sNPTEL

NPTEL ONLINE COURSE

CH5230: SYSTEM IDENTIFICATION

SAMPLED: DATA SYSTEMS 6

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Now we turn our attention to sampling which is a very important facet of identification of discrete time models from sample data, and it's a, obviously therefore the most critical operation, why? Because if I sample incorrectly, then I'll lose information, it's very simple, if I sample very slowly, if I sample irregularly and so on, all this can lead to loss of information, and therefore I have to be quite careful. Formally sampling is nothing but the act of obtaining the values of a continuous time signal.

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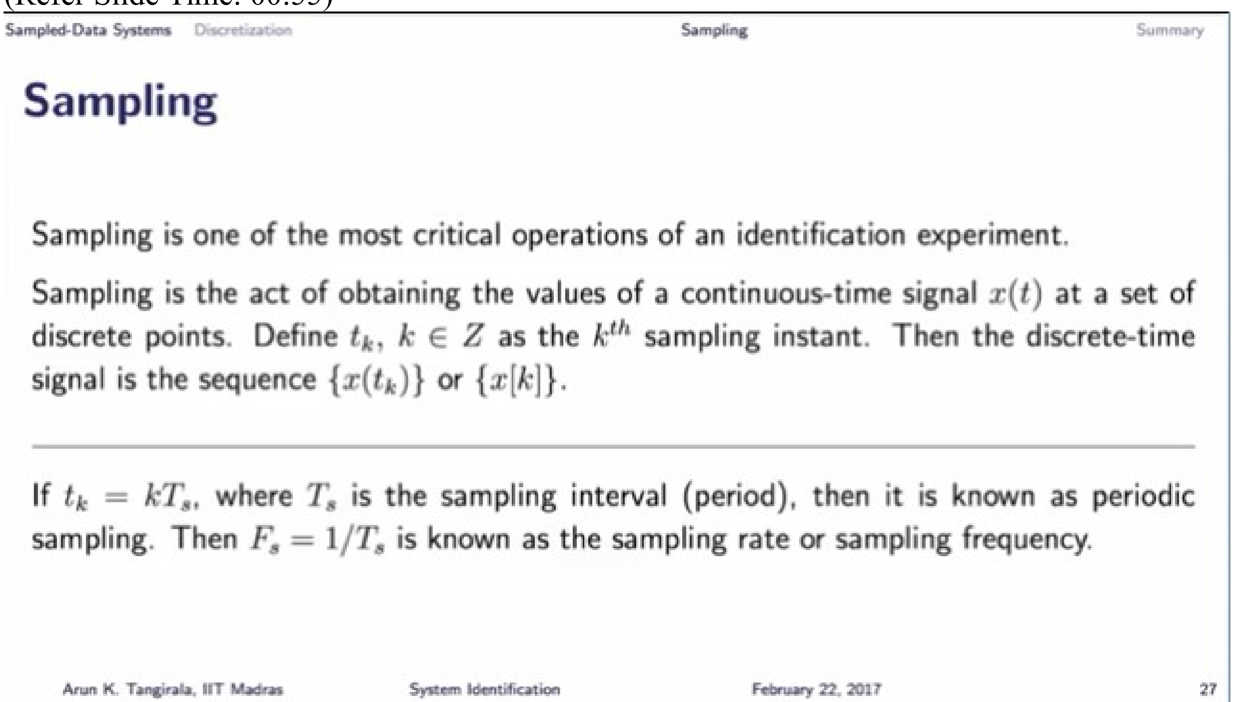

Now there are two possibilities I can sample regularly, I can sample irregularly, generally the math associated with regular or uniform sampling is a lot more easy to handle then the irregular sampling, but having said that there is a, the entire body of literature on irregular sampling and sometimes it may not be in your hands at all, okay, sometimes in many industrial applications sampling is event based, what do you understand by event based sampling? So suppose I'm a teacher right, and I just want to relax and I'm going to bring a novel and keep reading, the class is quiet, typical is use to happen in school days, I have some work I'll get to the classroom, children all of you sit quiet, okay, everybody sits quietly, I don't have to keep observing every minute, suddenly I hear a noise then I'll actually sample the class. If I don't hear anything I don't sample at all.

So if an event occurs in the process I will look up and see what has happened or in my mind if I can say, if the noise levels, obviously when I say quiet children are not going to be quiet, some conversations will be going on either about the teacher or about the subject or something else, these guys it's all of course quiet because they are busy texting using WhatsApp, but some discussion happens and then in my mind I can say, I can tell myself if the noise levels increase beyond the certain threshold, then I'm going to sample. All of these are actually event based sampling where there is no guarantee that you will obtain samples regularly.

There are other situations where you can run into irregular sampling when the sampling is manual, when you're going to collect samples and you have to take it to a lab to get the readings, there is no online sensor, again you will be let to your regular sample, okay.

Now as much as you may think that irregular sampling is not good for health, that means for our mathematics health here, today there is a complete revolution when it comes to sampling for about 60 years, 70 years people have been thinking that uniform sampling is one of the best ways to handle things and so on, but today there is a complete you know revolution on that matter where you have this theory of so called compressive sensing where the samples are not obtained uniformly at all necessarily, I can sample randomly and still recover the signal, of course there is a limitation, it's not that I can sample now and then sample few hours later and so on, but what that theory says is you don't have to sample regularly in fact sampling randomly is much more efficient when it comes to signal compression and so on, so they are even questions being raised whether when I'm looking as a human being when you're looking around with your eyes open, you're actually sampling the space, right, you're doing a spatial sampling, we don't know whether our eyes are doing or uniform spatial sampling or a random sampling.

And then there is theory that's available which says that from this random samples you can reconstruct the original image or the signal with an arbitrary degree of accuracy, so that is going to bring about a revolution that you will see perhaps in about 10, 15 years because already some of these things are being embedded into the technology, in this course we will restrict ourselves to the traditional uniform sampling, where we are going to sample the continuous time signal on a uniform grade, right, we will, because this theory is very, very well understood and most of the data acquisition systems rely on this concept as of now, when things change at the technological level we'll introduce changes in the curriculum and the syllabus, okay

As the convention goes we denote the sampling interval by TS,
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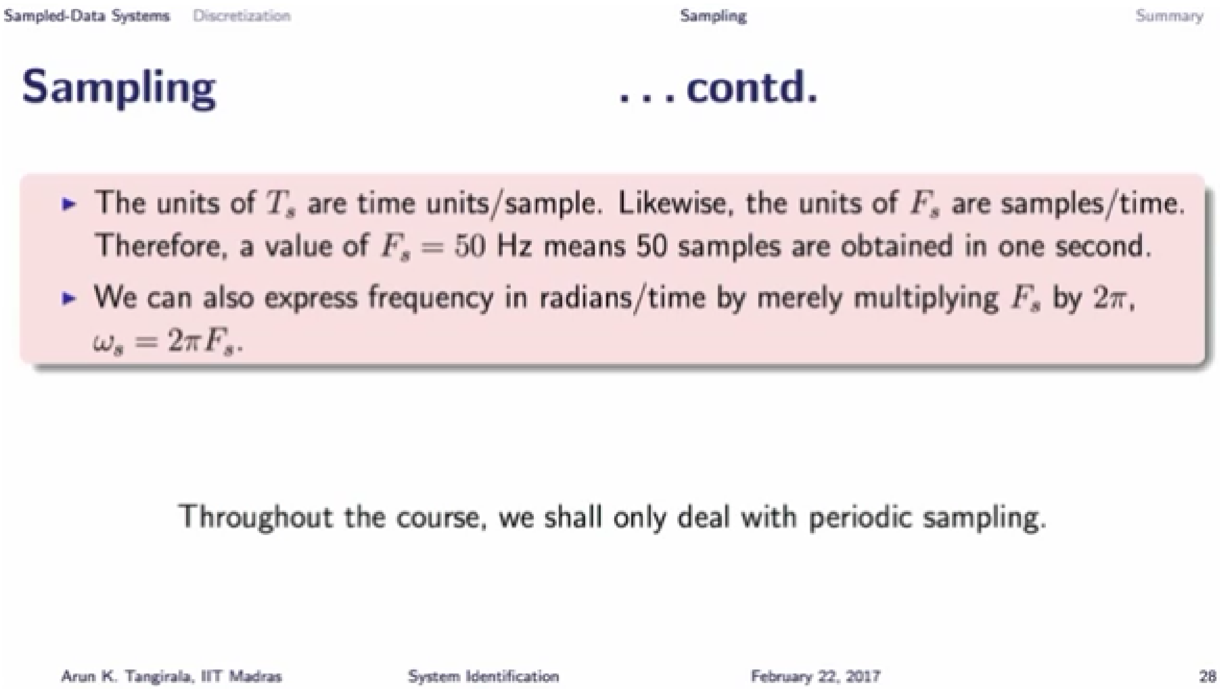

and the sampling instant by K and you should remember that the units of sampling interval generally we say time units, but strictly speaking the units of sampling interval are time units per sample, so if I say sampling interval is 2 seconds, it’s implicitly understood it is 2 seconds per sample, we must have never noticed that aspect but that is the hidden unit that is there in the sampling interval, so when we wrote TS is 1 second, it’s actually 1 second per sample, okay,

