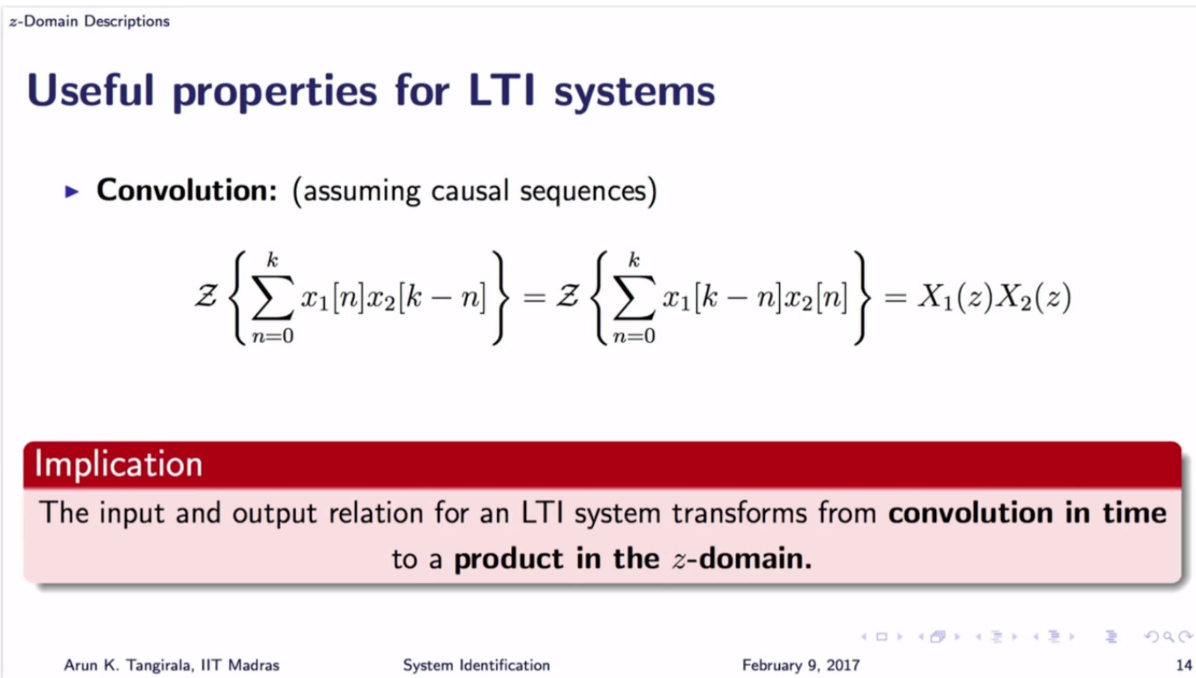
CH5230: System Identification

z-Domain Descriptions 3

Now there is yet another property which is perhaps the most useful property when it comes to system analysis. Until now we have talked about properties with respect to signal analysis. So this notion of signal and system has to be very nicely embedded in our minds, because always in any data driven analysis in any signal processing exercise there is a signal analysis and there is a system analysis and we need to be well versed with both. This convolution property allows us to analyse now. How systems operate in z-domain. Right? How does an LTI system operate in time domain? Mathematically, it operates with the help of convolution, right? Now the question is how does this LTI system operate in z-domain? What we mean, what do we mean by that? What do we mean by LTI system operating in z-domain? Well, in-- go back to time domain.

(Refer Slide Time: 01:16)



When the system LTI system is presented with an input u, what does it do to the input? It convolves the input with the impulse response and produces an output that is what we mean by operation in time domain. So it takes the input. It has its own impulse response which is a property of the system. It convolves the input with the impulse response and gives you the response that is what we mean operation time domain. Now we want to ask what is the system doing z-domain. What this means is that if I present not the time domain input, but the Z-transform of the input. What does the system do in this z-domain? These are all mathematical imagination because reality is that I always see the system in time domain. But since we have already decided that there are good reasons for looking at systems in z-domain. We want to ask this question, what does the system do in-- to signals in z-domain.

So we want to ask the question, here is the LTI system. Now, instead of u k, I'm going to present u of z. What do I get here, right? And what is the block here? In time domain we know very well that when the system is presented with u k, it performs this operation to produce y k. And you can say in time domain I can represent this using the transfer function operator, right? Convolution equation is compactly represented using the transfer function of it. We want to know answers to these two question marks here. And the convolution property of the Z-transforms answers this question.

What does it say?

