CH5230: System Identification

Discrete time LTI system 2

Today, for the first part of the lecture. What we'll do is we learn to rewrite this difference equation from using what is known as a shift operator. We have come across a shift operator earlier as well, when we were discussing the liquid level case study. But we look at it a bit more in detail today.

The shift operator itself, if you look at it, it's an operator. And when it's a shift operator I'm referring to the backward shift operator. Its role is to simply operate on a signal at an instant and produce an observation at the previous instant. So that is a result of operating or applying the shift operator to any signal or any sequence at a particular instant. And likewise you also have the forward shift operator, which advances the signal by one sampling instant, right. Before we proceed, it's good to ask, why are we now introducing this again. Already we have the difference equation forms. Isn't that enough?

Well mathematically, yes. That's enough, in the sense I can work with the difference equation forms without having to even introduce a shift operator. But rewriting the difference equation from using the shift operator helps us write the differential equation form in a compact way. That's the primary use of introducing the shift operator. The secondary users would be in the transfer function operator, that results as a consequence of applying this operator or as a consequence of writing the difference equation from using the shift operator. By looking at what are known as poles and zeros and so on. We can infer a lot of useful characteristics about the system. Of course normally these poles and zeros are discussed in the context of transfer functions rather than transfer function operators.

But as we shall learn today and partly tomorrow, the transfer functions and transfer function operate symbolically look the same.In fact even the coefficients would look the same. Of course, with a very important and subtle difference that we'll discuss later. So the bottom line is that we introduce a shift operator so as to write the difference a question form in a compact way.

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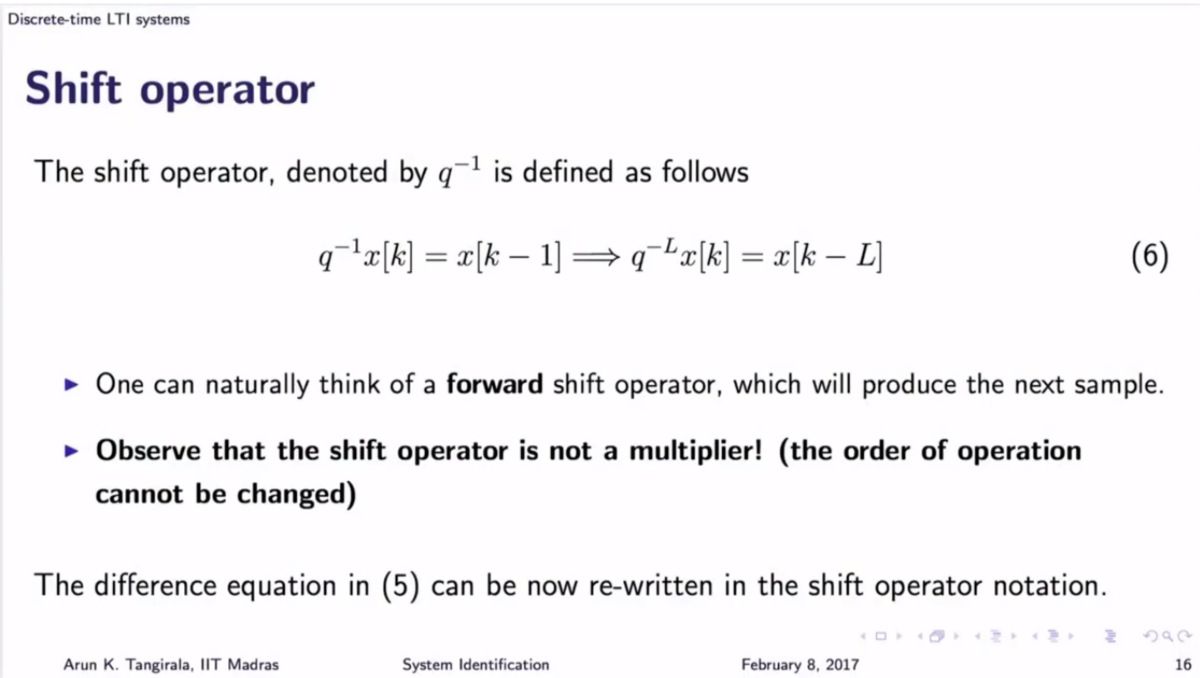

The application of the shift operator is fairly easy in differenceequations.

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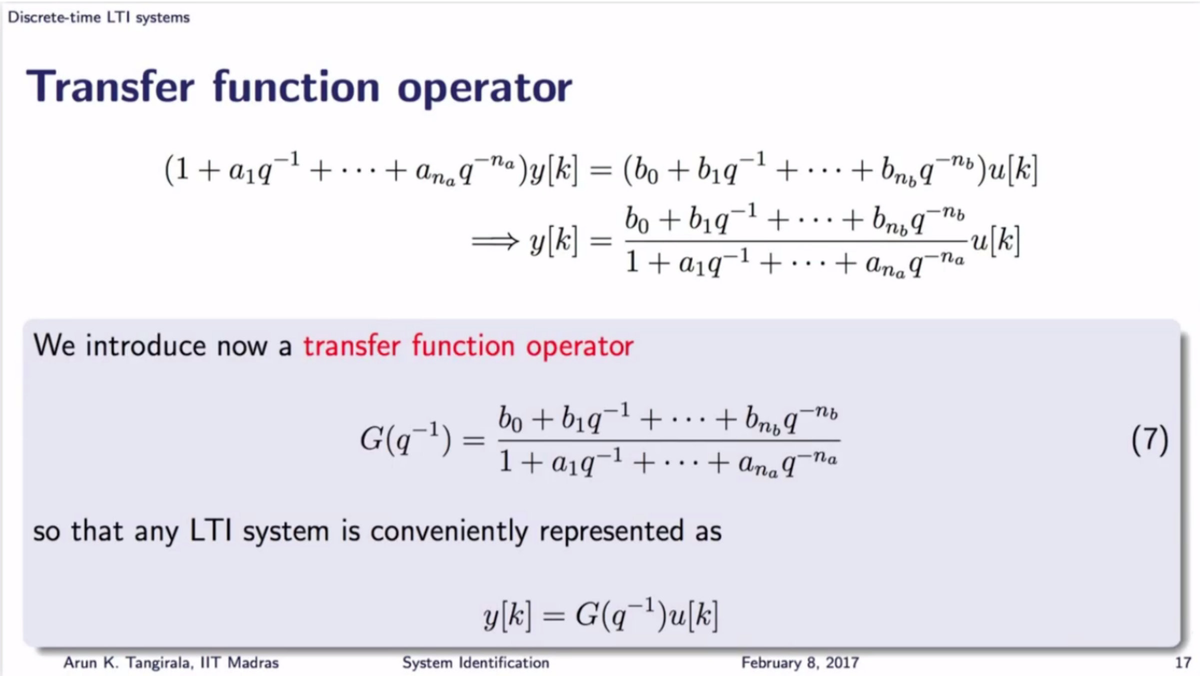

