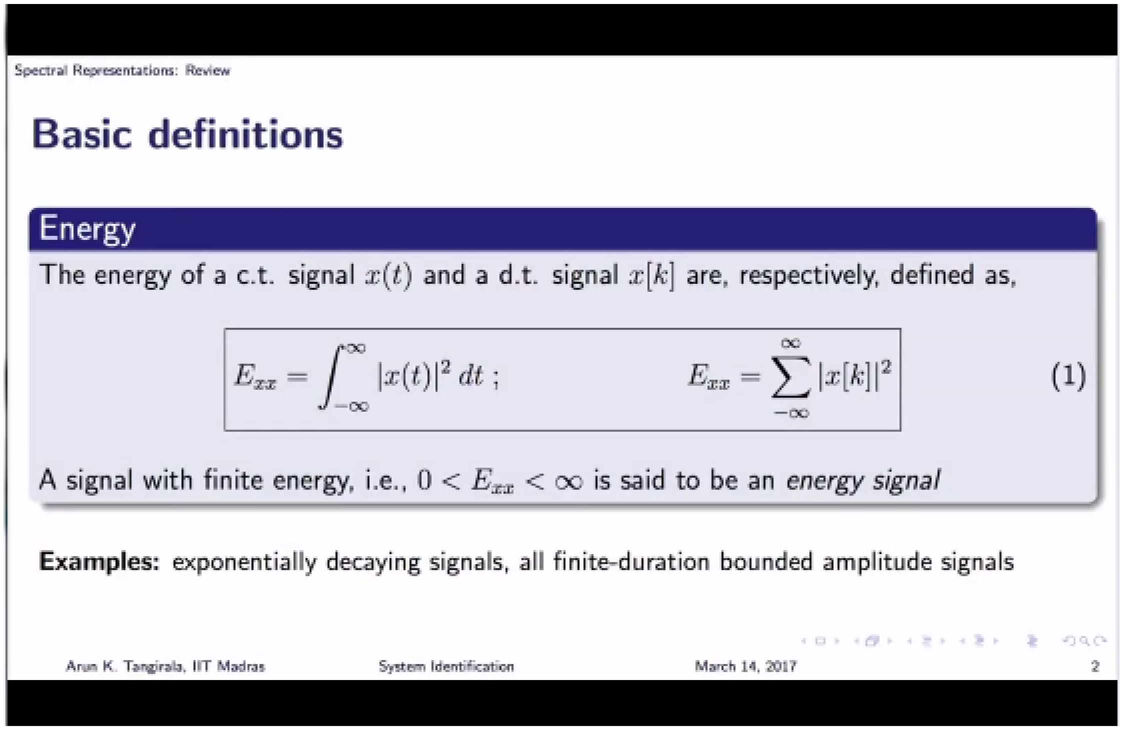

If they exist, then we say the signals are of finite energy. And, if you look at some of the examples exponentially, decaying signal, for example, is an energy signal. On the other hand, what about a sine wave? Is a sinusoid and energy signal by the definition that we have given? Yes or No? What do you think, Aruja? Okay. So sine wave is not an energy signal. Good. Suppose, I have a pulse of finite duration? Is it an energy signal, right? Because you know, the integral is fine. Okay, good. So you have an idea of what is an energy signal.

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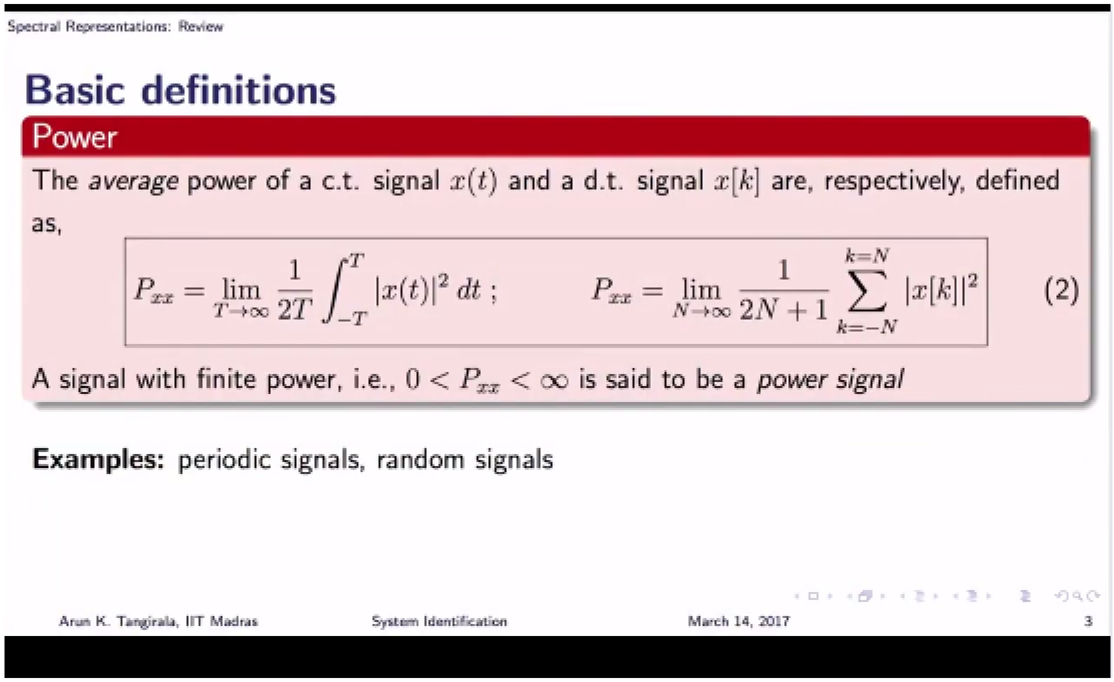