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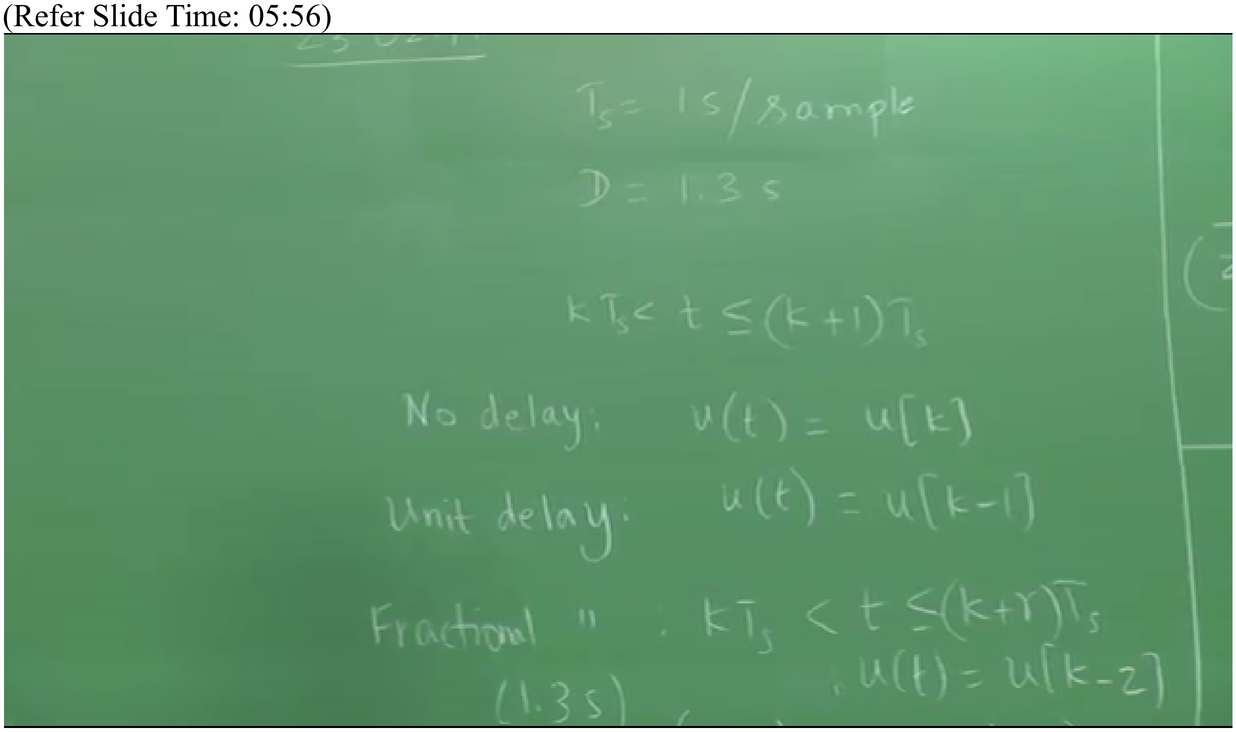

associated with the sampling interval is a sampling frequency which has the units of now samples per unit time which normally we express in hertz, but this hertz is a very generic thing, you should have to be careful when I use hertz for sampling frequency you should read hertz as samples per unit time, whatever units of time that's being used.

When I'm referring to continuous time signal and I say frequency is 40 hertz there the hertz should be read in a different way that is 40 cycles per unit time, hertz is a very generic frequency that is used for expressing frequencies, alright, so called cyclic frequencies. FS is your sampling frequency in samples per unit time and typically we use FS only, we don't use 2 pi times FS, you can also express frequency in radiance per time and so on, but we'll not do that.

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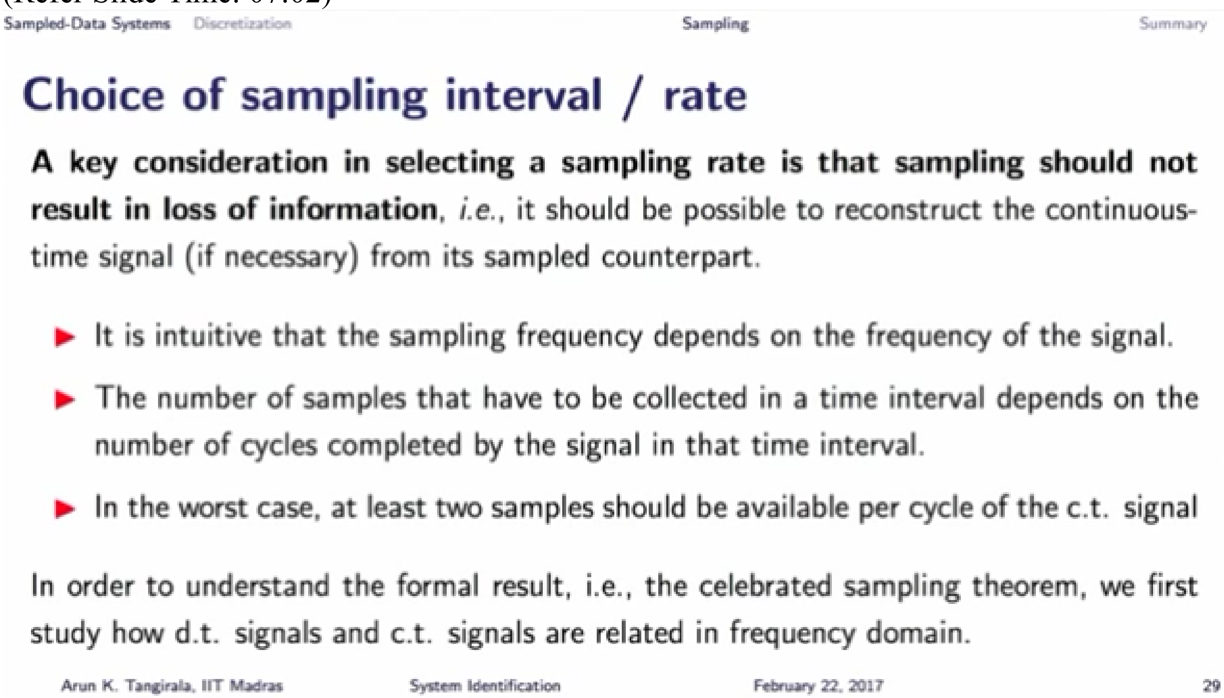

Now the big question that we have ahead of us is what is the appropriate sampling rate? Right, it's a big question, and the key consideration in answering that question is sampling should not result in loss of information, now that's a weird thing if you read at I mean on the face of it, obviously when I sample and I'm dealing with sample data, I have lost some information, which information I have lost? Between the samples I don't have information.

So what do you mean by sampling should not result in loss of information? I've already lost information, so the statement may not make sense on the face of a, however what the sentence actually means is if you were to recover the continuous time signal from your sample signal using whatever method that you have, you should be able to do it perfectly, and I'm vigorously and without any loss of information, we don't have a mathematical formula yet for doing that, we have been only talking of the forward thing, going from continuous time to discrete time.

And also I should tell you we don't do that, generally we don't try to recover the continuous time signal from the discrete time signal, but the theory for sampling demands that if you have to do it you should be able to do it perfectly, so that then you don't have this so called loss of information.

Now one can think of this in an intuitive fashion as well and arrive at this guideline here which is that if I look at the sine wave for example, not all continuous time signals be sine waves, but if I take a sine wave and I have to ask what is the sampling rate, appropriate sampling rate for a sine wave of a given frequency, and intuition says that I should at least obtain two samples per cycle of the continuous time sine wave right, we don't know if that is mathematically sufficient to recover the full sine, at least the cycle of the sine wave but somewhere our intuition says at least I need two points to interpolate, and it turns out that this intuition is not that bad, it's correct, okay, so this the question on how far should I sample is answered by the celebrated sampling theorem due to, at least attributed to 3 people Shannon, Whittaker and Nyquist.

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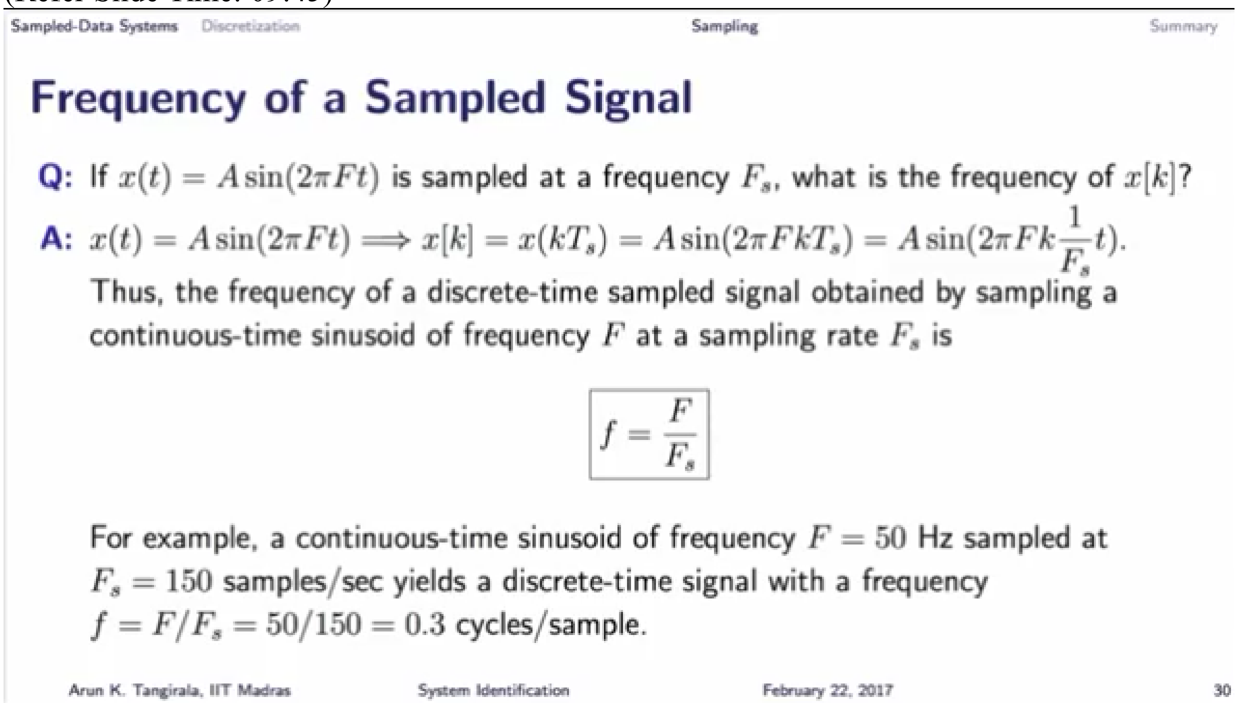