All you have to do is go to your difference equation, general difference equation that you see on the top. Remember we wrote this genetic difference equation from yesterday, which is of order n a and delay just being assumed to be 0 to keep things generic and that we have here n b past inputs affecting the system. Whenever you see a y k minus 1or y k minus 2 or a u k minus 1and so on,you replace those variables with q inverse, u k. So if I, q inversey k q, q inverse square y k n and so on.So if you see y k minus 1 you replace that with q inverse y k. And that's what we have done here. So taking you back to the generic difference equation form.

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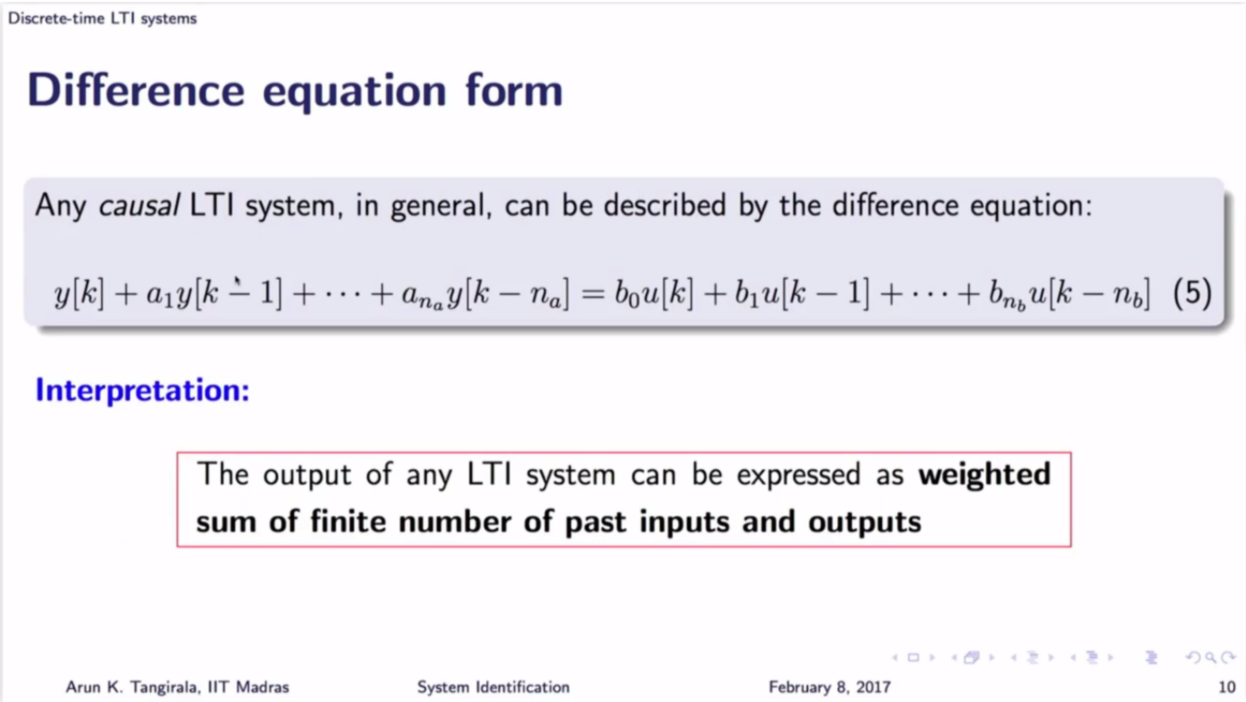

So here is a generic difference equation from, I replace y k minus 1 with q inverse and so on. Y k minus n a with q to the minus n a. Applying q inverse twice takes you to two observations in the past and so on. So that is something to be kept in mind. And applying a backward and forward once, that is in succession doesn't matter which order we will

keep you where you are. That is another thing to remember. Whether it is q inverse or q inverse qit will just keep you where you are. So that's it. So you just replace all this lagged variables with respectiveq to theto the minus operating on u k and then you arrive at this difference equation in terms of the shift operators.