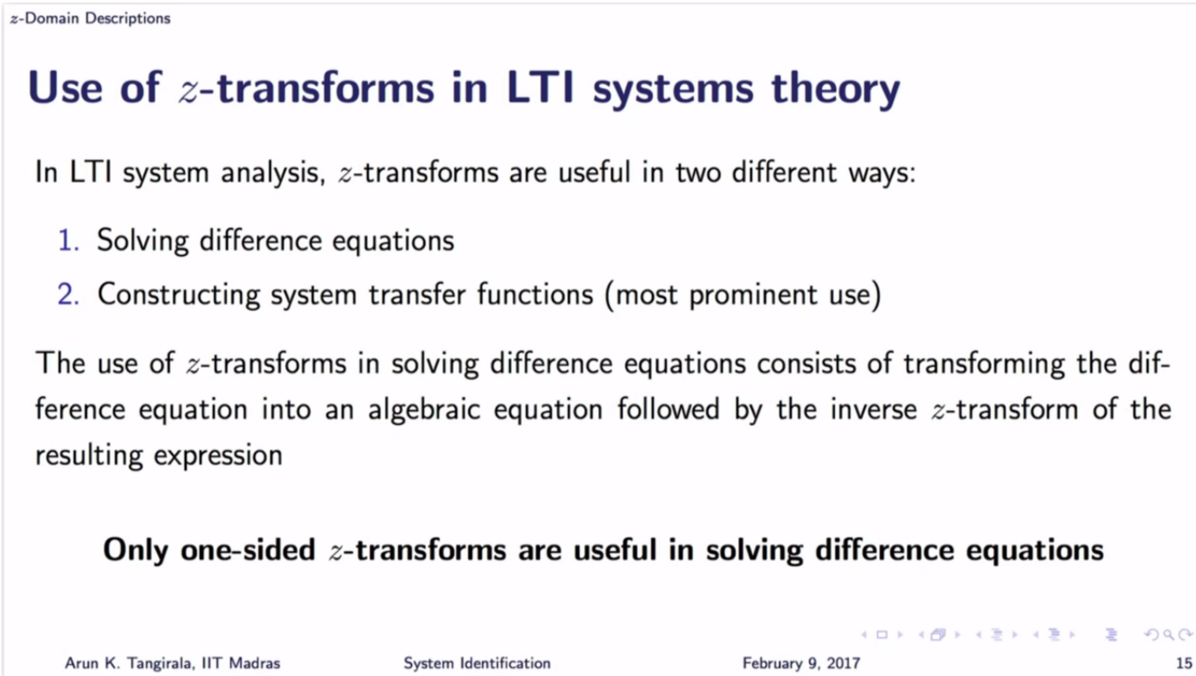
Is that the Z-transform of the convolution of two causal sequences? Why are we assuming causal sequences?

What are the causal sequences mean?

Signal is Detroit negative times and why are we assuming that here, because we are working with unilateral Z-transform. Remember we have already said that yesterday. We are working with one sided Z-transform. If the signal is not 0 at negative times 0 valued at negative times then we have lost some information. And this property may not hold. In which case you may have to use it two sided Z-transfer? Anyway, so assuming causal sequences the Z-transform of the convolution of two set sequences is nothing but the product in the z-domain. So a convolution operation in time maps to a product in z-domain. That's all you have to remember. And that's a beautiful thing because convolution if you come to think of it in, right from the beginning when I've learned this convolution whenever I learned it. I found it to be a kind of a painful operation. It's not a direct product, there is a summation. You know there are four operations involved in convolution if you break it down. First of all there is a flipping off the signal, of one of the sequences. Then there is a shift and then there is a product and then there is a summation. My God, I'm in convolution actually involved four operations and computationally also, it's a painful thing to do. It's not. It doesn't lend itself to nice efficient ways of, efficient computation algorithms. Whereas products are really easy, it's really simple. Yeah, of course, you may say well in computing x1 of z and x2 of z, you will have summation and so on. But if I'm as far as theory is concerned this is great. In fact this property also applies Fourier transforms. Fourier transform of a convolution of two sequences is a product of the respective Fourier transforms. If Fourier transforms exist for those sequences. This property is used everywhere. That means in theoretical analysis in computation and so on. Just know I mentioned convolution performing convolution in time that is computing it, can be quite inefficient, because there are four operations involved. So all convolutions almost all good convolution algorithms first compute Fourier transforms of the two sequences, because you have efficient ways of computing the Fourier transform. If given two sequences when you want to compute the convolution, step one compute the Fourier transform of these two sequences, step two take a product which is very easy and respect at each frequency and then simply take the inverse Fourier transform that will get you the value of the convolution that you want to now. So you see this property has not only got a big role to play in the theoretical analysis, but also in practical data analysis as well.