Now, let's look at the notion of power. When we think of power, we think of average power. In a certain, we don't define general power like that, we just define average power over an interval 2t. And again, for a continuous time and a discrete time signal, the power is defined as follows in a limiting sense. So we first define energy. So if you look at the definition here, what we are doing is we are computing energy over an interval 2t and then calculating energy per unit time, average power, because you're not calculating instantaneous power. You're computing energy one interval divided by 2t and then evaluating that in the limiting case. And likewise here for the discrete time signal as well. Now periodic signals will have finite power. Again, when signals have finite power, we say this, that signal is a power signal. Periodic signal is a classic example of a power signal. Why suppose you're looking at this integral here whether or the summation and let's say 2n plus one is a period. Then it'd be finite, right? Whenever I evaluate as like m goes to infinity or not, or 2n plus one, it will turn out to be a finite quantity. Random signals, not all class of random signals, but a large class of random signals are power signals. Now, when we talk of signals, one thing you should get used to is, particularly at this stage of your career, of your education, you should understand that whenever we talk of a signal, we thought we talk of a system as well. All right. So when I say energy of a signal, you should not just think of the signal, you should also think of the system that is generating the signal. Likewise, when we talk of a power of a signal, you shouldn't think of just a signal you should think of the system. The interpretation of energy or power of a signal is that it is the energy or the power expended by the system to produce that signal. You say, oh, that guy is so energetic. Yeh, he speaks all the time. So, if you were to record his audio signal, you say, "No, he has a lot of energy," but that energy is finite. He said that gay so powerful. Of course, that power is not the same as this power. But if you think of power, we are not talking of the power and the center and state and so on. That power is different. Power is the rate at which you're expanding the energy. If that is finite, then we say the signal is a power signal that is the rate at which the system is expanding is putting in that much power to produce that signal. So when we say random signal is a power signal, the random process is actually putting in that much power that is energy per unit time to generate the time signal. No signal exist without the system. Isn't that clear? It's obvious .They cannot exist a signal without a system that is generating it. It cannot just know be exist on its own. So random, when you say random signal is a power signal, we are saying random process is a power process it's expanding that much energy per unit time. All right. So now you'll see random signals fall into the class of power signals, but they're not periodic necessarily. Deterministic periodic signals are power signals. Now, the other thing to keep in mind is and in general, any signal can either be an energy signal in general. I mean, there are some weird examples, but if you have an energy signal, the signal can be energy or power signal. There are some special examples that you can give it an either energy not power will not get into that, but if you have a signal that is energy signal that means it has finite energy, then this integral will go to zero or the summation will go to zero. That is not summation, but the entire limit. One over 2n plus one.

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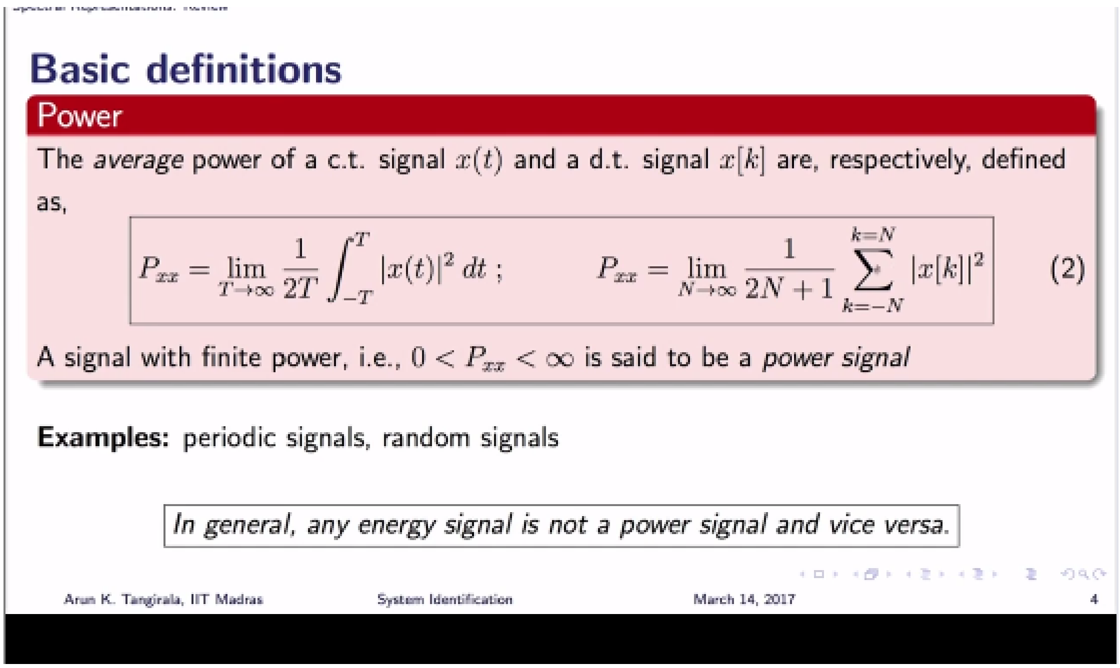

So imagine that the summation here in the limit as x goes to infinity, if this is an energy signal, the summation would be fine, but the denominator will run away. And as a result, the power will go to zero. So an energy signal cannot be a power signal because you're saying a signal is a power signal if it has finite power. Non-zero and bounded. Likewise a power signal cannot be an energy signal because power it is the rate at which you're expending energy. And if that is constant, that means at any time it has the power to generate the signal that means at any time the signal exists. And if a signal is existing forever, how can it be an energy signal? It's not possible, right? So a power signal cannot be an energy signal and vice versa. Just have to have this notion very clear in your mind. Okay, so associated with this energy and power we have these notions of energy and power densities.

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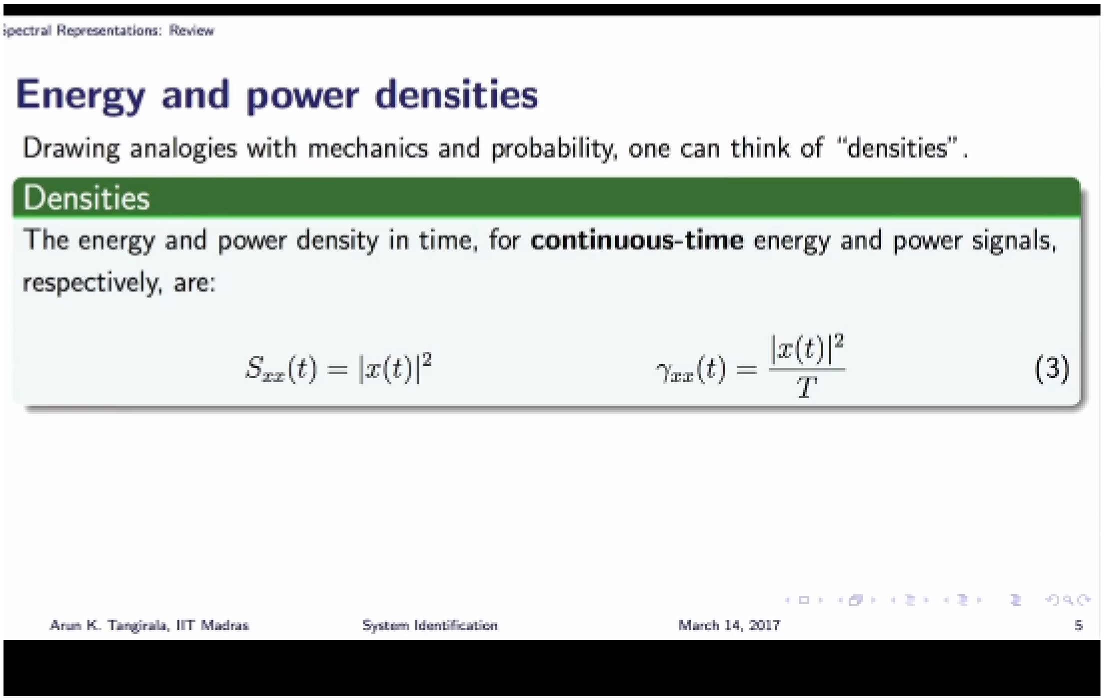