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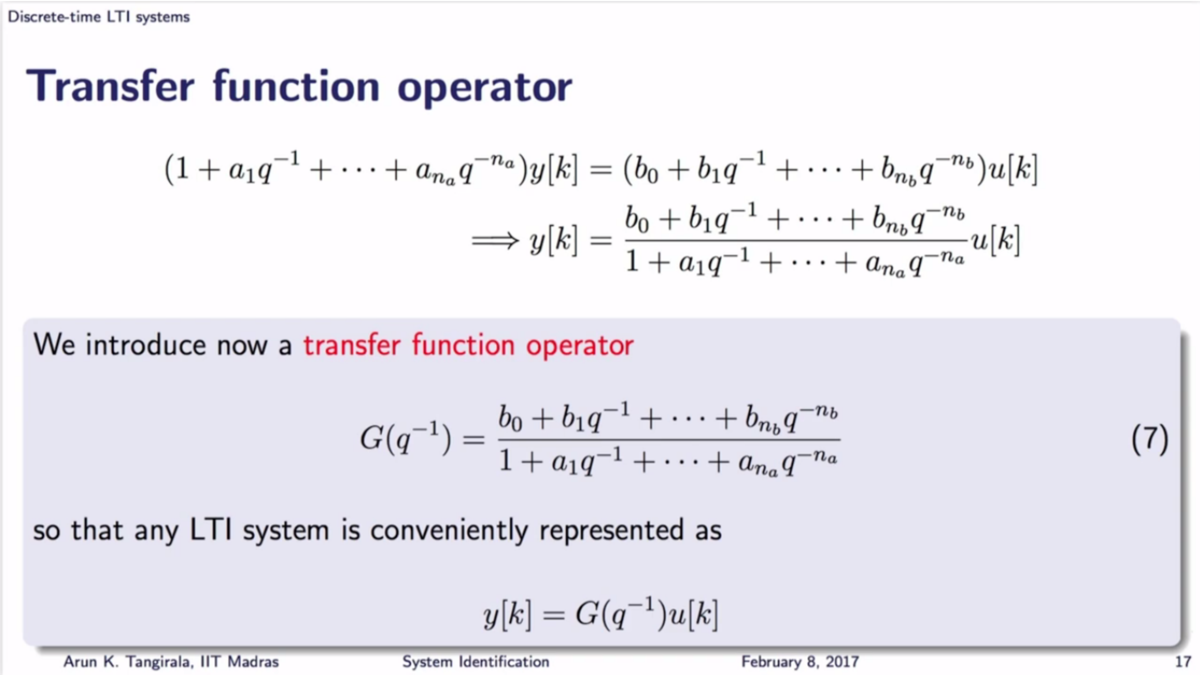

One should always remember that this q inverse is an operator and not a multiplier. That's very important, right. So when you're writing here 1 plus a 1 q inverse plus up to a, n a q to the minus n a although I have written here in parentheses, it's an operator operating on y k. And the algebra and the calculus of operators can be quite different from the algebra and calculus of multipliers. Later on when we talk of z domain presentations, we will see of identical looking expression with q inverse is replaces by z inverses. But the difference there would be z inverse, it would be would be a complex variable and then you can think of z inverse being multiplied with something. But the big difference would be that when we talk of z domain, we would be completely in the z domain.

Here, when we are using shift operators, we are still in the time domain. We have not gone anywhere. So here with the help of the shift Operator we can relay the relation between the input and output at an instant using this operator.

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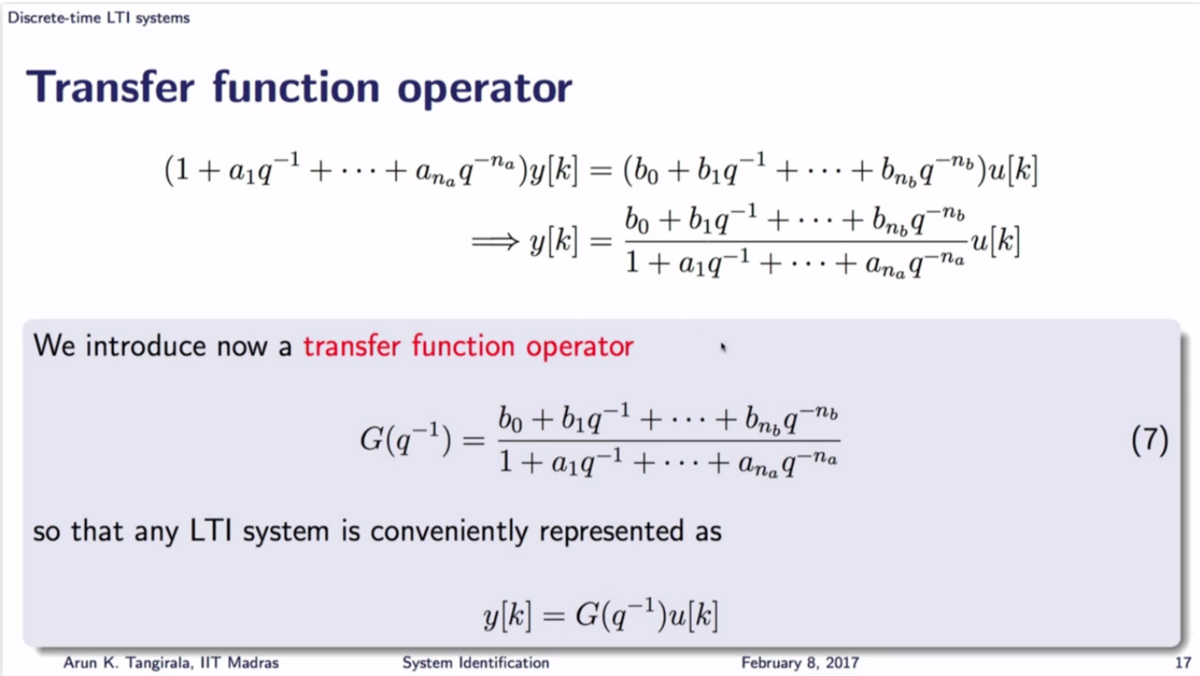

So you can say know y k is G of q inverse operating on u k.Many times you may see instead of G of q inverse, this notation G of q as well. At the moment we'll stick to G of q inverse and what is G of q inverse, expression is given here.