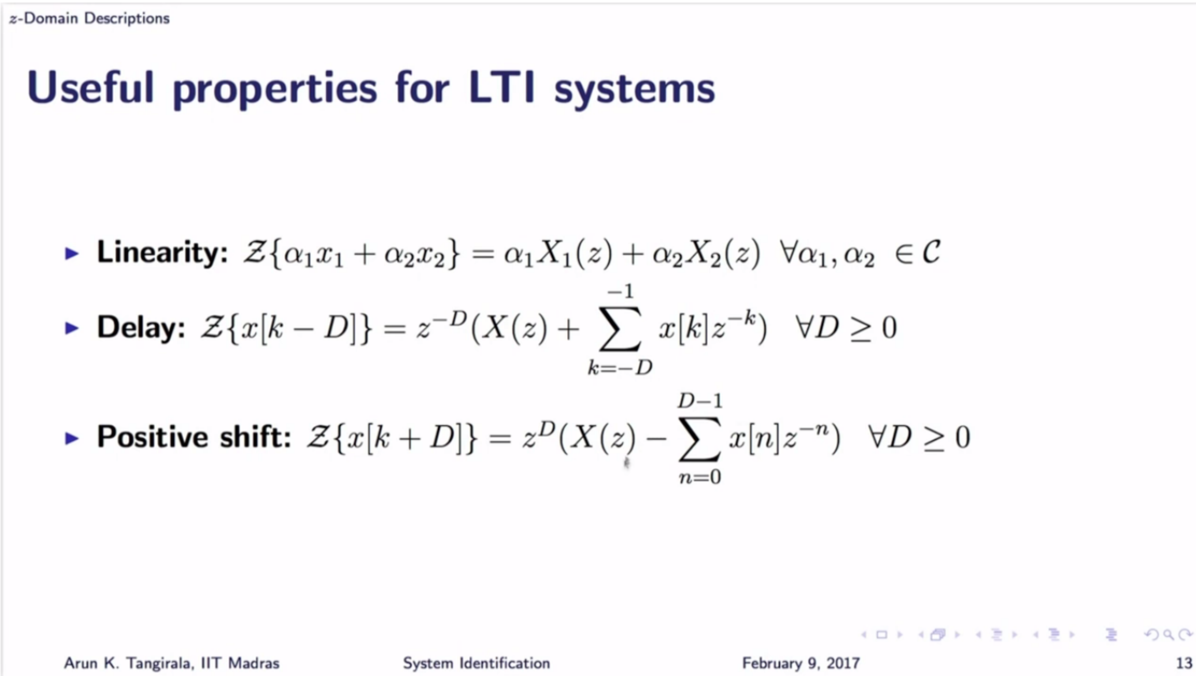
So whatever you learn of Z-transforms more or less is this applies to Fourier transforms as well, because as we have seen yesterday Fourier Transform is a special case of Z-transform evaluate it on the unit circle which we will talk about a bit later. So coming back to the point convolution it time maps to product in the z-domain. The other thing that you should remember are, is that there is a duality to these results. What we mean by duality is? Product in time domain would mean convolution in z-domain. So it becomes very easy to remember us. Which we don't use, we don't have a use to the property therefore I don't mention. But this duality property is very nice and the time and frequency whether it is complex frequency or just your natural Fourier frequency. They are their called, you know, in physics they're called conjugate pairs of conjugate variables and so on. Conjugate domains they are, they enjoy some very, very special relation. They just cannot live without each other. So it's like that. Anyway, so now how do we use this property? Bought this property that is the properties that we have studied especially the ones with respect to delay and so on and the most important one which is the mapping of convolution to product. We use this property in at least two different applications one isn't solving differential equations, right? When you're given initial conditions boundary values and so on, the other is to arrive at the notion of a transfer function. And third which is actually a corollary of the second one is in the computation of the response of a system to some given input sequence. So let's look at that and always remember that when it comes to solving differential equations it's the one sided Z-transform that is quite useful.

(Refer Slide Time: 08:46)



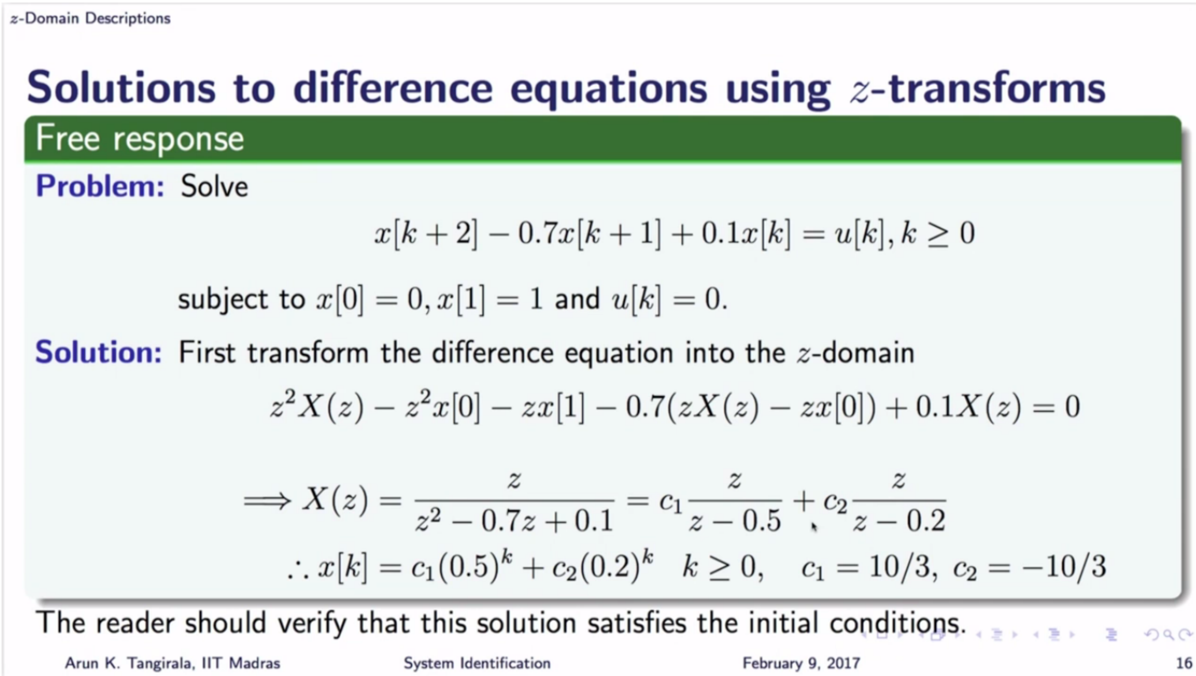
So let's look at an example here. I have here a difference equation look at how I have written the difference equation I have written it in a forward shift from, not like the ones that we have been writing. So I have written here as x k plus 2 minus, point 7 x k plus 1 plus point 1 x k is u k, but this is the difference equation. There is a system that's being driven by this equation and we are given the initial conditions. Remember this is a second order difference equation. I need to be given two initial conditions which I had given here. And input is 0 which means we are looking at free response or so-called natural response, just response to change in non-zero initial conditions. So the first step is to, that is now I want to solve these using Z-transforms. That's the objective. The first step is to take the Z-transform on both sides of this difference equation, right? And remember Z-transform is a linear operator. Which means on the left hand side, when I take the Z-transform of the sum of terms it would be some of the Z-transform of the respective terms. The first term is x k plus 2. We know from the property of Z-transforms. Z-transform of x k plus 2 would be z square. Go back to the property here, z square times x of z minus this, right?