And in order to understand the sampling theorem it's very important to quickly understand what happens when you sample a continuous time signal and the difference between the frequency of a continuous time and a discrete time signal, so let's work through this example quickly, X(t) it's a continuous time sine wave, some amplitude A is being sampled at a frequency FS, and the question is what is a frequency of the discrete time sine wave? Answer is pretty straight forward, all you have to do is see you have to evaluate the continuous time signal at the sampling instance, so if X(t) is this then X(k) is nothing but value of X(t) at KTS, so you have here A sine 2 pi FKTS which I can rewrite as in fact there is a mistake in the slide, I'll correct that, so it should be X(k) A sine 2 pi F/FSK, there is no T, that's a mistake,

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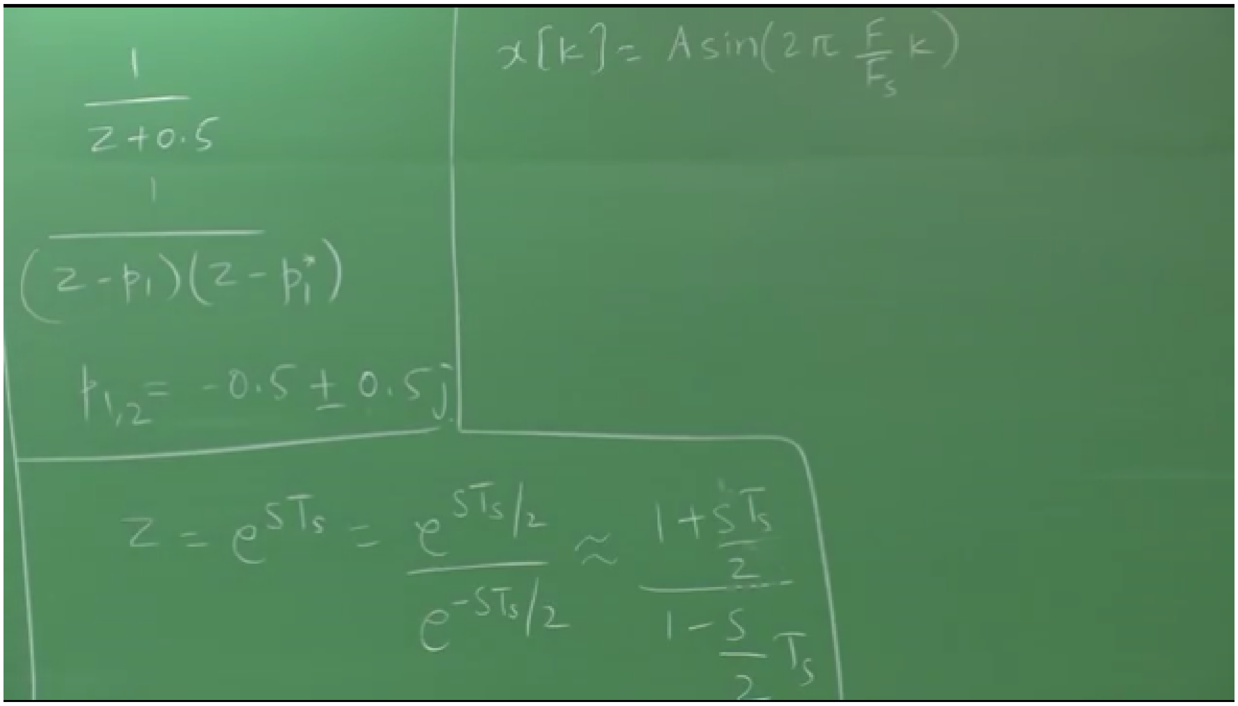

so that I can write this as A sine 2 pi FK,

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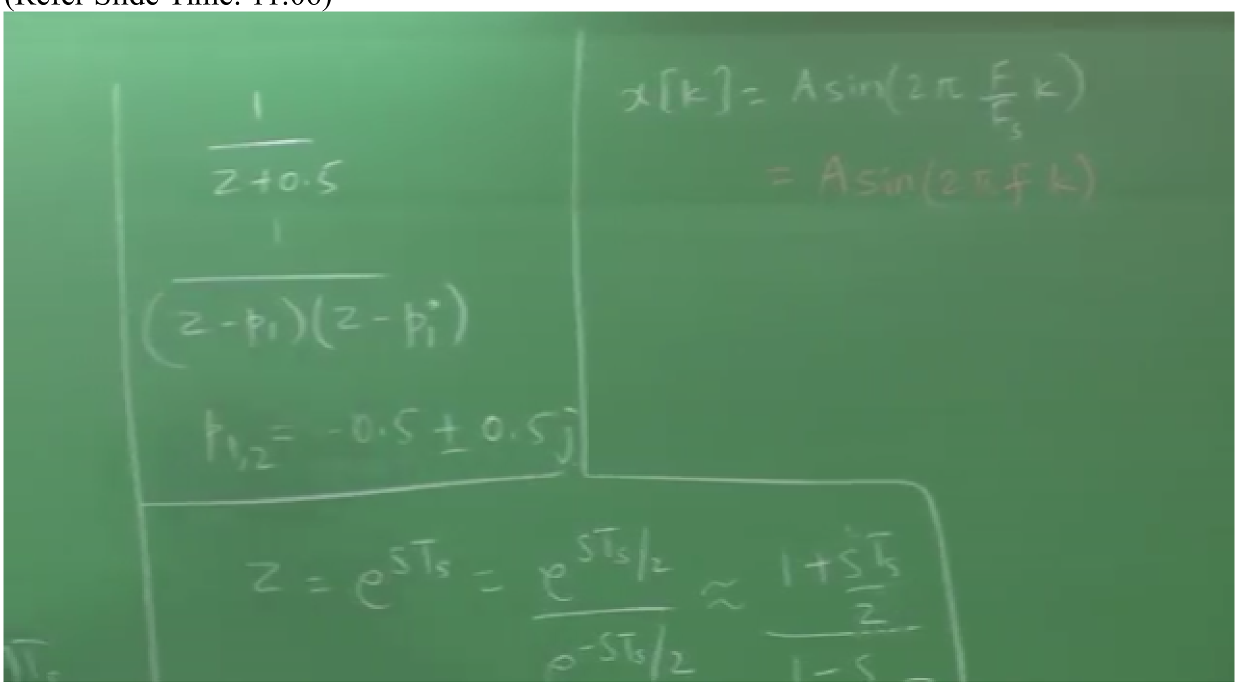

notice that we have used upper case F for the frequency of the continuous time signal, and lower case f for the discrete time signal, and this is the convention that we'll follow throughout the course. Likewise if you are looking at cyclic frequency, we'll use the upper case omega for continuous time and lower case omega for discrete time.

So now we have a relation between the frequency of the discrete time sine wave and the frequency of the continuous time sine wave, again I should tell you here not all discrete time sine waves are a result of sampling a continuous time sine wave, we have set this for systems also, right, same story, but what we are studying here is sampling, therefore we are looking at a discrete time sine wave that has come about as a result of sampling, continuous time sine wave, so here I have F = big F over FS, that's the connection.

For example as I said if the, as I say here,
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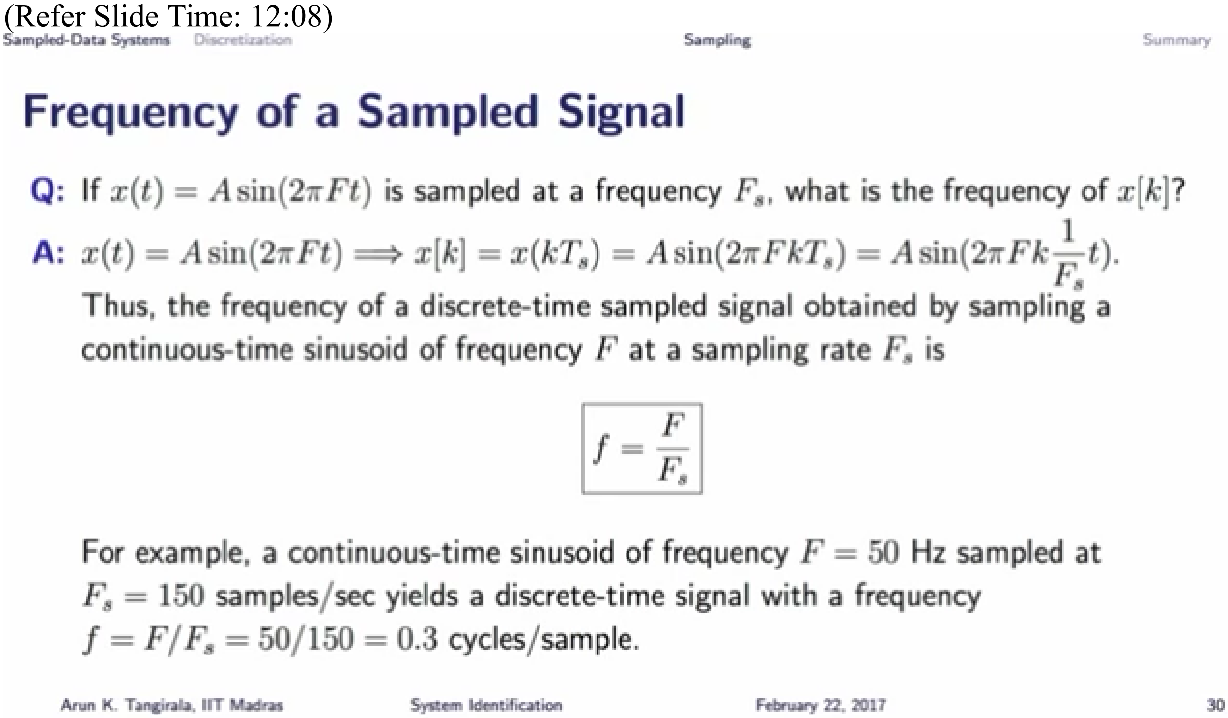