We have heard of densities before? Where? In probability theory, right? What was the role of the density there? When could we define density? This, you should be able to answer. For continuous and analytics. Good. So what was the interpretation of the density or use of density there? The area under the probability density gives me the probability. Correct. That X will take on values with their interval. Energy density is also similar to that. The only difference is you are not necessarily associating this density with a random phenomenon. It is a deterministic phenomenon. So I can talk of mass density. Likewise, I can speak of energy density. When can I think of energy density when the domain is continuous? So for now, we are talking of energy density in time. See, when I talk of mass densities, I can think of mass density along the length or on a surface or in a volume. Different dimensions I can think of. Likewise here, I can think of now, as of now, energy density in time, that is per time. It is almost like your instantaneous power, right? It has the units of power. How do you get this? Well, go back to this definition of energy and look at the integral for example. Don't look at the discrete time case. Why, why shouldn't we look at discrete time? Because the time is not continuous there.

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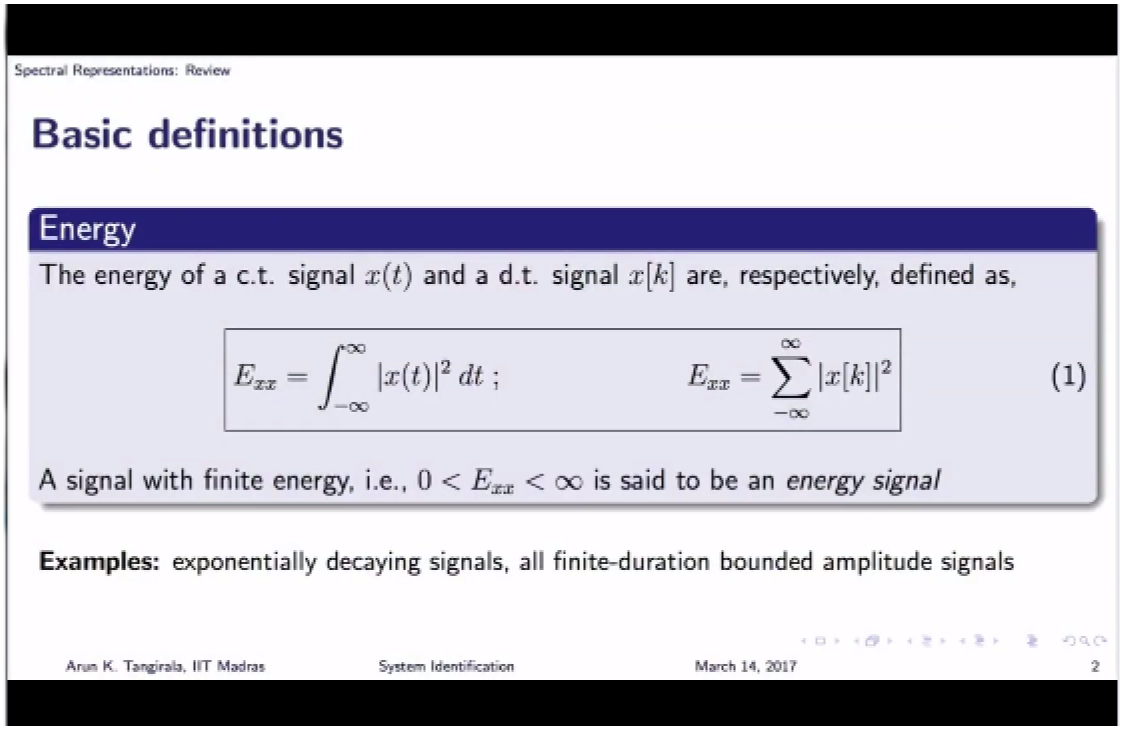

So, there is no notion of density there. So, we think of energy density is in time only for continuous time signals. For discrete time signals, there is an alternative and we'll talk about that. That will be the center of our discussion. So, if you look at this integral here, the area under mod of x(t) Square. Gives me the energy, in a certain interval, correct? So, I can attach your interpretation of density in time two mod of x of t square. Right? Magnitude square of x of t. And that's how the notion of energy density is one. Is it clear? Likewise, I can define power density. These are the units of energy per time or power per time. Now, what do we do for discrete time signal? So what are we going to do with these densities by the way? They're going to, I'm going to do the same thing I did for, with a probability densities. What did I do with probability densities, compute the probability. Anything else did I do? Did I use the probability density function in any other way? Not estimating. Sorry? Did we estimate everything as of now? No. What did we do? With the density function what else did we do? Did we have a party? What did we do, man?

[19:11 inaudible].

Sorry.

Compute that may joined [19:10 inaudible].

No, define the moments. So, we never say estimated, we defined the moments. Mean, variance and so on. What is the mean and variance tells me? Mean gave me the center of outcomes. Variance gave me the idea of spread, right? With this energy density is also, I can define a center of activity of the signal. Okay. We say that the meantime of the signal. Around what time did the signal really exist? Right. An average time. For how long roughly existed that is a measure of the spread, that is what we call as duration in time. What we call a standard deviation for random variables, we call us duration for these signals. All of this can be defined using this energy or power densities. All right. But we don't get into that. If you want to know more you can go and listen to my online course on introduction to time frequency analysis and so on. So there, we work with these moments of density functions extensively. So, what do we do for discrete time signals, right?