(Refer Slide Time: 10:18)



So minus z square x 0 and minus z x1. Is it clear? That's the Z-transform of the first term. Then likewise minus point 7 times Z-transform of the second term.Once again the same story. Now we use the same property but the shift is by only one unit. So you have z times x of z, minus z times x at 0. And then this third one is simply point 1 time x of 0. The right hand side is given to us as 0. So you don't have to evaluate anything. Now when you put together everything collect like factors you straightaway get X of z as this expression here. Is it clear? Now what do you do? You want x of k. You want the response, right? From here on it is simply a matter of taking the inverse Z-transfer. That's all. Followed? That's all it is. And c1 and c2 values are given here and x k now the solution turns out to be, I recognize the first one to be the Z-transform of point 5 rise to k and Z-transform of the second one here as point 2 rise to k. So I have c1 times point 5 k rise to k plus c2 times point 2 rises to k as a solution. And you should verify that this solution satisfies the initial conditions. That's a good check for you, right? For example at k equals 0. The initial condition is 0. Does this solution satisfy that, right? Because c1 and c2 are identical here at k equals 0, x [0] is 0 and likewise you can check for x of [z]. Any questions on this? Fine?

(Refer Slide Time: 12:27)



Straightforward, just apply this Z-transform property get to x of z and apply in the Z-transform. That's all. Let's look at the second example for the same system. Now we are interested in force response. That's why the initial conditions have been set to 0. I only, I want to know only how the system responds to a change in the input, assuming that the system start from a relaxed state. Now, what kind of input I mean injecting into the system, what is this? A step. Good. Story is the same. What is the difference between the previous and this case? Of course, the initial conditions are 0. So the left hand side is a lot more simpler. What about the right hand side? Simply the Z- transform of a step, which is 1 over 1 minus z inverse or z over z minus 1. Either, whichever way you look at it. Once again you get here. By the way it is a mistake there should be z minus 1 not minus 1, I will correct that. This factor here should be z minus 1. You're only able to see or it's too small. Now I have three terms in the denominator, right? So I do a partial faction expansion involving three terms. Once again I find c1 c2 c3 all of you must be familiar with the partial faction expansion that's one of the basic things we learn in 11. That's it, so now you get the solution. Notice that now I have a constant. That is I have c1 times point 5 rise to k plus c2 times point 2 rise to k plus c3. And the values of c1 c2 c3 are given here.

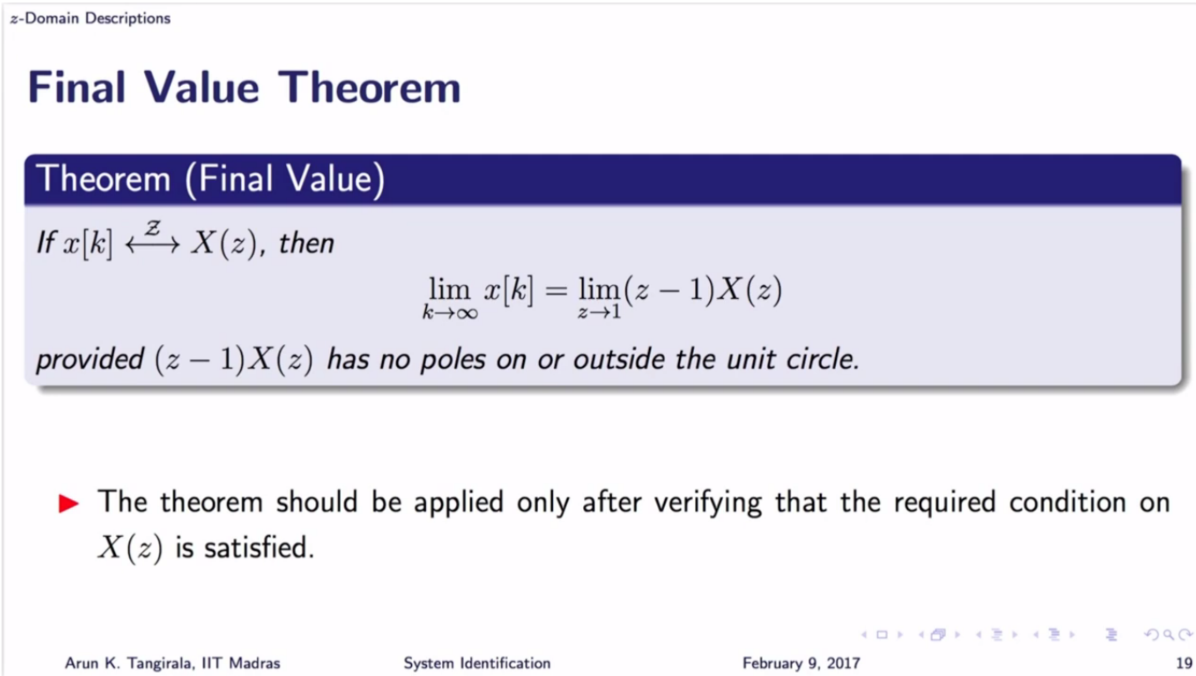
Do you notice a similarity in the solution to this problem in the previous case? What is the similarity? This party here c1 times point 5 rise to k plus c2 times point 2 rise to k, correct? They are common to both forced and free response. Why is that? In linear systems you can do a lot of nice analysis like this. So you can see that there is some commonality between natural response and force response. And why is that a commonality?