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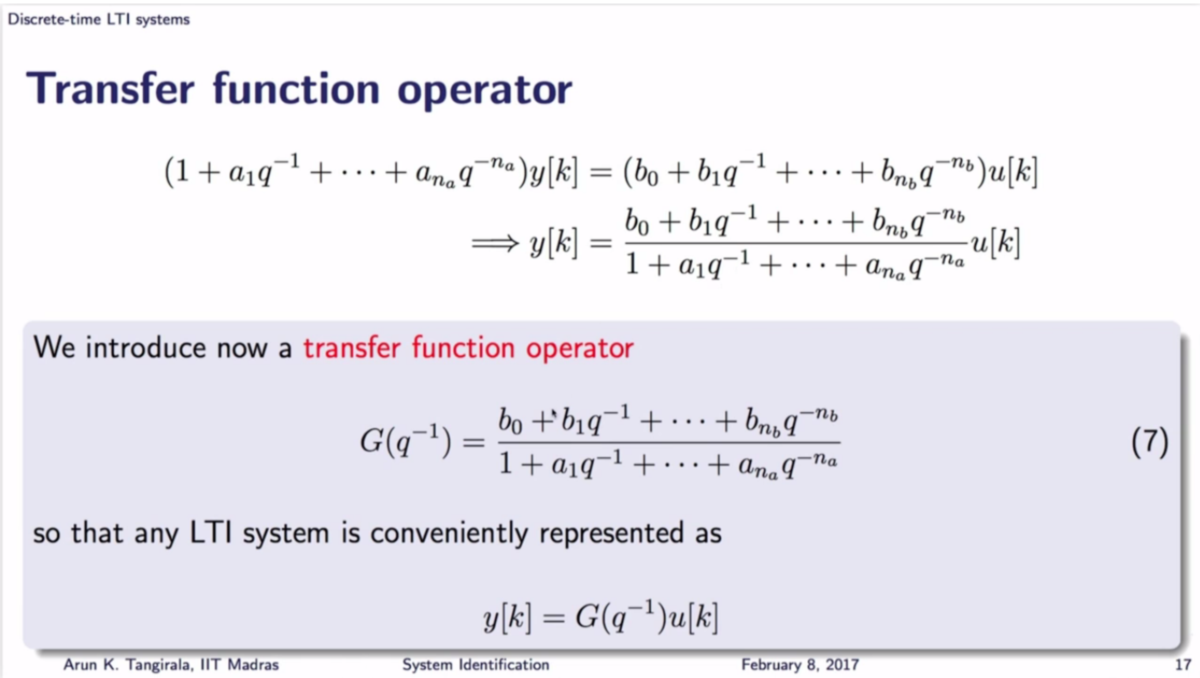

It is simply the numerator of G of q inverse is simply b0 plus b1 q inverse and so on and the denominator is one plus a1 q inverse and so on. Sometimes we may have a question as to why the coefficient on y k is always unity. I mean in a difference equation form, right.

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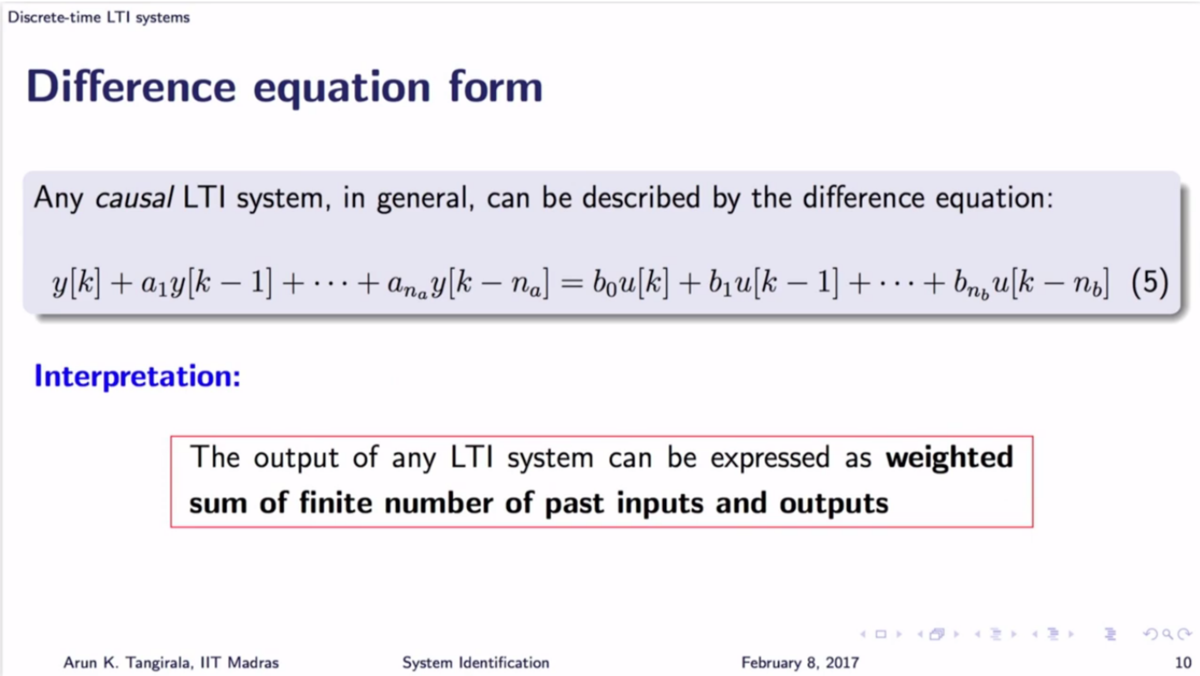

If yougo back to the difference equation from, while all the other terms in this difference equation have some coefficient which could be necessarily different from unity. Why is it that only y khas a coefficient of unity.The question may seem trivial but it's good to think over it. What do you think could be the answer?