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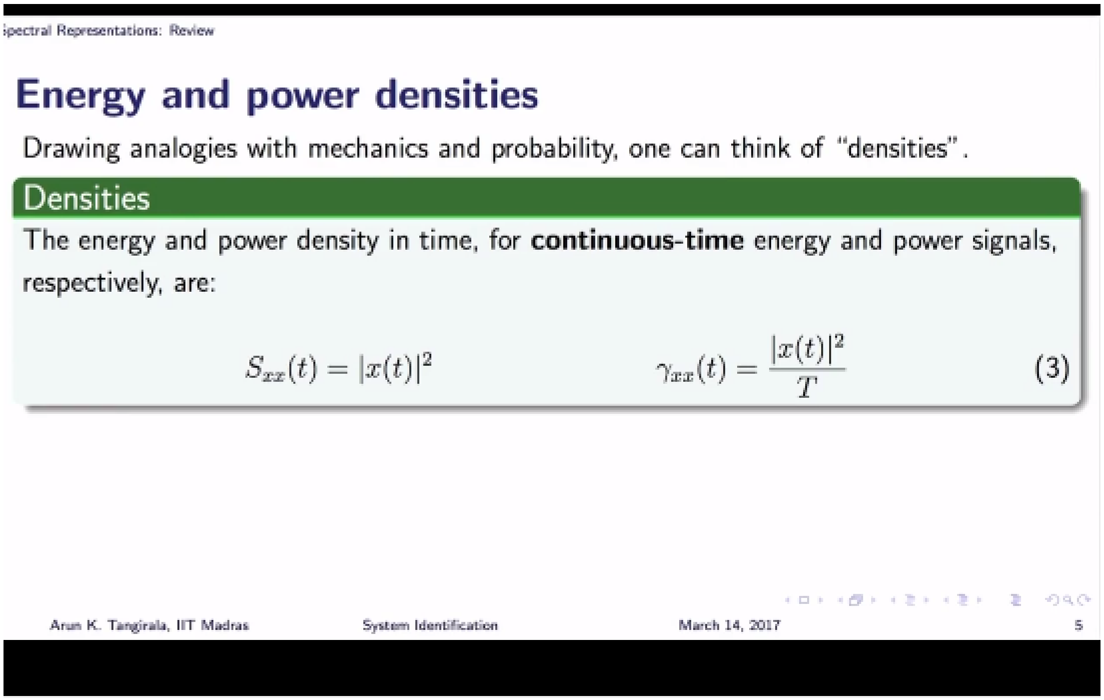

We cannot. We cannot think of this energy or power densities in time. However, the good news is that, I can think of this energy density or power density for discrete time signals in some other domain. That domain should be continuous and we call that as a transform domain. And that transform domain is nothing but the frequency domain. You can look at some of the domain also, it need not be frequency, it could be wave length. For example, wavelength domain. But we will look at Fourier domain because we know Fourier domain or frequency domain analysis gives us a lot of insights into how the systems behave. Ultimately, this is system identification course, there's not signal identification course, right? But without knowledge of the signal analysis we cannot do a system analysis. And that's why, every now and then we go back to signal analysis and then come back to systems. So that, when we move to this frequency domain, we will encounter this notion of energy spectral density and power spectral density and so on. Alright. So let's move on now. It turns out that, yes, right.

What will the capital T over here?

Capital T is the same T, oh, sorry. Here, this is the T.

So, if T tends [21:48 inaudible]. Infinitely or [21:49 inaudible].

No, T doesn't tend to infinite. It is the power density over the unit time that powers the, yes, I'm sorry, there should be a limit there, t going to infinity. But then, yeah, that's perfect. Correct. So you observe here? No, you don't let the table to infinity. I take back that. You look at this here. Let's say, this is not 2t but some T prime. T prime is 2t. Okay? The symbols, unfortunately, notation is confusing. You absorb into the integral. So let's define a T prime. It's a good point that you made. I should have noticed that. So let's define T prime as 2t, so that I can write the power as, you know, I can even shift from minus T to T to zero to T prime that doesn't matter. X of t whole square by prime, of course, then I will [22:47 inaudible] keep it here DT, that's fine. Now the area under this, so what is this interpret? [22:55 inaudible] correct? Of the signal. The area under this quantity is going to give me the power. All right? Normally we don't talk of power densities. Because it depends on this T. And that power density keeps changing as I keep changing T. So the, the power density itself now is depend on the interval that you're analyzing. Therefore, we don't refer to power densities in time. All right? But that's a very good point whereas the energy density is independent of that. So, gamma of T is rarely used in practice. S of T is used widely.

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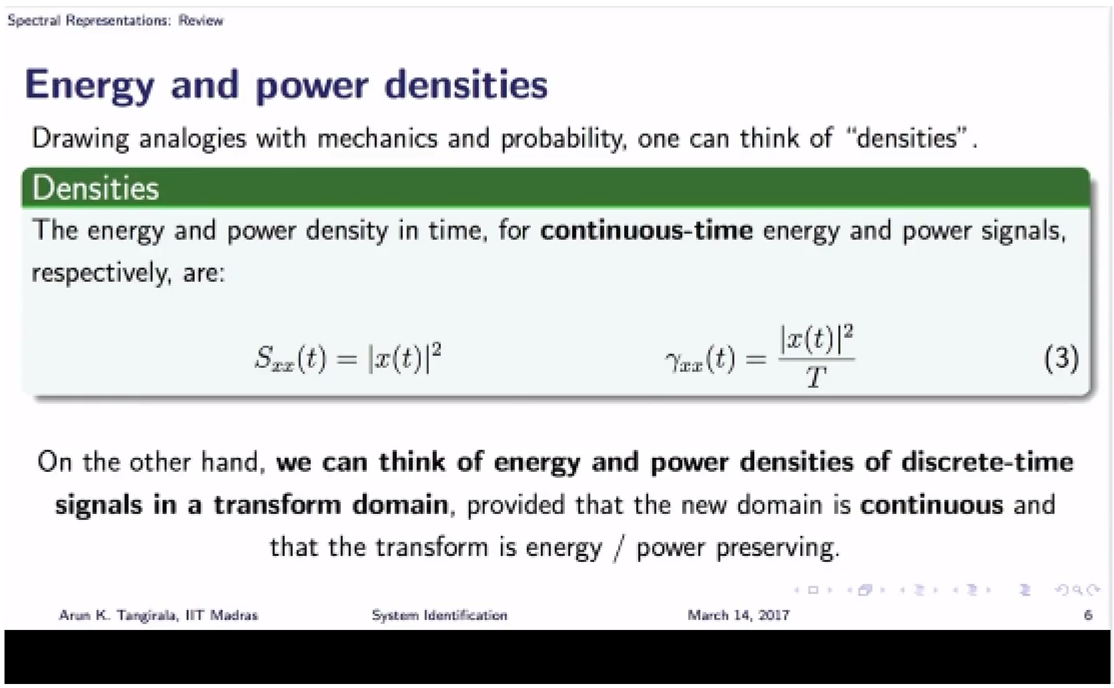

But you should keep telling yourself we are looking at deterministic signals.