Okay. That's way of. That is a mathematical way of stating it. It is the system's characteristics. Whether it's free response or forced response the systems characteristics will always come into play, right? The only difference is in the forced response, the input characteristics also appear in the response and that is why you have a c3. And why is it a constant here, because the input is a step. If it was some other input then the third time would have been different. That's what we mean by particular solution and so on in our theory of difference equation. That is. But that is a mathematical way of stating it, but a physical way of looking at it is, this contribution c3 comes from the input. And we have always said this when it comes to LTI systems whatever the input you inject after sufficiently long time the input shape will appear in the output. So when you let k go to infinity or become very large what happened to the first two terms? They go to 0 or become negligible. So it's only the input signal that dominates. Always, whether it is a sine wave, whether it is an impulse, whether it's a step ramp whatever it is, if the system is stable. How do I know now the system is stable? That's where these so-called poles that we'll talk about very soon. We will come into play. If the poles are within the unit circle that means if these values here point 5 and point 2 which happened to be the roots of the characteristic equation.

If they are less than 1 in magnitude then the system's characteristics will die down and the input characteristics will take over. Clear? So always in a linear system you can break up the total response as the sum of two responses. Whenever if you hear we have only a force response suppose the initial conditions were non-zero, then you would see very clearly this form of a solution as well, same solution that c1 c2 values would be different, right? Now you can safely say that the total response of a system is always the free plus force response. What you see here is force response. What we saw earlier is free response. But suppose you clapboard these situations together at a homework problem. Change the initial conditions in this problem to the one that we had earlier and see what difference does it make? You'll see it makes a difference in terms of c1 and c2. All right.So many things that we can learn from this simple example.Any questions? Good. The Z-transforms incidentally can be also used to know what is the initial value of a signal and the final value of a signal. Which find again use in system analysis? All of this we are learning in bits and pieces but slowly we put them together in, when it comes to system analysis. At this moment you may say what is a use I'm just going through one property after the other, but at least with respect to some properties we have seen some uses. For example the positive shift one you are able to use it to solve difference equations.