if the big F is 50 hertz and it's being sampled at 150 samples per second, then I have discrete time signal that has a frequency of 0.3 cycles per sample, look at the units of small f, right, it is cycles per sample, whereas the big F has a unit, has units of cycles per second or cycles per time, so you have to remember that, and you should also recall the discussion that we had earlier about periodicities of discrete time signals, not all sine waves in discrete time are necessarily periodic, only those which are rational, which have rational frequencies are periodic, okay.
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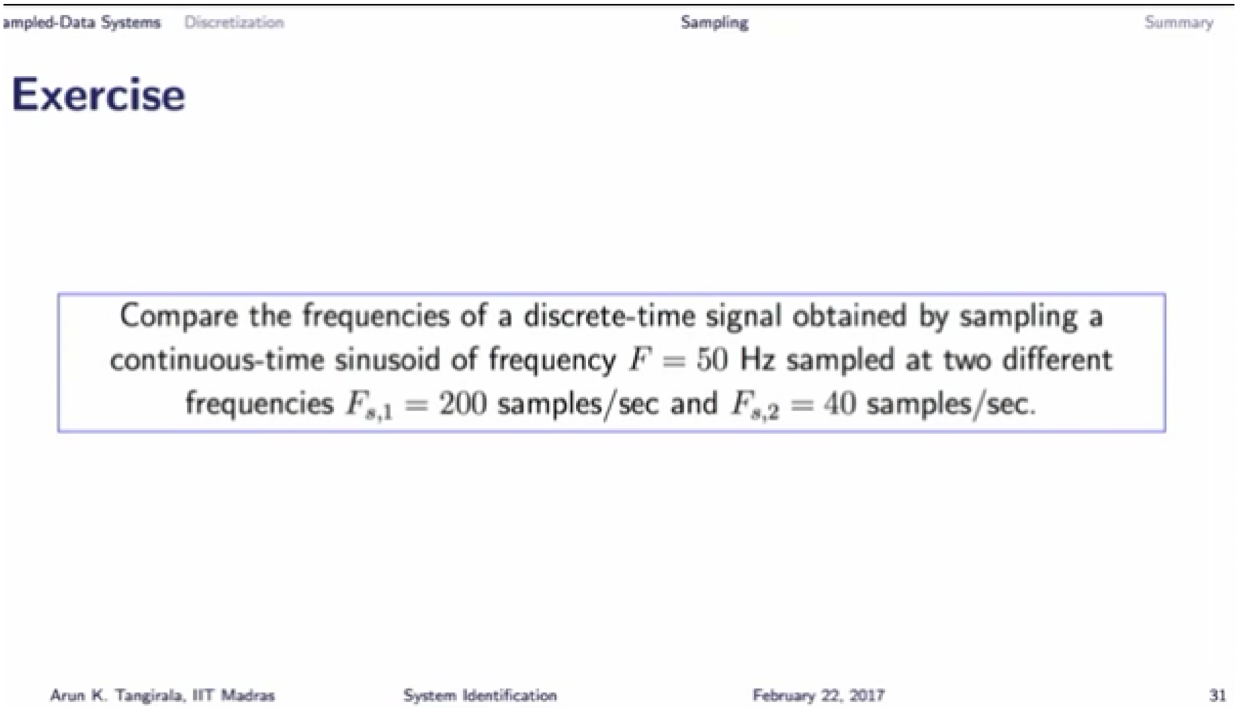

So as a simple exercise you can go back and ask yourself what are the frequencies that you obtained for the same continuous time signal when you sample a 200 samples per second and 40 samples per second, just as a simple homework exercise and I'm sure answers will come out in the discussion to follow.

Now since we are saying a key consideration in choosing the sampling rate relies on reconstruction,
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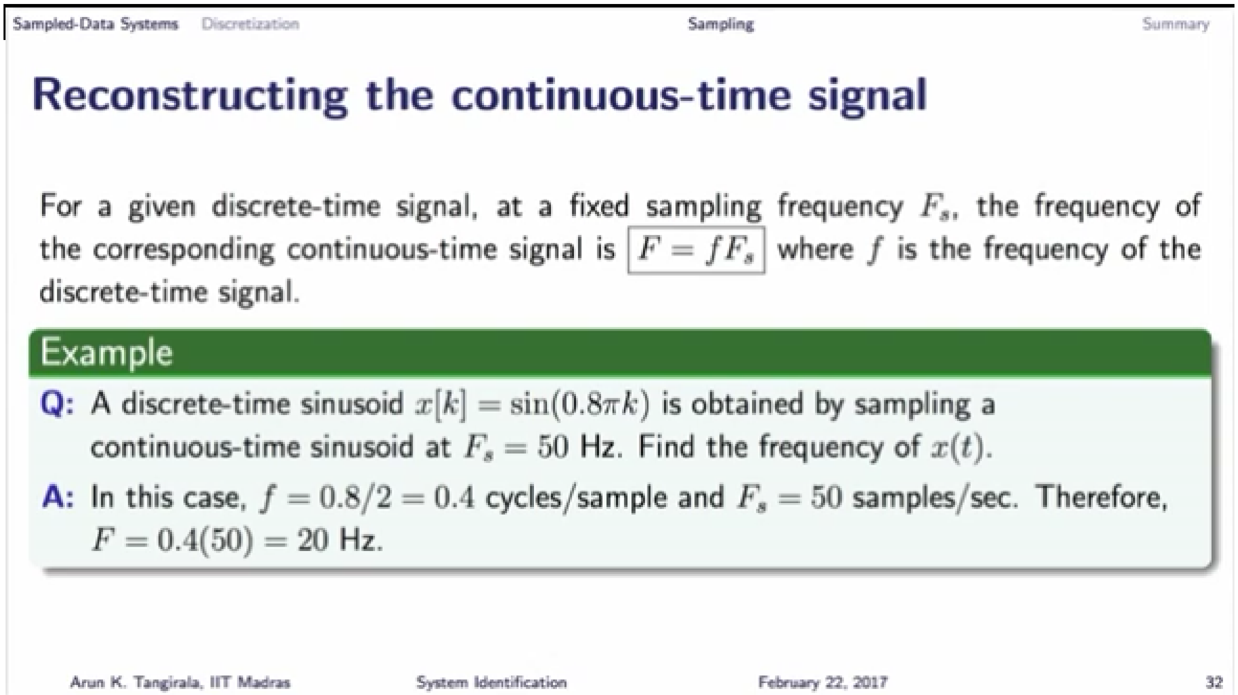

our ability to recover the signal an ambiguously, let us look at a simple example on reconstructing the continuous time signal, we are not going to use any complicated mathematics here, we are just going to use the same result that we've used here. This equation here allows you to determine small f if I give you big F and FS,

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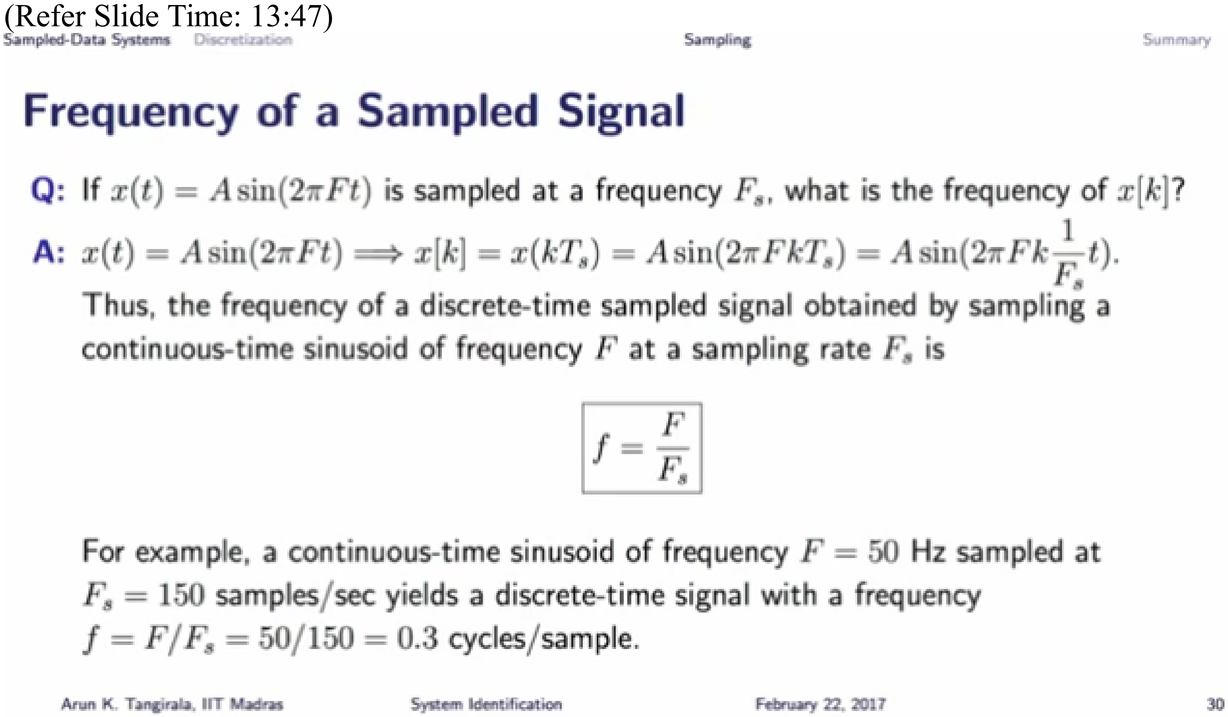

likewise if I give you small f that is the frequency of the discrete time signal and the sampling frequency you should be able to recover or calculate the big F that is a frequency of the corresponding continuous time signal, correct? Any two should be given to determine the third one, and that's what we are going to use here, so I have a discrete time sine wave which has a frequency of 0.8 pi K, so which means small f is 0.4, correct?