Sorry.

Normalize the output.

Is that the reason?

Sorry.

[7:35 inaudible]

So why can't I have y0, y k?

[7:47 inaudible]

That's okay. I can do it. Need a stronger answer. The reason is when I have a0(not)y k there, thereis no unique way of writing this difference equation. That is I will run into identifiable issues. In other words, if I have a0(not)there you may give me a solution with some a's and b's, I can claim that there exists another solution which has alpha a's and alpha b's, right. If I haven't a a0(not) there and you claim that there is a solution with a0,a1 up to a n a and b0, b1 up to b n b, I can also say that alpha times a0, alpha times a1up to alpha times a n a and alpha times b0 and alpha times b1 and so on, it is also a solution. Which means that is not unique answer. In other words we cannot afford to have a a0 or a coefficient on y k because we cannot have a unique answer to it. So that means we run into identifiability issues. Therefore for identifiability reasons, we freeze the coefficient on y k to a0. Of course, the other reasons that you've mentioned do apply. For example, every time I have to compute y k, I have to divide a0, that's all secondary reasons. Primary reason is identifiability. There is no way I can write a difference equation uniquely, if I have a coefficient a0 on y k. All right. You can also say well it is natural to write an expedition for y k rather than a0 y k, all that is acceptable. But from an identification viewpoint, we do not have a coefficient for identifiability reasons.

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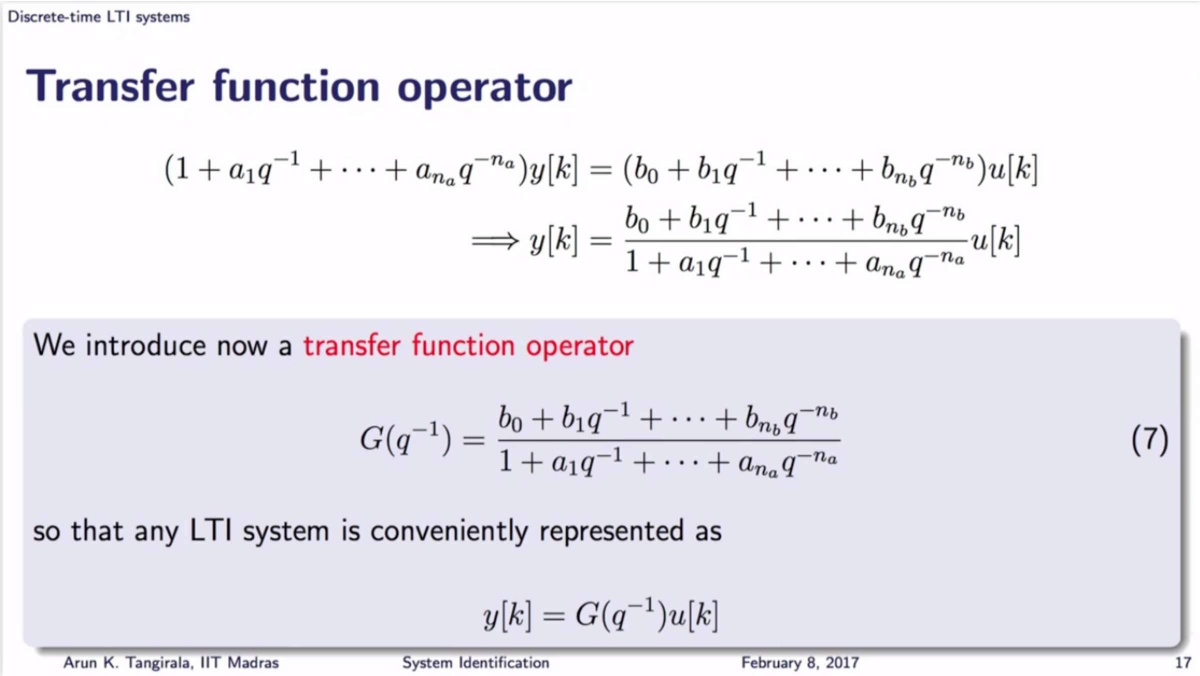

So the reason I point this out is because if you look at the G of q inverse, you'll notice that the numerator, the leading coefficient on the numerator is different from unity.

Well, in general it could be unity. Whereas the leading coefficient in the denominator always has a unity coefficient. You don't have to estimate it, in other words. Okay. So, that's a default. You can assume that always the first coefficient is unity. Okay. So G of qinverse is a transfer function operator. And again it is not a multiplier. If I give you the difference equation form you should be able to straightaway write the transfer function operator and vice versa. If I give you the difference transfer function operator, you should be able to visualize the difference equation. It's a very straightforward thing to do. Okay.