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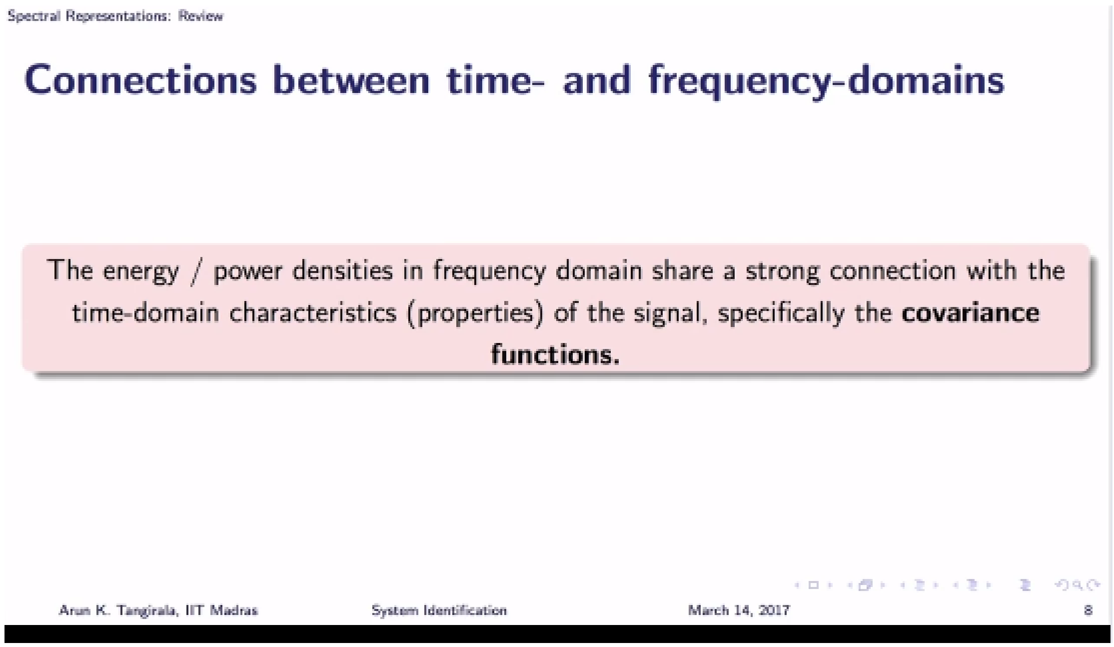

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Now, the interesting part is that we just know we said for discrete time signals we can and we will see shortly we can define densities in frequency domain. We couldn't define density in time domain. The interesting part is that these densities in frequency domain have a strong connection with so-called covariance functions. We have already learned what our covariance functions. But we learned for random signals. But we don't know what how covariance functions are defined for deterministic signals. But it turns out the expressions are similar, their roles are similar. What was the role of covariance, auto covariance function in the random world, random signal world, it told us what is the dependency what is a correlation structure, what kind of a linear model can be built and so on. The covariance functions in the deterministic world also play the same role except that you don't have expectations. You will now work with signals in time, purely. Whereas with random signals, we looked at the on ensemble. So if you look at, now the cross covariance function definition, there are two definitions, one for periodic signals, and the other for a periodic signals, finite energy, a periodic signal. So let's look at the periodic one. That's given an equation for, the cross covariance, for zero mean periodic deterministic signal, sorry between two zero mean periodic deterministic signals is given by this expression.

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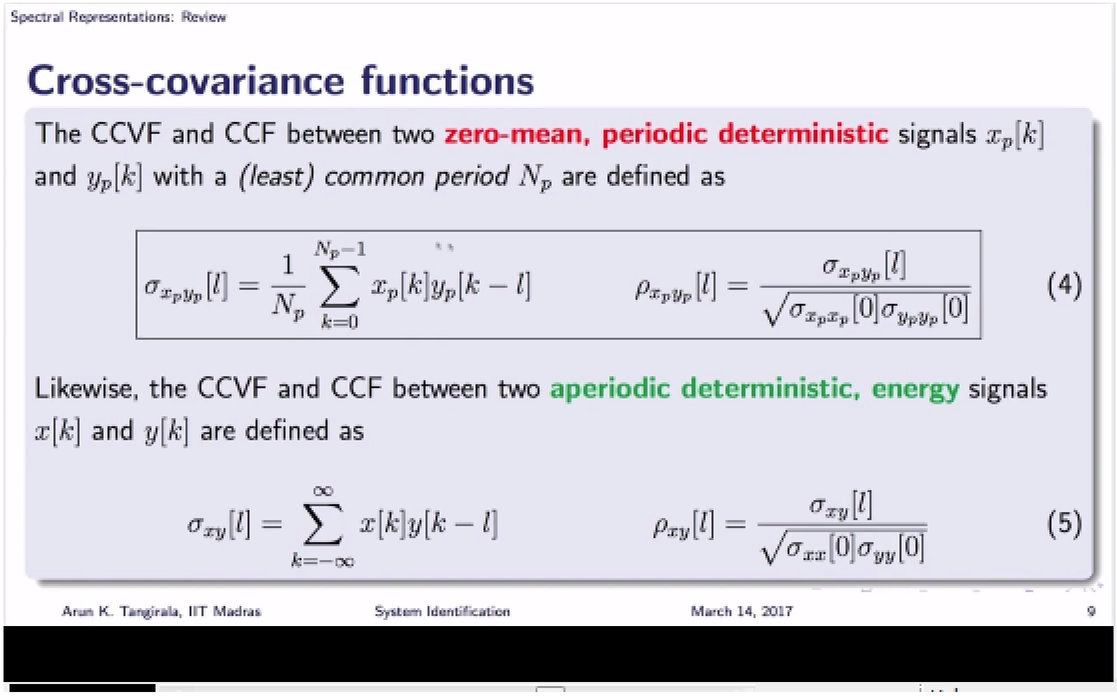

Doesn't it look similar to what you saw for cross covariance between two random signals but there are some differences. The first differences is, I'm not averaging across an ensemble instead, I'm averaging in in-time. Because now a single realization, there's no notion of realization, whatever signal I have is the truth of x and y. So I'm taking this pair of products, I should have mean-centered, but I didn't mean center it. Why? Because have assumed to be zero mean. Okay? So I'm just taking this pairs and then actually walking averaging in time. But since this is a periodic signal, they both are periodic signals with the common period we only need to evaluate it over that common period and then take the average that is what we are doing, all right. And as usual, instead of working with covariance function we work with correlation functions. So this is the cross covariance and correlation for periodic signals. When it comes to periodic signals, I'll have to consider the entire time of existence. But not all a periodic signals qualify or, you know, can you can define a cross covariance function for all a periodic signals. They should have finite energy. That's important. Because this summation has to converge. For periodic signals, there is no such qualifier. So qualification that aperiodic signal should have is, like we say B-Tech, PhD and so on, it should have energy. Next to it, it should have energy. If that is satisfied, then the cross coherence function is defined now, there is no averaging in time, and so on. Because if you average in time, it will go to zero. Here, we are not worried about averaging. They're just saying over the entire existence, compute this product and that is cross covariance and then cross correlation is as usual, the normalized version. You should quickly recognize that when you evaluate at lag zero, and also set x equals y, what do you get in the in the second case here, let's say x equals y and you evaluate at lag zero? What do you get?