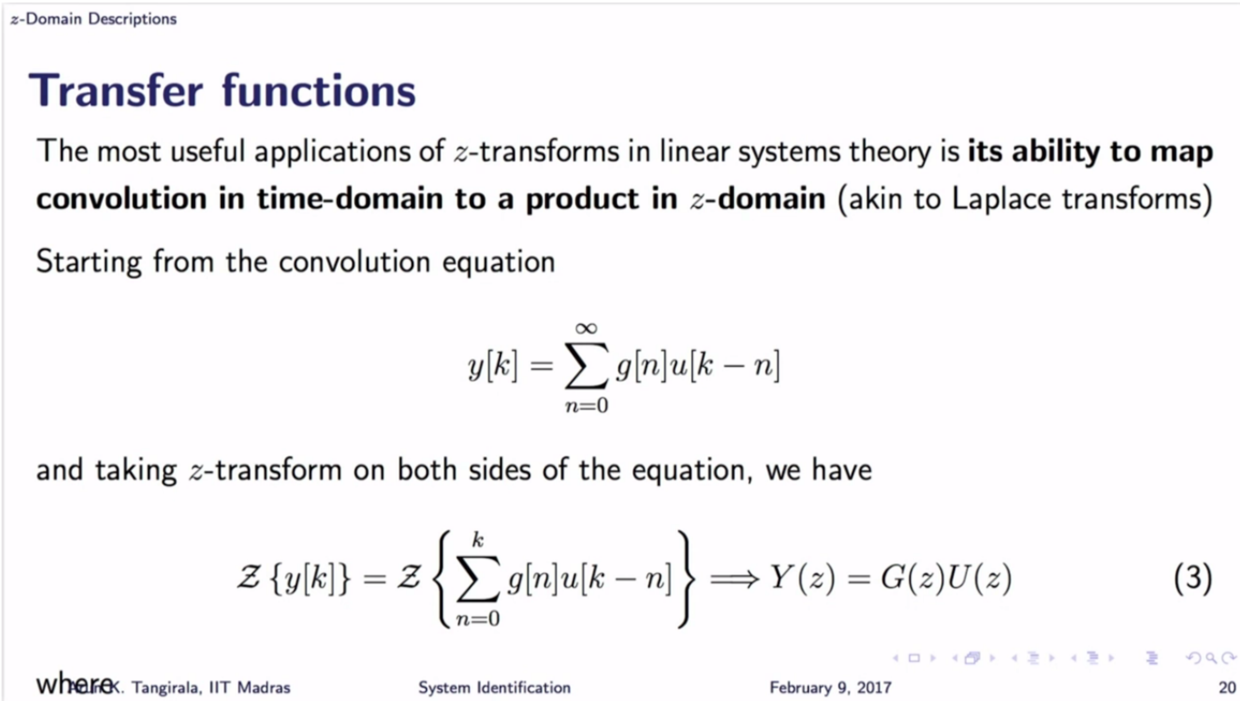
The initial value theorem says that I can figure out what was the initial value of the signal by performing some operation on the Z-transform. It should not come as a surprise, because remember whether it is Z-transform or Fourier transform it encodes the history of the signal from 0 to infinity, correct? When I am computing the Z-transform, what am I doing? I'm actually taking the Z-transform X sigma k equals 0 to infinity x k z to the minus k, which means x of z are Fourier transform encodes all the information right from the beginning to end. So I should be able to recover. So you should not come as a surprise. The question is what operation should I perform? And that operation is a limiting operation. So it says that limit k going to 0 x k is limit z going to infinity x of z. I'm avoiding the proof but you can either derive it by yourself or you can refer to any standard text. Provided x k is castle. That means at negative times the signal is 0, that that remains all through our properties. All right. The more important one that's of use to us is this so-called Final Value Theorem, which says once again that I can discover or I can in fact calculate the final value of the signal again by performing some operation on its Z-transfer. Now here when you apply this Final Value Theorem, despite my repeated cautioning every semester year, year after year, whether it is Laplace transform or Z-transform. Students tend to apply this blindly. Without checking if the signal indeed has a final value. What do we mean by that? Signal can blow with time, then there's no notion of a final value. That means his final value theorem should be applied to those signals only to those signals that have reached a steady state. For example a step, does it have a final value? Final finite value. Yes. What about a ramp? No. Right. So what about a sinusoid? What about an impulse? Yes. It has a finally value. Final value is 0, don't worry. It should not be infinite. That condition here is encapsulated in this is in the form of a restriction on the Z-transform which is provided, z minus 1 times x of z has no poles on or outside the unit circle. What we mean by poles is the roots of denominator. That is another way of saying that provided x k has a final value. So, for example, take a step what is the Z-transform, z over z minus 1. So what does this condition say z minus 1 times x of z should not have any poles on or outside the unit circle? When x k is a step x of z is z over z minus 1. Which means z minus 1 x of z would simply be z. It has no poles on or outside so you can apply. What about a ramp?

(Refer Slide Time: 22:00)



We just wrote down this morning, this Z-transform of a ramp. What was that? Z-inverse by 1 minus z inverse square write it in terms of z what would you get? Z by z minus 1 to the whole square. Apply this condition. What would you have? You have z minus 1 times z over z minus 1 to the whole square. Situation is that now, this has a pole on the unit circle, so you can't apply. If you ignore this condition or even the common sense thing that the signal doesn't have a steady state you will still find a limit. You'll still be able to find a limit that limit has no connection with the physical nature of the sector. So you said confidently. Okay. This is the final value of the signal take it3.5 there's no meaning at all, be careful.The same thing will appear in the system analysis as well. Okay.So, we will apply that final value theorem to compute what is known as a game and now comes the much awaited notion of transfer function. We'll put together all these properties to analyse the system. Ultimately this is a course on system identification not as pure signal analysis.So to arriveat the transfer function, we use this convolution property. Remember, now we won't ask this question. Come back to this question here. Where we returned to convolution and the property that Z-transform of convolution is a product,straight away we are able to see this result that you see in equation three,y of z is g of z time's u of z,provided all the Z-transforms exist, but that’s okay. It's not like, it's not as restrictive as Fourier transform. We were able to write this in the Fourier domain provided.What does it mean?For what class or systems we were able to ride this?

(Refer Slide Time: 24:06)



Stable.

Stable systems. Correct? Here I have no such restriction g of z exists even when g k is not absolutely convergent. What is g of z?Same story g of z is this. I've written g of z inverse, but please read it as g of z. The right hand side the summation converges, even when g k is not absolutely convergent. That means that exists a region of convergence.In fact as I said yesterday that is one of the reasons why Z-transforms was introduced, you'll see this in many textbooks that look Fourier transformcannot be defined for signals that are not absolutely convergent. So, another transformwas needed to handle signals that do not meet that requirement and that happens to be Z-transform. Why does the Z- transform exist and not the Fourier? For signals that are not absolutely convergent. Sorry. Any quick thought on it. What makes it so special?