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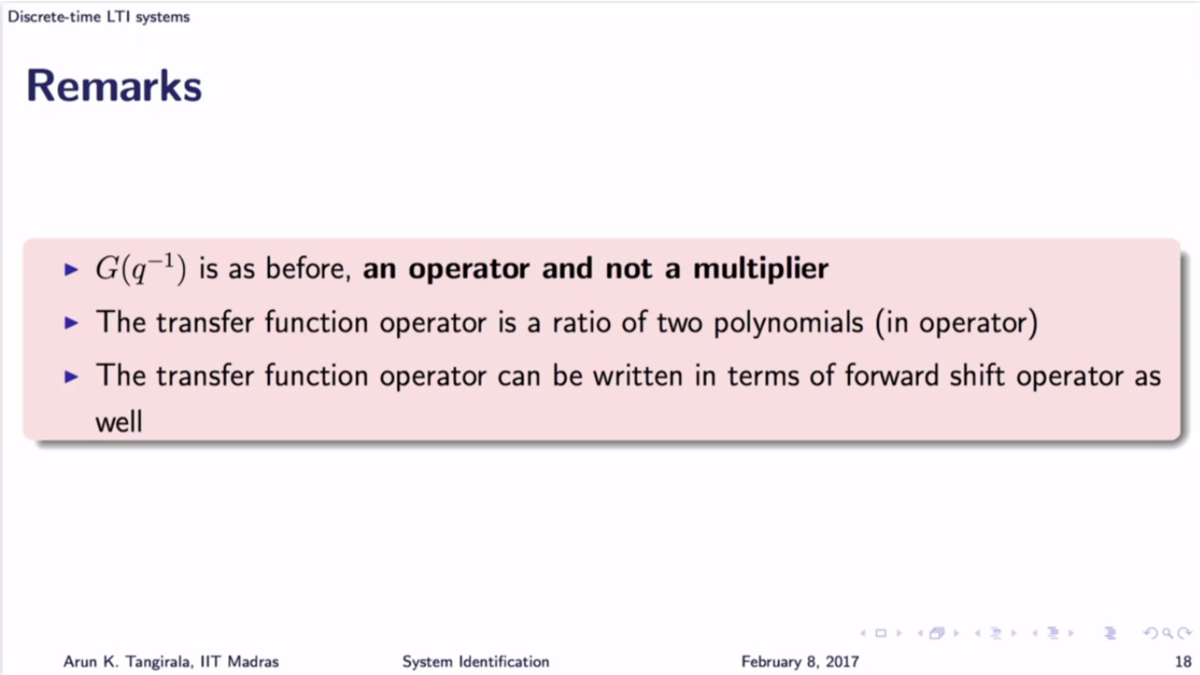

So it's a ratio of two polynomials but polynomials in operators, not polynomials in variables. You could also rewrite this in terms of the forward shift operator. Nothing prevents you from doing that. Right. You could write for example, y k as q to the n a, y k minusn a. But it's not so useful to us. What we wanted to relate is the input at this instant and output at this instant. That is what we want to relate. As a simple example here of converting a difference equation form to a transfer function operatorit's very straightforward and self-explanatory.

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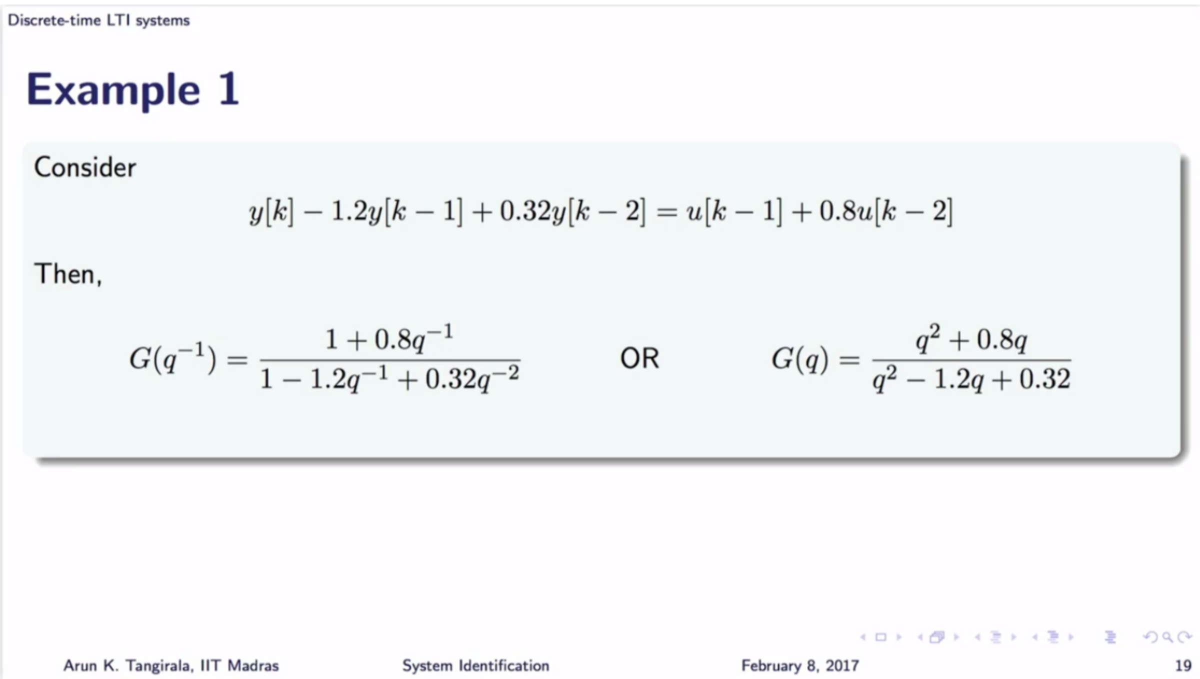

Here is a differential equation given to you,no initial conditions have to be specified. We don't need any of that to write the transfer function operator. It's very easy. Of course I'm showing you both forms here. Transfer function operated return in terms of backward shift operator as well as forward shift operators. Nothing mysterious about this.Any questions on this example? Okay.