[27:54 inaudible].

That's what. What we call as variance for random signals, here we are, the variance for the is the auto covariance at lag zero. There, we use auto covariance, right? Of course, auto covariance is a straightforward specialization of this, cross covariance function. All I have to do is replace y with by x. Now that is what leads us to the auto covariance function.

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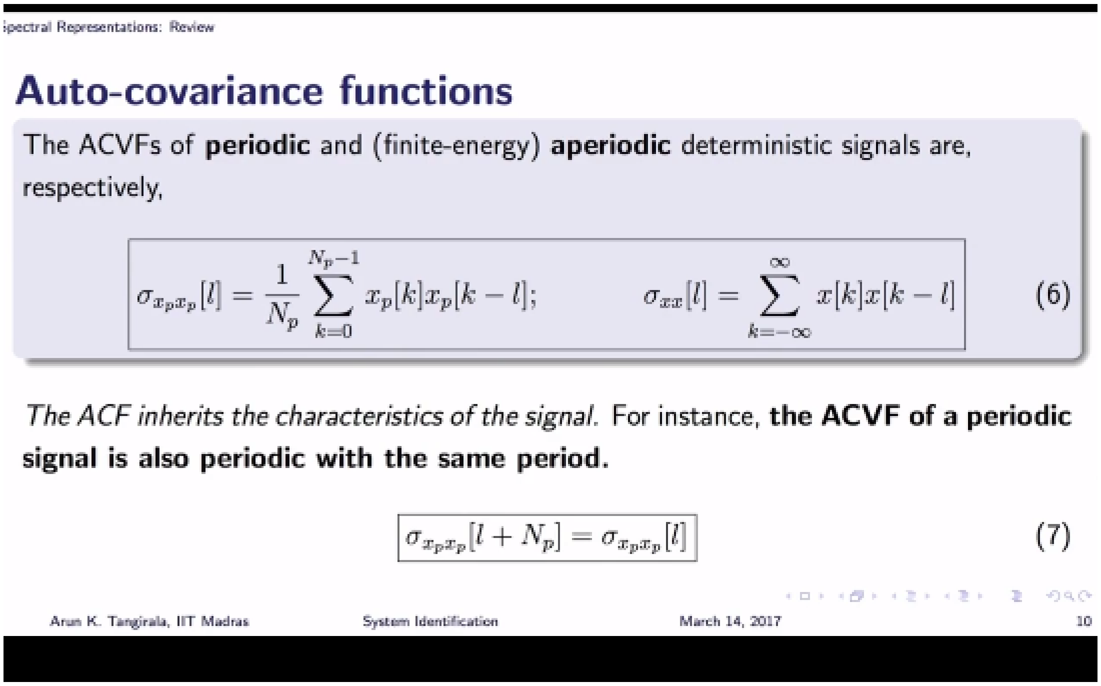

Now, an important feature of this auto covariance function for at least deterministic signals, very interesting and this is used widely in signal analysis is that, it inherit the properties of the signal, features of the signal. For example, if the signal is periodic, you can hope for the deterministic signals only. So now you should be well versed with the notion of the definition of covariance functions for both deterministic and random signals. And you have to keep asking, why on earth am I being tormented with this? That's because we're dealing with composite systems. We are dealing with a deterministic, plus stochastic. I need to understand the definitions of covariance of both parts. Then, tomorrow when we look at unified system we will figure out how to unify these definitions. All right. Already, one unification we'll learn today. So if you look at the deterministic signal, the auto covariance is periodic with the same period as the signal.

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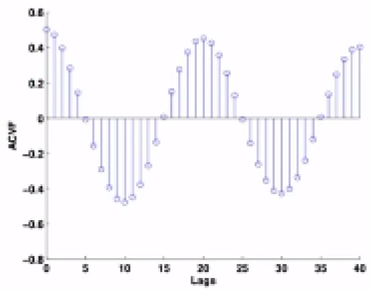
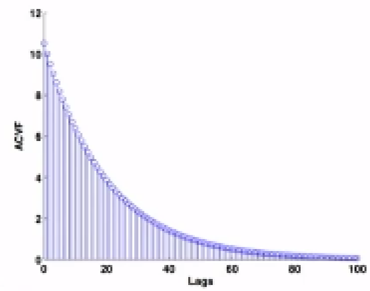
Examples

$$x_p[k] = \cos(2\pi f k)$$
$$\implies \sigma_{x_p x_p}[l] = \frac{1}{2}\cos(2\pi f l)$$

$$x[k] = \begin{cases} e^{\alpha k}, & k \geq 0 \quad, \alpha < 0 \\ 0, & k < 0 \end{cases}$$
$$\implies \sigma_{xx}[l] = \frac{e^{\alpha l}}{1 - e^{2\alpha}}$$





So, let me show you examples here.

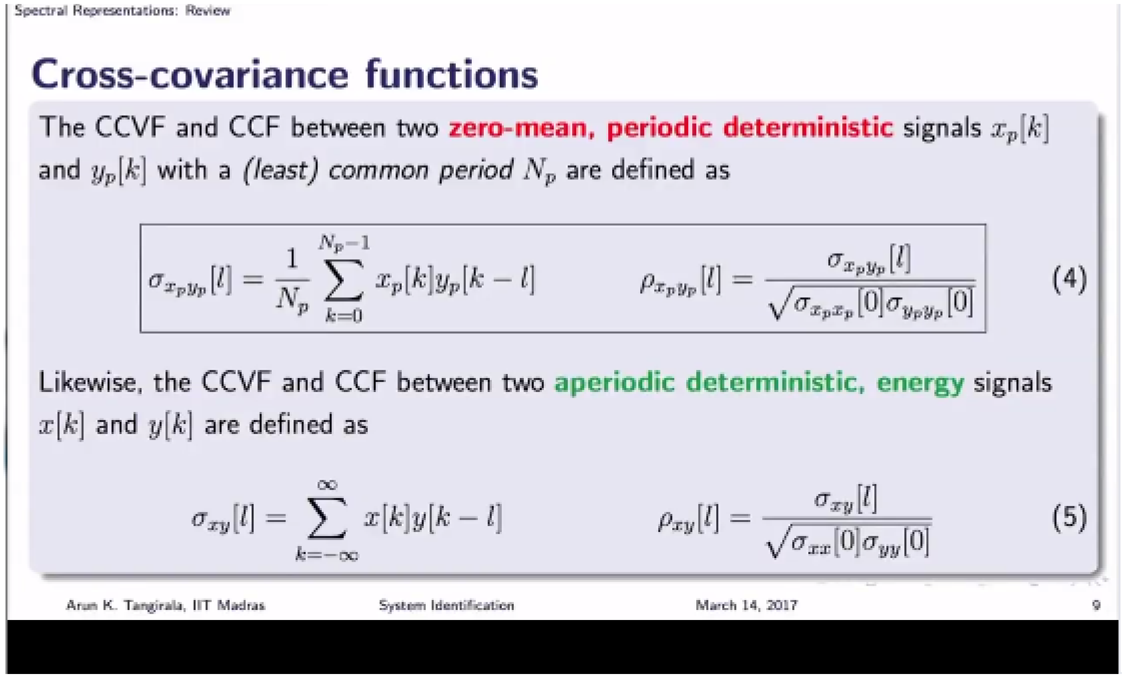
Auto-covariance functions

The ACVFs of **periodic** and (finite-energy) **aperiodic** deterministic signals are, respectively,