And I'm saying that this discrete time sine wave has been obtained by sampling at 50 hertz, is that correct? Yeah, sampling at 50 hertz, now we have to figure out what was a frequency of the continuous time sine wave, this is very straight forward. Now I've given small f, I've given FS and I just have to figure out what is big F, so it's simply F time FS, correct? Small f is 0.4 cycles per sample, and FS is 50, therefore the continuous time signal has a frequency of 20 hertz, this is how if I've given sine waves and the sampling frequency, and the frequency of the discrete time sine wave I can straight away reconstruct the continuous time sine wave, no complicated math here,
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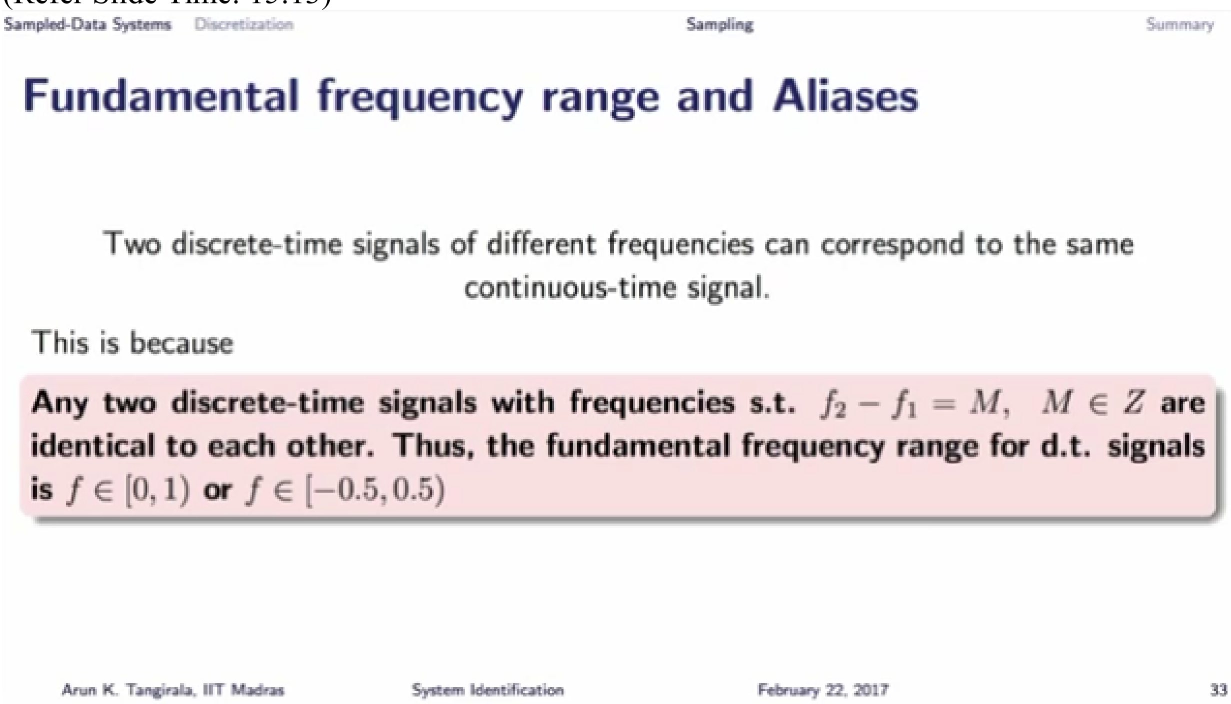

but there is this issue with discrete time sine waves that we have spoken about, that discrete time sine waves are unique only in a certain frequency range, if you look at angular frequency discrete time sine waves are unique only between –pi and pi or any length of interval 2 pi, or if you talk in terms of cyclic frequency any length of 0 of 1, -0.5 to 0.5 or 0 to 1, this can post some difficulties and this is the main thing that actually possess difficulties for us when it comes to recovering the continuous time signal an ambiguously, right.

So let's look at an example and understand how this can actually present a situation, we'll come back to this definition of aliasing, let's look at the simple example, I have a continuous time signal that is being sampled, so what is the frequency of the continuous time signal here? 10, 10 hertz is being sampled at 50 hertz, excellent,
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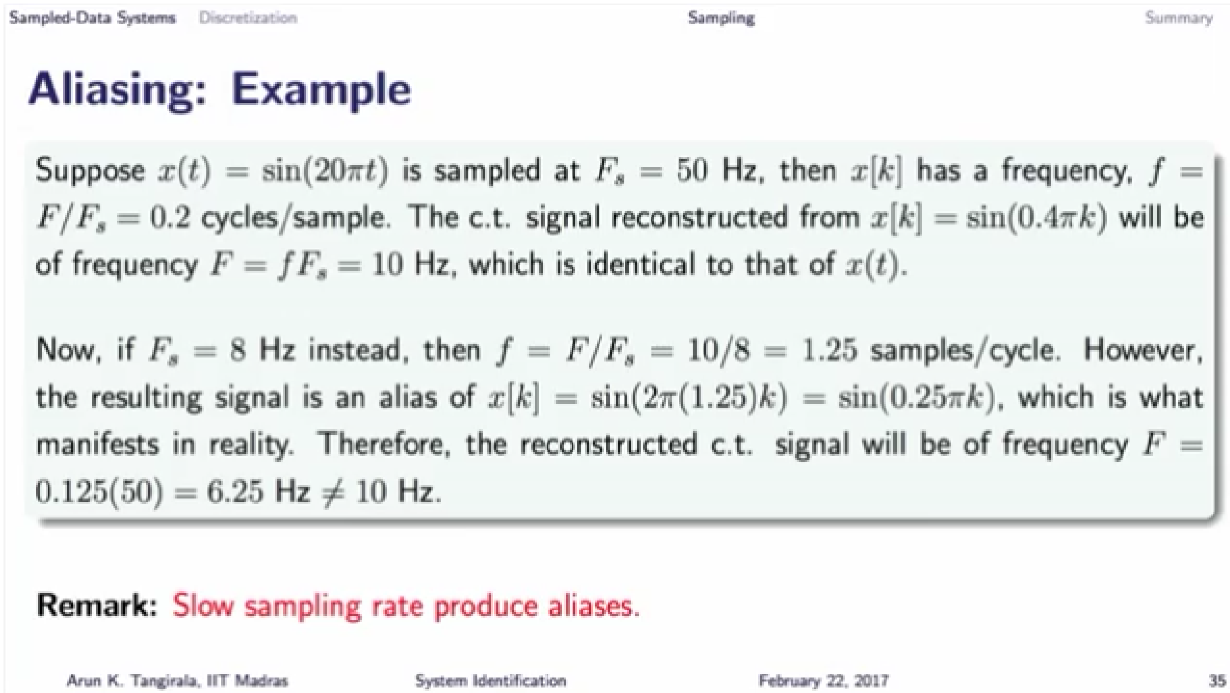

so I get small f as 0.2, therefore I'm able to reconstruct it uniquely no problem, if I reconstruct given only this 0.2 and sampling frequency I recover exactly 10 hertz.

On the other hand had I chosen a sampling frequency of 8 hertz, just for the you know heck of it I choose 8 hertz, and then I obtained F as 1.25 samples per cycle, small f, however because of the non-uniqueness beyond this fundamental interval this sine wave of 1.25 cycles per sample now manifest to you as an observer as a sine wave of frequency, what do you get? 0.125, right, if you had, earlier when we sample at 50 hertz I got 0.2, correct, if I change the sampling interval the frequency has to change, but now the frequency has changed to 1.25 which is mathematically correct, but this 1.25 samples per cycle now manifest as, what do I get here? In fact there is a mistake here it should be 0.25, I'll correct that, it should be sine 2 pi times 1.25K, there is a mistake in the slide, it should be 0.25 it seems so I get sine 2 pi times 1.25K which due to the periodicity of sine wave manifest as,
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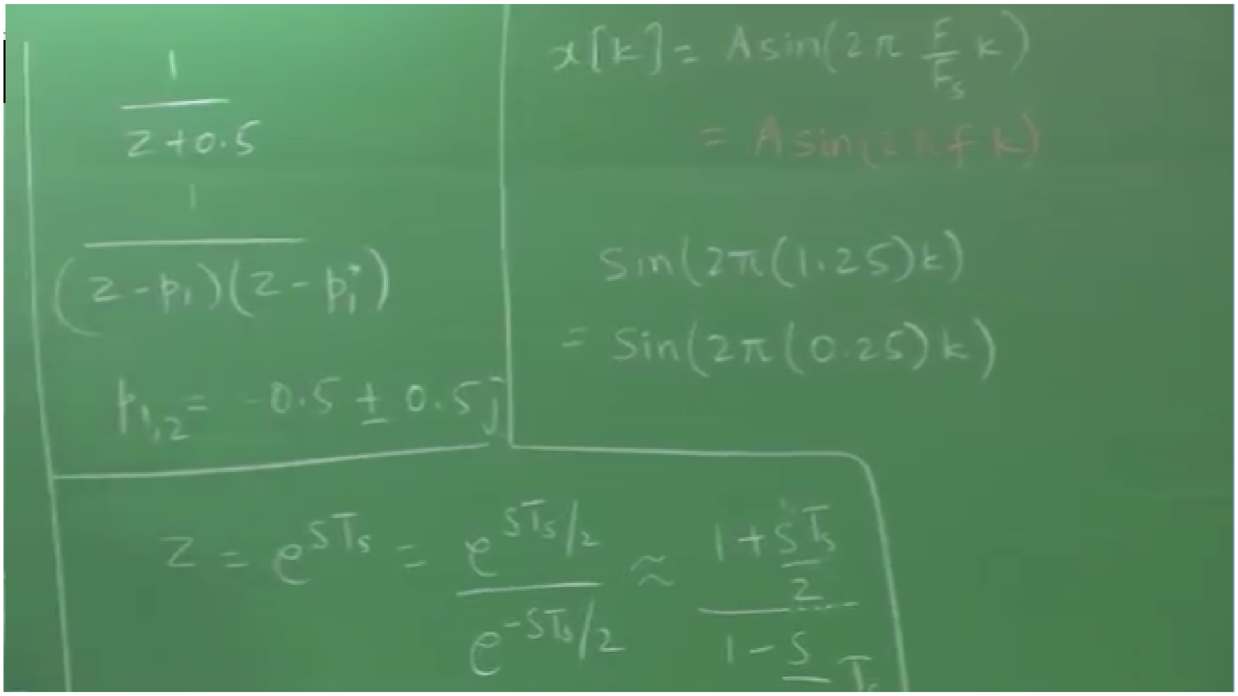

because sine are periodic with periodicity 2 pi, which means what should a manifested as frequency 1.25 cycles per sample now manifest as 0.25 cycles per sample, so this is what you will see, you will never figure out that it was 1.25.