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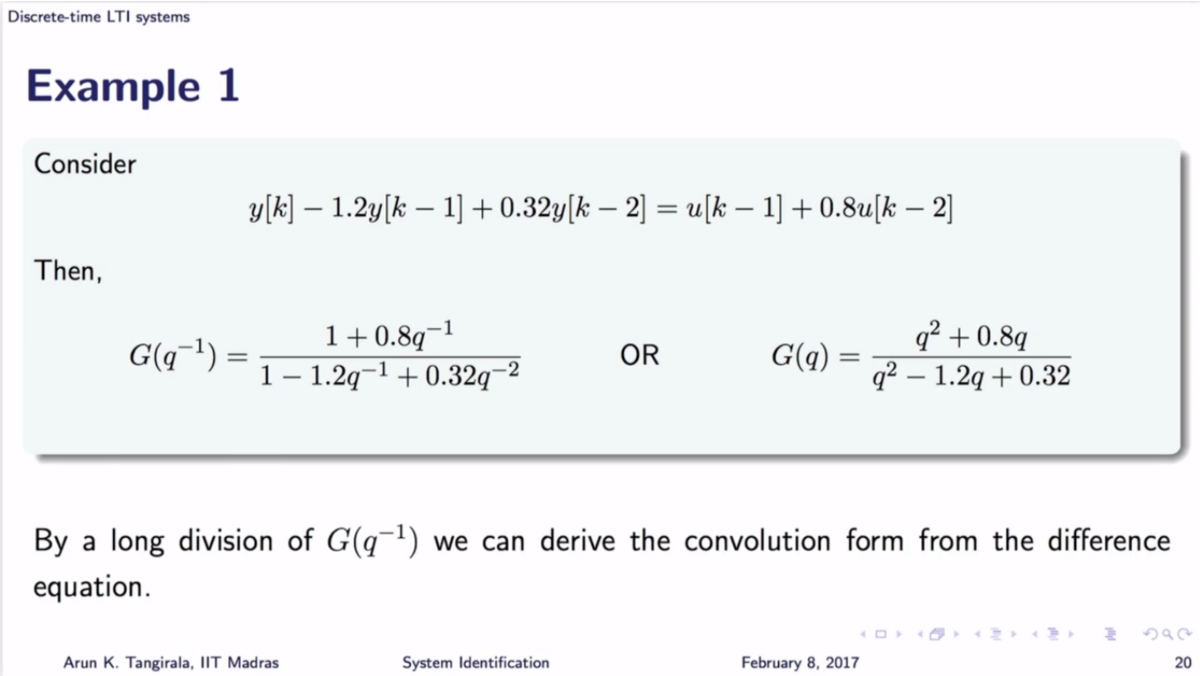

So one of the interesting things about using the shift operator, one of the advantages of using this shift operator is, by a long division method I can recover the impulse response coefficients without having to solve the difference equation. I asked you to obtain the impulse response coefficients. The natural ways to inject an impulse into the difference equation and keep solving recursively. But fortunately there is another way of arriving at the impulse response coefficients from the difference equation form. To understand that, first let us look at the convolution equation. We'll go back to the convolution equation, we'll assume causal systems. Here also I can rewrite this as G of q inverse u k. What is G of q inverse now? In this case what would be G of q inverse? Why is it so much difficult? Again you apply the same for a backward shift operator. Sorry.

Summation of--

Summation of--

[13:27 inaudible]

Q power minus n and --

[13:34 inaudible]

Very good. That's it. So you can say, n or k dummy variable, it doesn't matter. Is it clear. How do we arrive at this. Simply replace your. You can rewrite this by the way. I mean, how did we arrive at this. We could first rewrite this as g k minus n, u n. Okay.Remember k and n are dummy variables.Don't get confused with this interchange of k and n and so on. They're just dummy variables. So if you don't like k here, I can write n doesn't matter. So when you write here, by the way what would be the summation limits here?

[14:31 inaudible]

0 to infinity.

[14:42 inaudible]

Minus infinityto 0.

So, just, I mean if you're confused just replace n, right. If you don't like, you know k here or

[15:00 inaudible]

Sorry. K to infinity. Have you arrive at a consensus. So what is the final answer.

[15:19 inaudible]

Yousaid, k to infinity, minus infinity.Minus infinity to k. Is that is the final answer?

Let me help you with this. So here suppose I had minus infinity to infinity. Then what would be the case? What would be the limit here?Will the same limits may preserved. Correct. So the same limits would be preserved.

Now bring in causality. What is causal, what is the property of a causal system, the impulse response coefficients are 0 at negative instances, correct.Now modified this summation limits.0 to infinity.

Sure. Why there's so much confusion. K to infinity.

If it is k to infinity,you will have terms containing G at negative things, right.

What about minus infinity to k?No. You just expand this summation, right. When you expand the summation you do not want to have terms that involve G's at negative times. Right. If you go back to this, when we say the system is causal, we say that y k does not depend on any input beyond u k. It can depend on uk, uk minus 1 up to you minus infinity. And that is why, I mean one way of looking at this we say that we restrict here, they freeze the lower limit to zero. Because if n is negative then y k would depend on a future input, right. There would be a term involving u k plus 1, uk plus 2and so on. The other way of looking at it is, I have a system causal system and its impulse response coefficients at zero at negative times. Whichever way you look at it, it's the same thing. Is that clear? So you would apply the same logic to the summation here. Minus infinity to K correct.

In general to avoid confusion many times you may see in many books G of q, this convolution equation return from minus infinity to infinity, so that you don't get into any of this and you bring in the notion of causality at some other time, right. For example, again if I had minus infinity to infinity here, then as I mentioned earlier there would be no issues here. The same limits would be preserved. In which case, G of q inverse would be simply running from minus infinity to infinity. At this point you can bring in causality, right.

What does causality imply?G is 0 at negative times. And therefore we go back and say well G is simply this. So this is another way of defining the transfer function operator. What is the difference between this G of Q inverse that you see and the G of qinverse that we gave here, earlier.