$$\sigma_{x_p x_p}[l] = \frac{1}{N_p}\sum_{k=0}^{N_p-1} x_p[k]x_p[k-l]; \qquad \sigma_{xx}[l] = \sum_{k=-\infty}^{\infty} x[k]x[k-l] \tag{6}$$

The ACF inherits the characteristics of the signal. For instance, **the ACVF of a periodic signal is also periodic with the same period.**

$$\sigma_{x_p x_p}[l + N_p] = \sigma_{x_p x_p}[l] \tag{7}$$



On the left you have a cosine and I'm just giving you, you can plug in this cosine into the definition here of the auto covariance function in equation six and show that the auto covariance is half cosine two pi FL.

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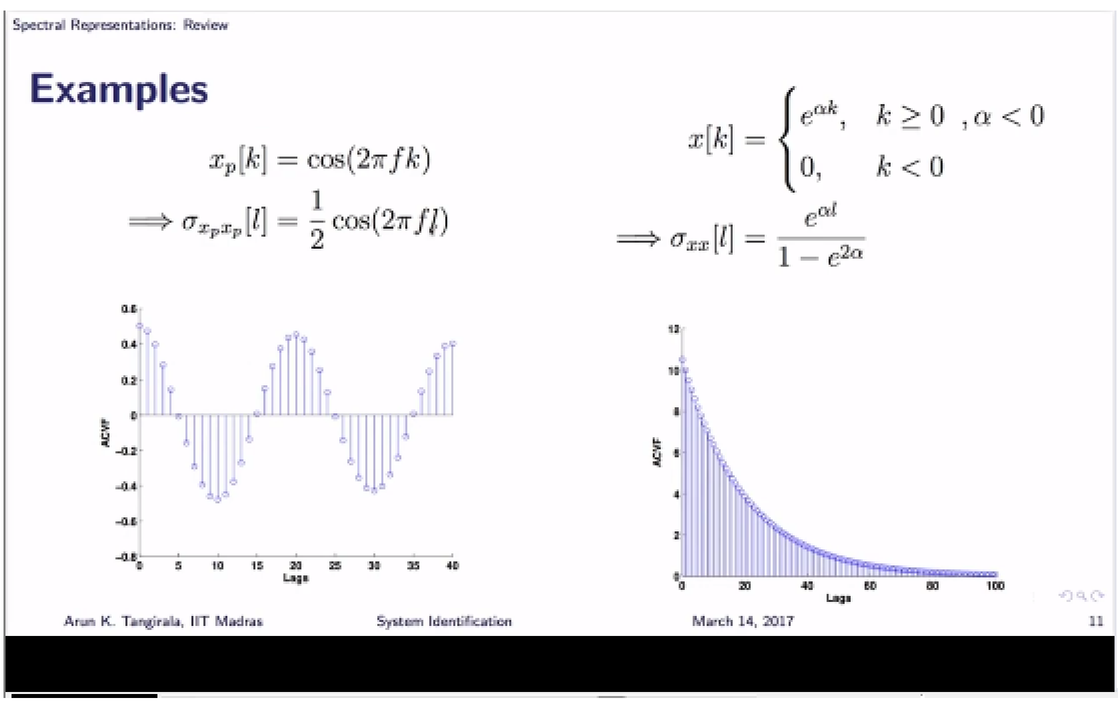

You see that it has the same period as a signal and I'm just showing it auto covariance here. I'm not showing you the signal, but this example is taken from the text you can look up the text for more details. On the right hand side I have an exponentially decaying signal. So it qualifies for, that is, it has this qualification for me to apply the notion of auto covariance. Although I've not defined for auto covariance for energy signals. This auto covariance that I've given here is for periodic signals. How do you get the Auto covariance of energy signals? Just use the same definition here, that's all. Replace y with x. That's all. When you do that you get the auto covariance theoretically as this expression and when you plot, it looks like this. It has exactly the same the decay rate. [30:46 inaudible]. Okay? As the original signal. This is used widely in many, many applications. So what this means is that if the signal is a sine wave, I'll get an auto covariance also with sine wave with the same period. There are some advantages to doing this, which we'll talk about hopefully later or maybe you can refer to the lectures on time series where I talk about the advantages of this. The more important result for us is the relations between this auto co variances or cross co variances and this energy then densities in a transform domain.

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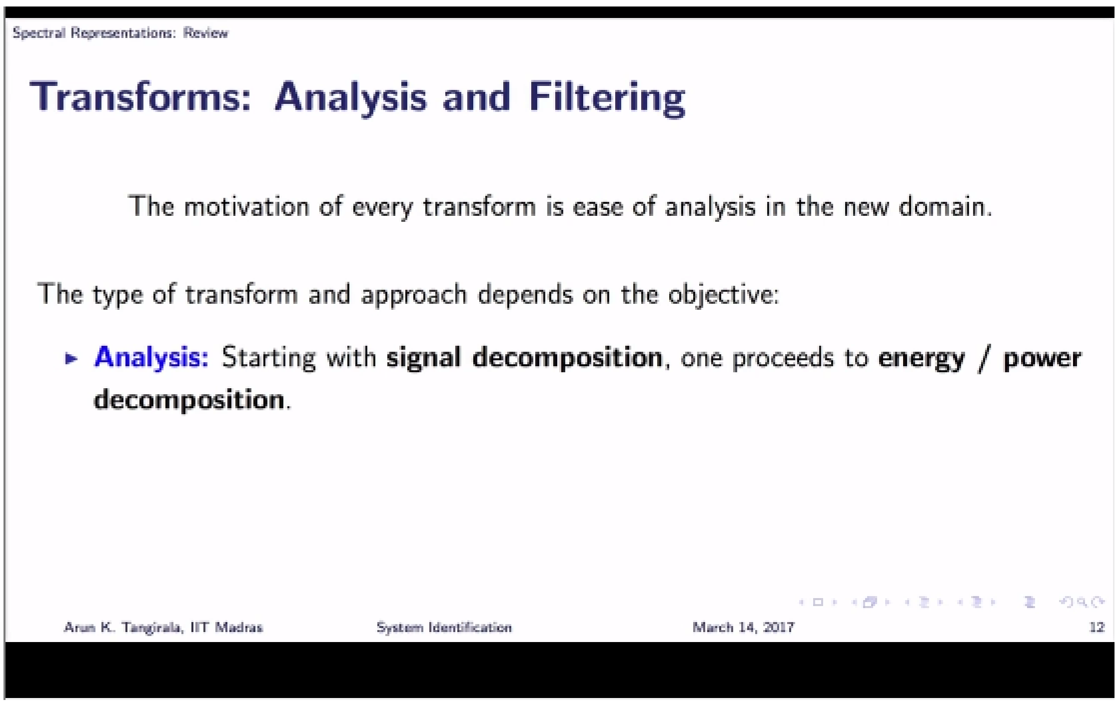