In other words you generate two sine waves, in fact one with sine, with frequency 1.25 and other with 0.25 you will not be able to distinguish at all, you better not if things are, if your software is correct, because that's a mathematical property of sine wave, therefore when you try to recover the continuous time sine wave you will see the discrete time sine wave as a 0.25 cycles per sample, not as 1.25, and therefore when you recover the continuous time frequency, what do you get? 2 hertz, is that correct?

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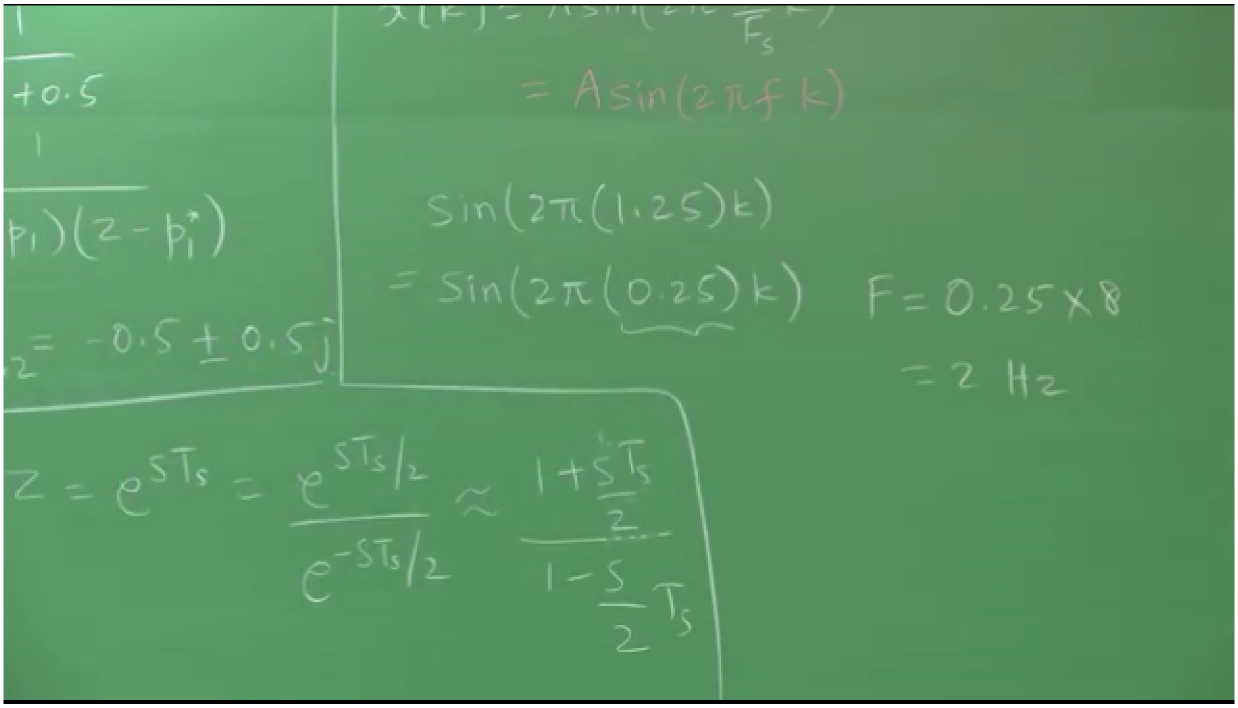

It's not, I have actually generated this discrete time sine wave from a signal that has frequency of 10 hertz, why? Now what has happened? I've not recovered the original signal, I've recovered something else, I've recovered a 2 hertz frequency, whereas I have supposed to recover a 10 hertz frequency, why has this occurred? Intuitively why has this occurred? This 6.25 is not correct, oh have I used 50? That's wrong actually,

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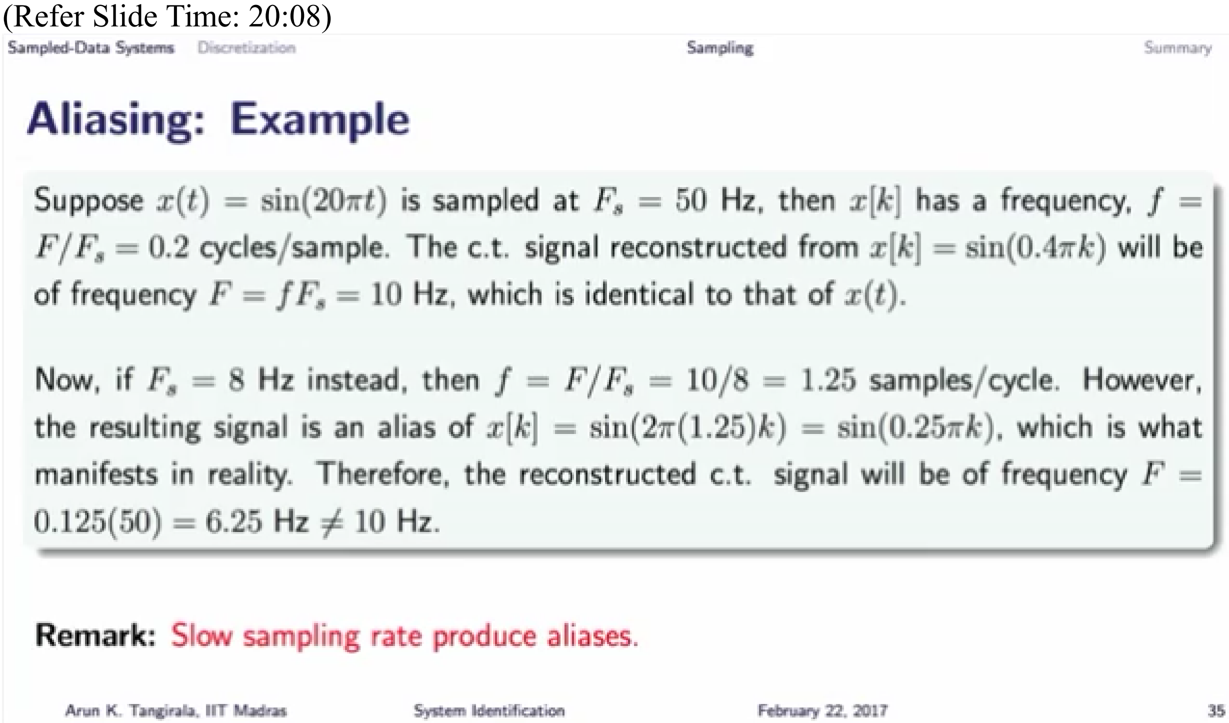

there is a mistake, in the slide I'll correct quite couple of mistakes there, ignore the second half of the slide, whatever you see on the board is correct, I'll correct when I post it, things will be okay.

So the point is I have here a situation where a 10 hertz frequency signal in continuous time is being recovered as a 2 hertz signal, why has this occurred? Whereas with the earlier sampling frequency I was able to recover correctly, right, when F is the say, when I choose FS as 50 hertz, no issue I was able to recover F as 10 hertz, so this is correct, but this is wrong.
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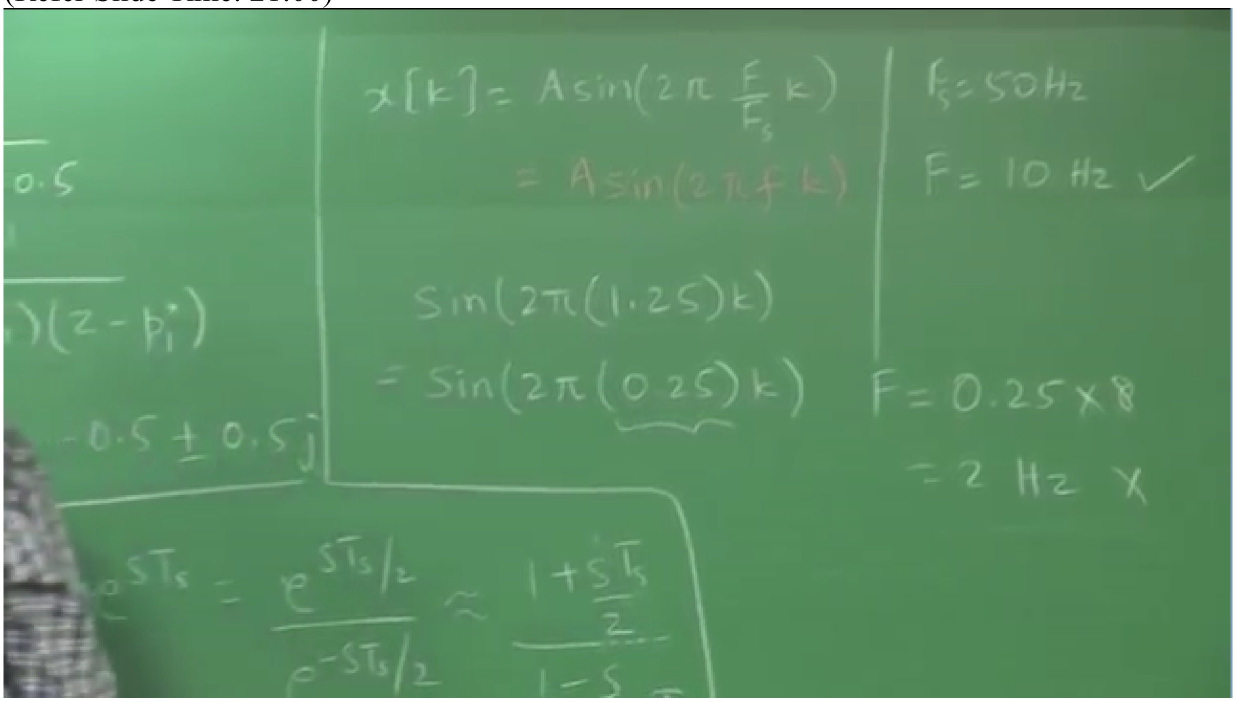

Wrong with respect to the original signal that I have used, so this has occurred because, yeah that is happened in fact the strict definition, I mean the requirement is that it should be twice that, that’s what we are coming to, qualitatively this has occurred because I have sampled very slowly, we don’t know slowly then what, but definitely I’ve sampled it much slower and therefore you see this as, you see when you look at the continuous time signal it appears as a very low frequency signal, which means the sampling frequency that you have chosen as 8 hertz is good for a 2 hertz signal, not for a 10 hertz signal, you see this in many movies also, right, when the car is going very fast in the frame you see the wheels are actually going in the opposite direction slowly, have you ever noticed? Right, in fact in many houses when you have the ceiling fans and there is tube light, you see that there is a signal that’s going in the opposite direction very slowly, whereas the fan is rotating very fast in the other direction, so I was wondering is the car really moving in the frame, because the rest of the frame shows that the car is moving very fast, but the wheels are moving slowly in the opposite direction, this phenomenon we call as aliasing.