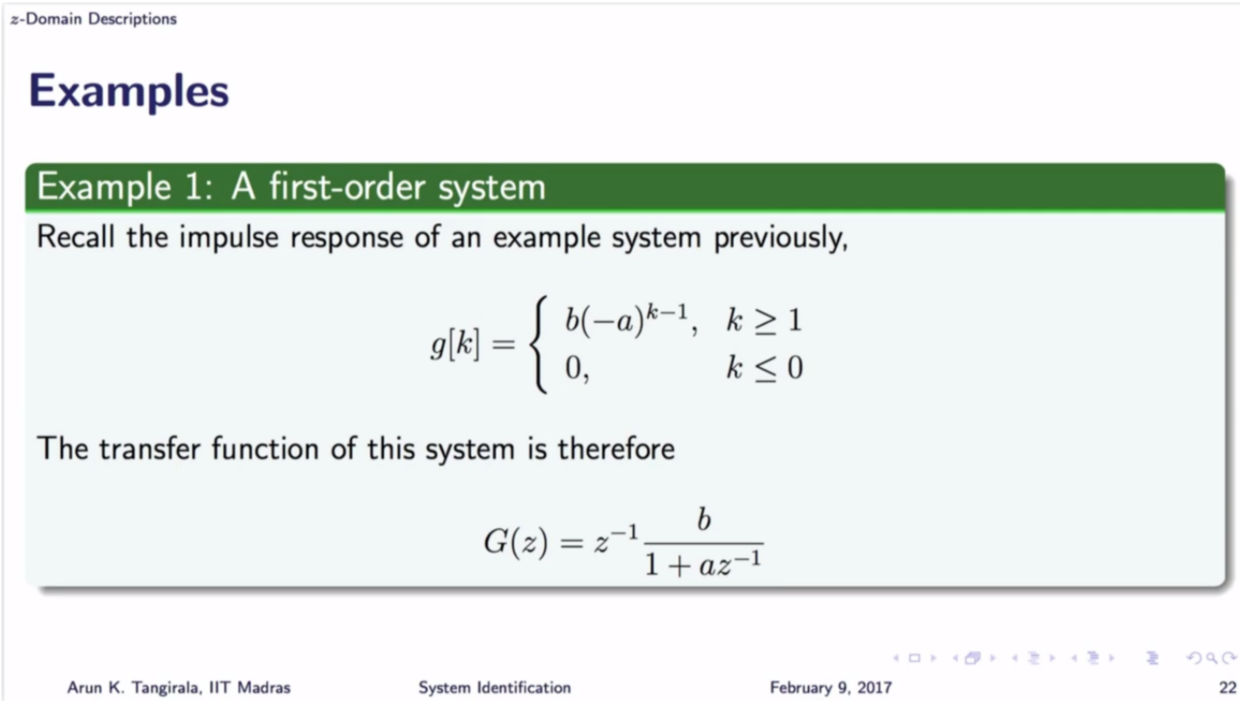
[25:18 inaudible]

Okay. Correct.

So the z has this e to the minus. So if you look at it, we have sigma plus g omegabut if you look at from a Fourier transform viewpoint. Sorry, the discrete in by replacing or evaluating this as Z-transform on the unit circle, right?Sorry, not n, j omega. This is gives you're DTFT, but in general for the Z-transformyou have e to the minus alpha.Sorry, times e to the minus j omega.This e to the minus alpha actually gives you the handle or the ability to accommodate signals that are growing with time.You just have to find a region corresponding to Alpha being greater than zero or you can write e to the alpha itself, if you wish.And then compensate for the growing nature of g k. Anyway, so, the beauty is that while frequency response function is only valid for stable systems, defined for stable systems.The transfer function is defined for all classes of system stable and unstable.So, like we have seen in the case of Fourier transforms the transfer functioncan be defined, sorry, in two different ways. One straight away from this result it follows that the transfer function is the ratio of the Z-transforms of the output and input.That is one result. The other result is that the transfer function, again from the same definition here is a Z-transform of the impulse response they are one and the same. Usually, in the first definition we do say that with zero initial conditions. Why? Because the transfer function is only telling you how the system responds to inputs, it doesn't tell you how the system responds to non-zero initial conditions, very important.It tells you how an input acts on a system, how does it respond. It doesn't tell you straight away. How does a system respond to non-zero initial conditions and no input?The free response part is contained but not fully g of z. You have to remember that all right.