So now we'll move on to the transform domain. So now that we are moving into transform domain, just a few words on why one should look at transforms have already spoken about this long ago. What transform world can get you? It can give you a convenient way of analyze, it can give you convenience of analysis in the new domain certain features that are not so obvious in the raw domain become highlighted very quickly. And the classic example that you will always encounter is this periodic signals embedded in noise. And, when you look at the power spectrum of such a measurement, then the periodicity gets highlighted very prominently in the power spectrum as a peek at a particular frequency. So what you couldn't see in the raw domain with the naked eye, you will be able to detect that image spectral domain, that, yes, there is a periodic signal embedded in noise. That's because of the way the time domain features map to frequency domain. So analysis becomes easy and in this analysis, typically, you should always watch out for this. It's not necessarily that all texts and articles will point this out clearly, there are some good texts that will find this out clearly. And that point is that in most of the cases that transforms begin with, begin with the decomposition of the signal. Analysis of the signal and quickly move on decomposition of the energy or power. What we mean by decomposition is, every transform involves some analyzing function. Like in Fourier transform, you have sinusoids. So what you're doing by transforming the signal is, you're breaking up the signal into these atoms called sinusoids. So you think, in physics, in science, we say the entire world is made up of atoms that has proton, electron and neutron. In signal world, we think of the signal being made up of some basic elements called atoms and it's my imagination. For your imagine, that the signal is made up of sines and cosines. So what is the advantage? There are advantages, as we all know, but what you're essentially doing by doing a Fourier transform or constructing your career series expansion is that you are breaking up, decomposing the signal into its constituent atoms. Then you say, how many of these items are there, right? If I give you a compound in chemistry, you'll say there is a carbon atom, there is hydrogen atom, there's oxygen item and so on, if you take an organic compound, right? So you have decomposed and then by looking at number of carbon atoms, number of hydrogen atoms, number of oxygen atoms you can comment on the nature of the compound. Likewise here, when I look at the signal and I break it up into sines and cosines of different frequencies which depending on what I get, after breaking it up, I will be able to say yeah, this frequency is present, the rest of the frequencies are not present or maybe these frequencies are dominant and so on. That is what we mean by signal the competition. But rather than looking at signal decomposition, it is advisable to look at and that is what is used in all analysis, to look at energy or power decomposition when you want to analyze the signal. What we mean by energy or power decomposition, the story is the same. In signal decomposition, we ask the question, "How is each frequency contributing to the signal?" "To the amplitude of the signal?" In energy or power decomposition, we'll be asking the question, "How is how much is each frequency contributing to the overall energy or power of the signal?" The same question is the same, right? Depending on the contributions, we will say, yeah, this is you know, a low pass predominantly these frequencies are present and so on. The other application of transform domain analysis is filtering. As we know, I've explained this long ago, you again break up the signal into their constitute atoms and say, yeah, you know, these items were not supposed to be there, they have come because of noise or some undesirable features, I only want to focus on these sinusoids, these frequencies, so you retain only those and reconstruct the signal back that is what is filtering. And this concept applies to every transform that is used in filtering, not just to Fourier. In Fourier, the atoms are characterized by frequencies. In wavelength analysis, the atoms are characterized right scales. And the time location center we say.

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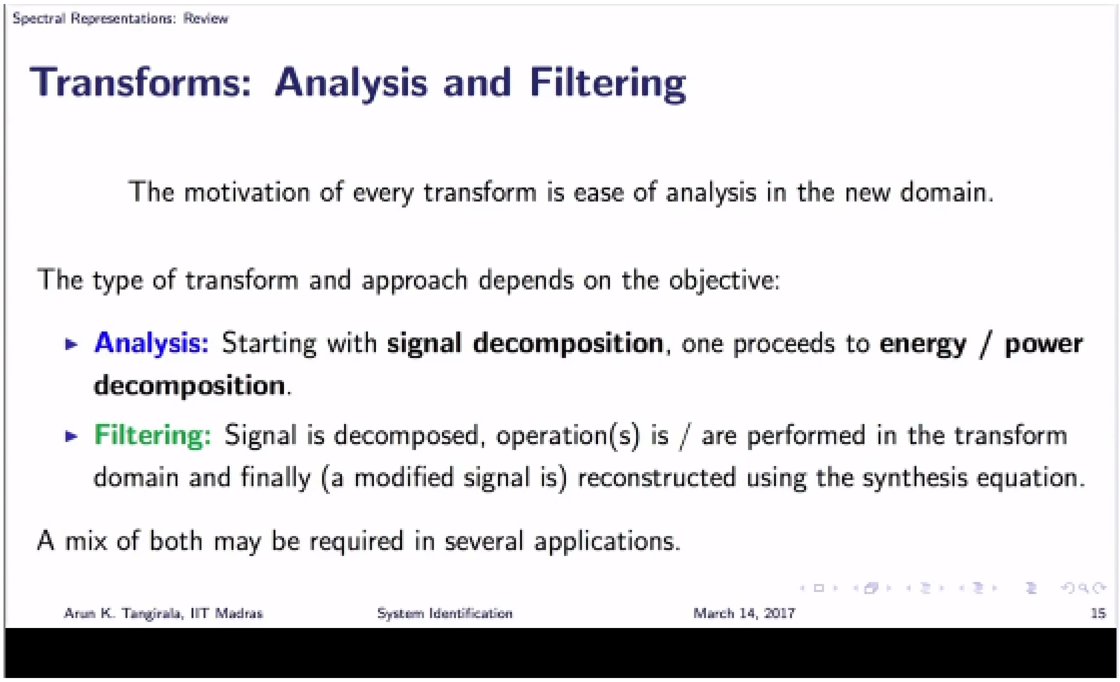

In some other transform, characterizing parameter is something else. Okay. So that is the point that you want to keep in mind that transforms are using analysis and filtering. Sometimes a mix of both is required in several applications.

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