The aliasing here occurs because the 1.25 samples per cycle has an alias of 0.25 samples per cycle, any signal, any sign wave with frequency outside this fundamental frequency has an alias within the fundamental frequency, right,
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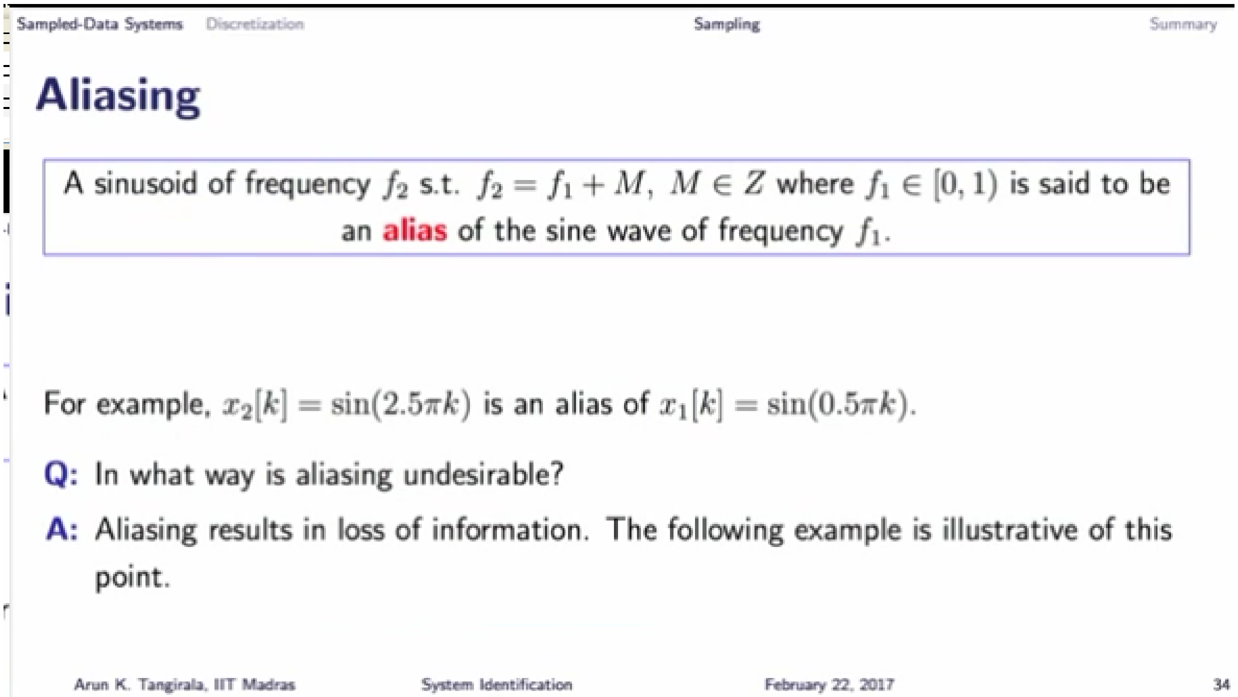

so that is what we mean by aliasing, any frequency F2 which is F1 + M, where M is sum integer, M could be 1, 2 and so on, F1 is a value within the fundamental frequency range, any other sine wave with frequency F2 will manifest as F1, that's because of the periodicity of the sine wave, we say then all those frequencies F2 have an alias here or you can say this F1 has aliases either way, which is the alias of the other is your choice, okay, so that is the reason why this sampling,

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Aliasing: Example

Suppose $x(t) = \sin(20\pi t)$ is sampled at $F_s = 50$ Hz, then $x[k]$ has a frequency, $f = F/F_s = 0.2$ cycles/sample. The c.t. signal reconstructed from $x[k] = \sin(0.4\pi k)$ will be of frequency $F = fF_s = 10$ Hz, which is identical to that of $x(t)$.

Now, if $F_s = 8$ Hz instead, then $f = F/F_s = 10/8 = 1.25$ samples/cycle. However, the resulting signal is an alias of $x[k] = \sin(2\pi(1.25)k) = \sin(0.25\pi k)$, which is what manifests in reality. Therefore, the reconstructed c.t. signal will be of frequency $F = 0.125(50) = 6.25$ Hz $\neq 10$ Hz.

Remark: Slow sampling rate produce aliases.

Sampling Theorem

Now, we return to the question: **how fast should a continuous-time signal be sampled so that there is no loss of information?**

Answer: The sampling rate should be such that it does not produce aliases, i.e., the frequency of the d.t. signal falls in the fundamental range $[0, 1)$

$$|f| = \left|\frac{F}{F_s}\right| < \frac{1}{2}$$

A continuous-time sinusoid of frequency F should be sampled at least as fast as twice its frequency so that there is no loss of information.

this phenomenon is occurred and we say slow sampling rate produces aliases, so the simple thing that we want to now guarantee is sampling should always produce f, this small f such that the small f is always between -0.5 to 0.5, that's a fundamental frequency range.

If the sampling results in a small f that is outside this, what will happen? You can find an alias in the fundamental frequency range that's of lower value and as a result you will reconstruct the wrong continuous time signal, and you substitute for F as small f over, I mean big F over FS and which clearly brings you to the sampling theorem, this is actually a way of proving based on the one that is given in the excellent book by Proakis on signal processing, Proakis and

Manolakis, this kind of proof was not given necessarily by Shannon, okay, if Shannon were to see this proof he'll shed the tear or two, but his proof was based on in a more formal setting and so on, but I don't want to get into that formal setting, you will find the proof in many text, this proof is sufficient for our understanding, all we are saying is I should sample in such a way that all the discrete time, whatever discrete time signal that results of sampling should have a frequency in the fundamental frequency range that's all, very simple, and straight away you'll get this result that FS should be at least 2F, right.

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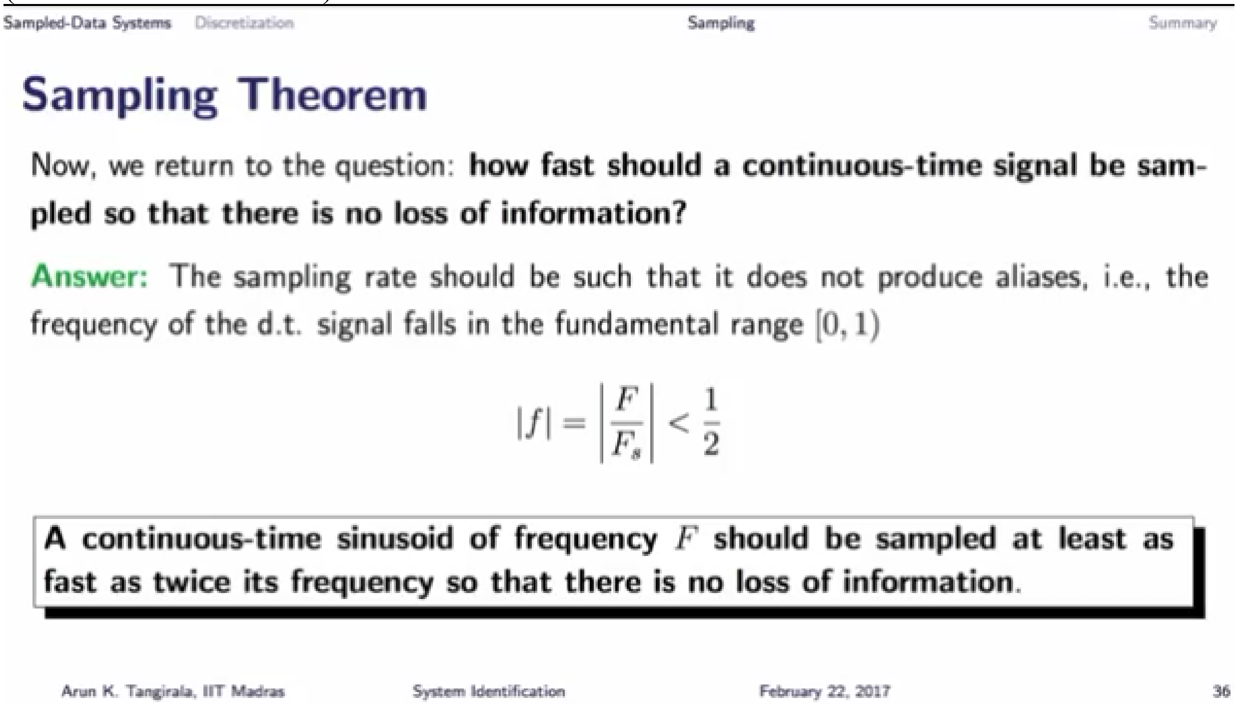