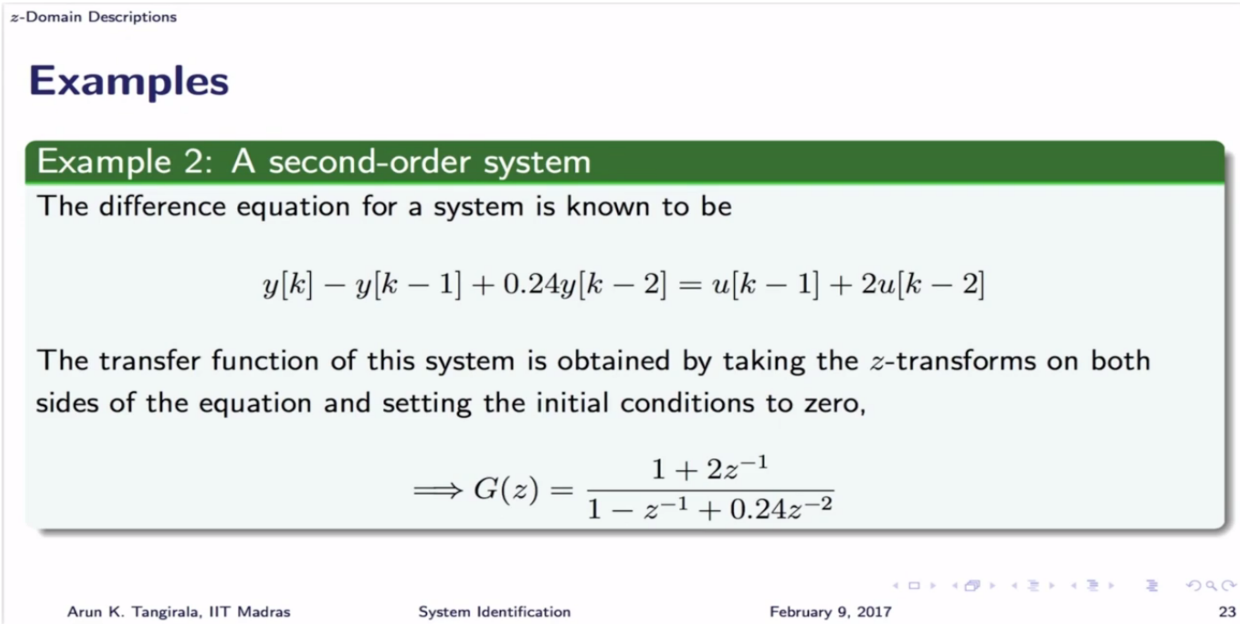
So as a simple example here I have a first order system, same first order system that we have seen earlier. The Z-transform has been written in this way, of the impulsive response that is your transfer function.Now notice very quickly that I have factored this into z inverse o time's b of b over 1 plus a z inverse. Why have it done that? Yeah, to indicate that there is a delay.Remember in the definition of the impulsive response that you see for the example, there is a unit delay.As a quick homework you have to ask what if there was no delay. That means it was b times minus a rise to k. With k starting from 0 then you wouldn't find the z inverse, you will simply obtain the transfer function as b over 1 plus a z inverse. It's always a good idea to factor that out to keep telling yourself, yes, there is a delay there. If there is no delay then you shouldn't see that, right. Likewise if there is a delay of two units you would have seen z to the minus 2.

(Refer Slide Time: 29:00)



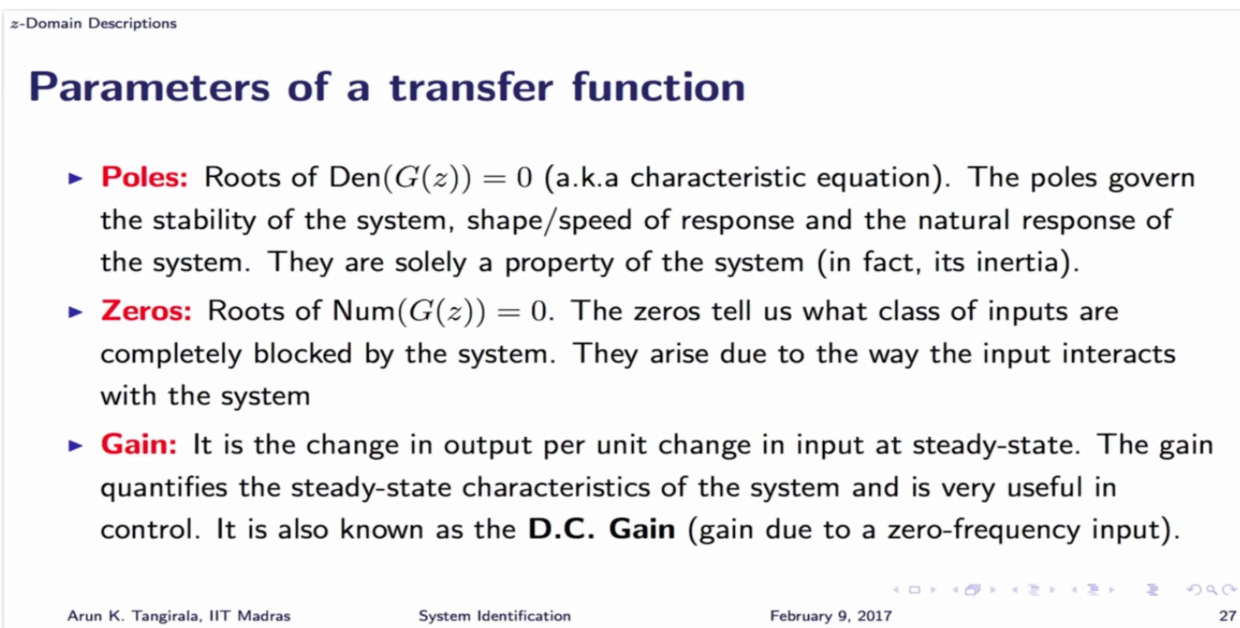
And here is another example of arriving at a transfer function straight away. You just up apply the Z-transform properties. Remember now, the difference the question is in a backward form. And we always derives the transfer function assuming initial conditions to be 0.Which means until the delay nothing happens? So you can use the property of the Z-transform of delayed signals and straight away write the transfer function. So here is where now all the properties are coming into play. We have already used a convolution property. Now we are using the delay signal property and so on. So that's your, it's a good idea always to write a transfer function in terms of Z inverses for certain applications. But when it comes to pole and zero calculations, it's a good idea to rewrite in terms of z. Remember this, okay.

(Refer Slide Time: 29:49)



So when we come back on Tuesday, we will close this discussion on how to use this transfer function to analyse the systems characteristics,particularly we look at Poles,Zeros, and Gain.We'll talk about this in detail with an example.

(Refer Slide Time: 30:12)



And then go on to state space descriptions, which will be the final set of descriptions in the deterministic LTI one. Thank you.