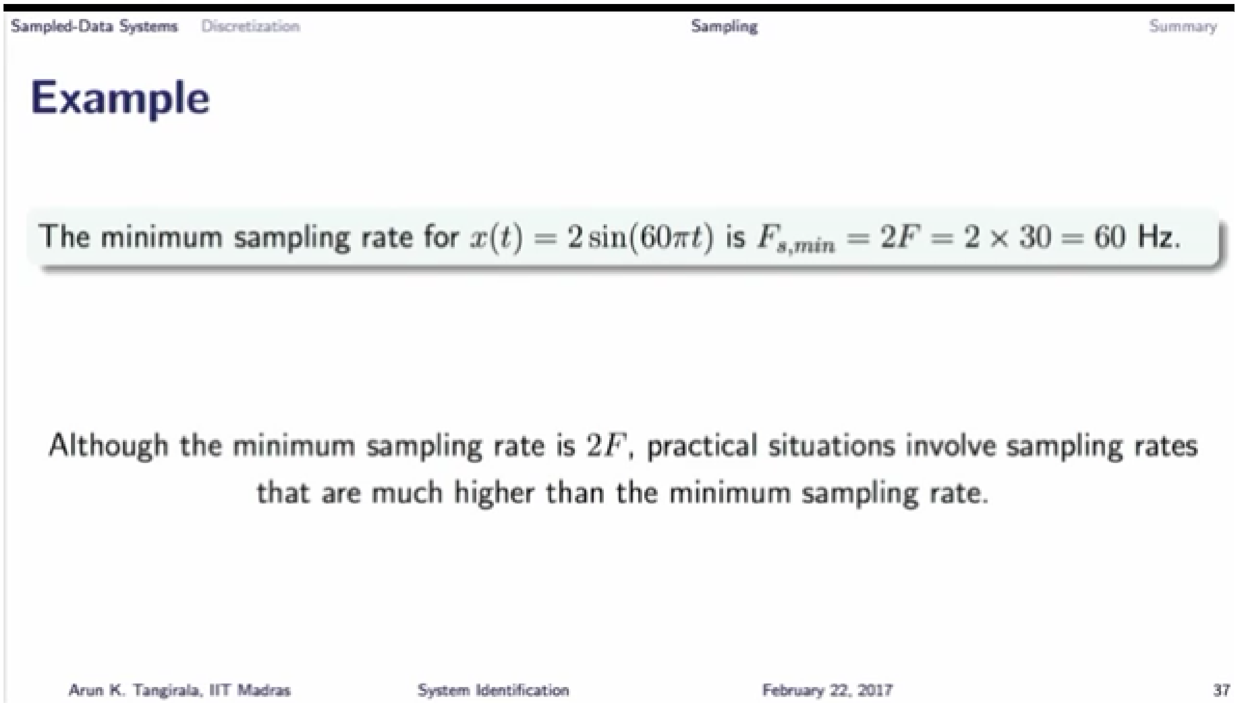

For example here if I have a 2 sine 60 pi T, amplitude doesn't matter, the frequency of this continuous time signal is, what is it? 30 hertz, therefore the minimum sampling frequency is 60 hertz, you know although the minimum sampling rate that is given out by the sampling theorem is 2F. In practice we use much higher sampling rates, okay, suppose I have mixture of frequencies in the sine wave, then I'll set my sampling rate based on the maximum frequency obviously, it's like your time constant thing, so whatever is the maximum frequency my sampling rate will be based on that,

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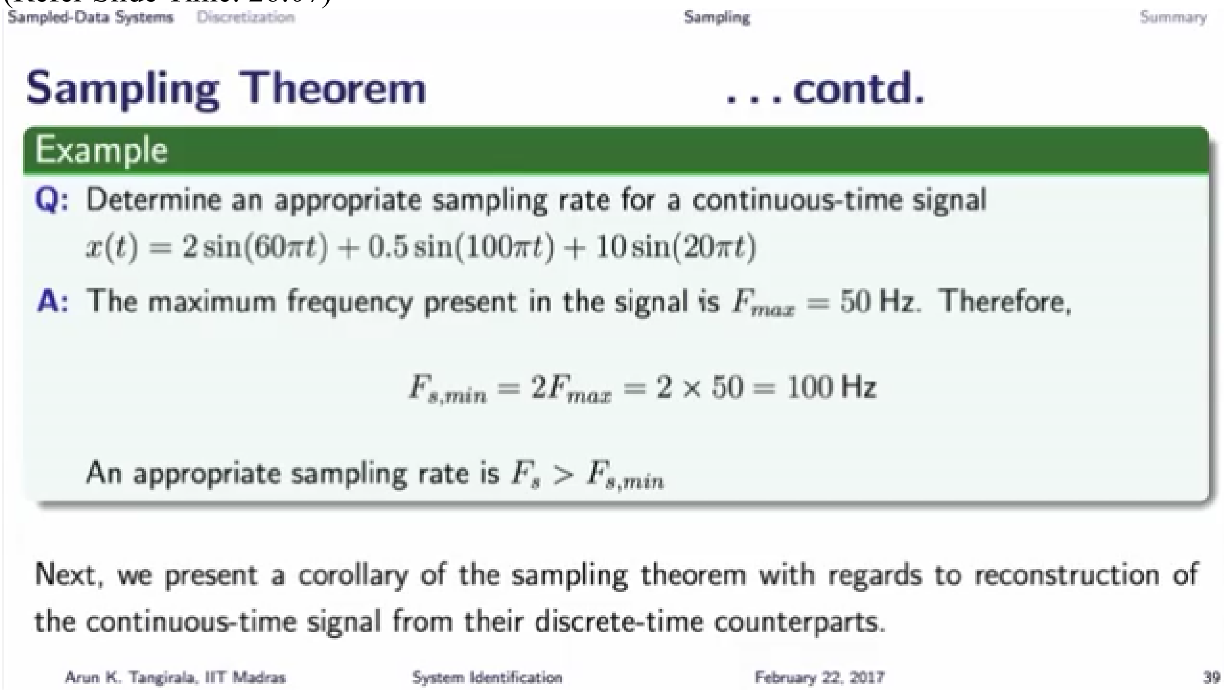

and as a simple example here I have a mixture of sine waves among which the maximum frequency of that component, the three components that are present is 50 hertz, therefore I choose a sampling frequency of 100 hertz, minimum sampling frequency generally as I said we choose much above then this.

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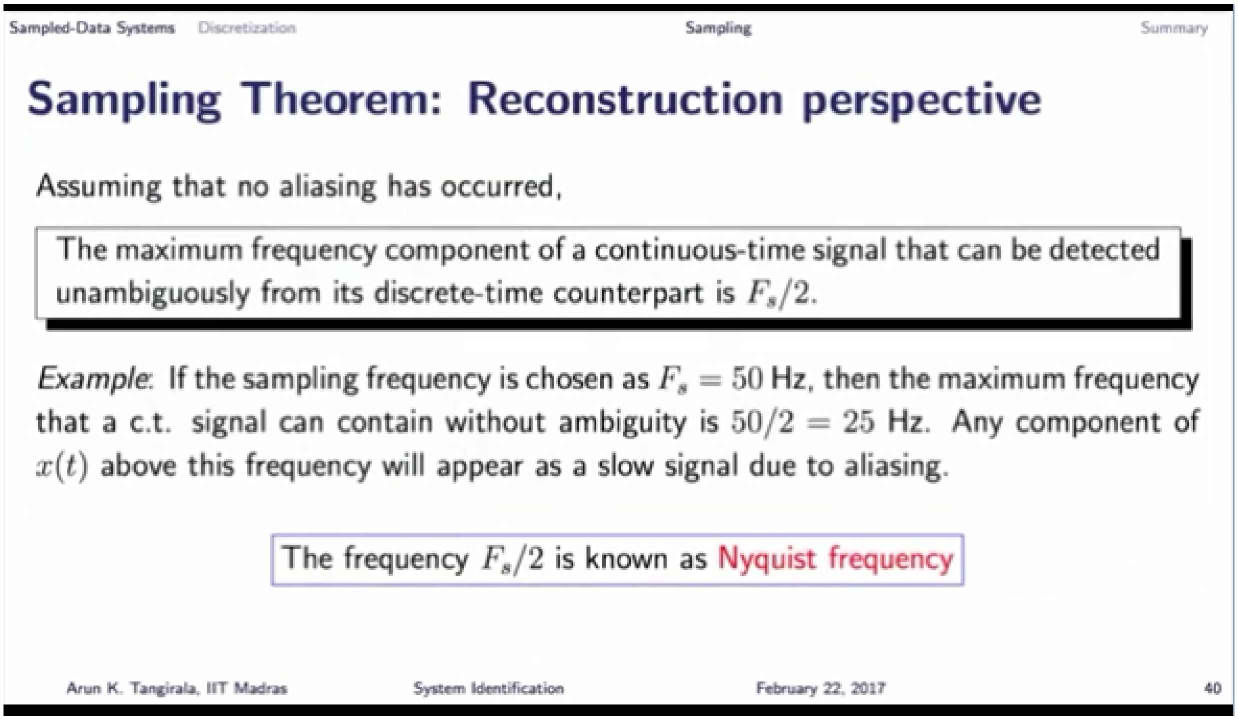

And there is one point that I'll conclude the class with, which is the reversing, if I give you a sampling frequency, what is the corresponding continuous time signal frequency that you can handle, so that you don't end up recovering a wrong continuous time signal, so that is the, suppose I tell you sampling frequency is 50 hertz, for what class of continuous time signals is it suited? I mean up to what frequencies? Half of it, right, 25, up to 25 hertz I can handle, if the continuous time signal has beyond 25 hertz I have an issue, then I'll end up with aliasing, right, aliasing cannot only result in lower frequency recovery, but also in a phase reversal and that is why I said many wheels moving in the opposite direction when the car is moving that's a phase reversal also, you can get a negative sine. This FS/2 is known as a Nyquist frequency, it's a maximum frequency that you can recover uniquely, or it is a maximum frequency of a continuous time signal that you can handle, so if FS is 1, Nyquist frequency is half, if FS is 2 the Nyquist frequency is 1, it's named after Nyquist.

So tomorrow when we come back we'll complete this discussion by giving general guidelines, what we have learnt is a formal result on what should be the minimum sampling frequency, we will briefly talk of perfect reconstruction and then talk about practical consideration, so that's just about 5 to 7 minutes of discussion, and then we'll start out with the review of probability and random variables, okay, see you tomorrow.

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