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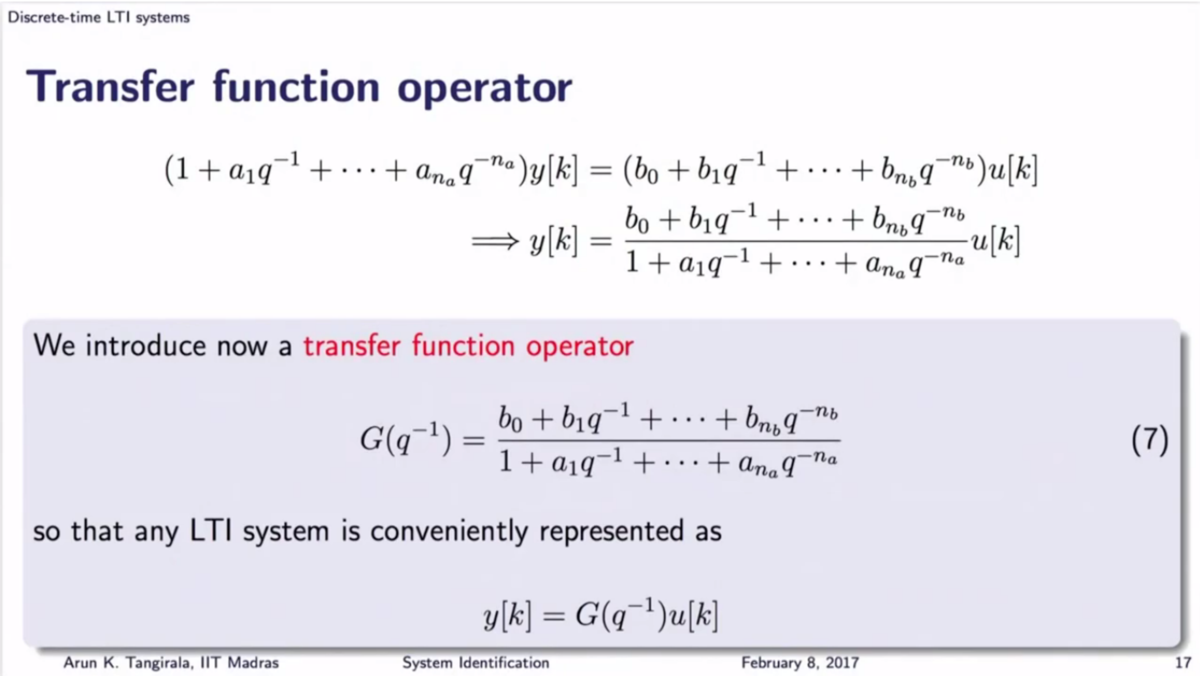

Thank you. But a bit more.Correct. So if you are thinking of a non-parametric form then this is good. Because I don't assume any structure on G. If you have already parameterized, if you already have a structure or a difference equation from, then G of q inverse has this rational polynomial from.

Now what I was trying to tell you earlier through this example is that for this difference a question for example, I can drive the impulsive response coefficients by performing a long division of G of q inverse.

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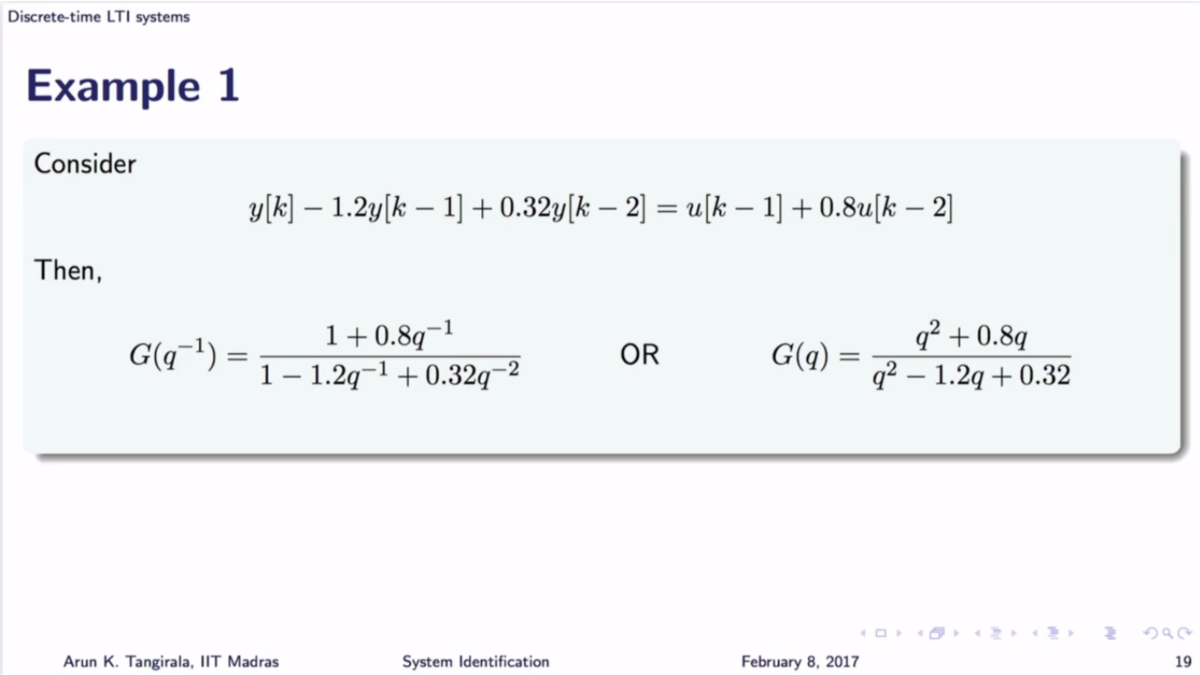

What is meant by long division?

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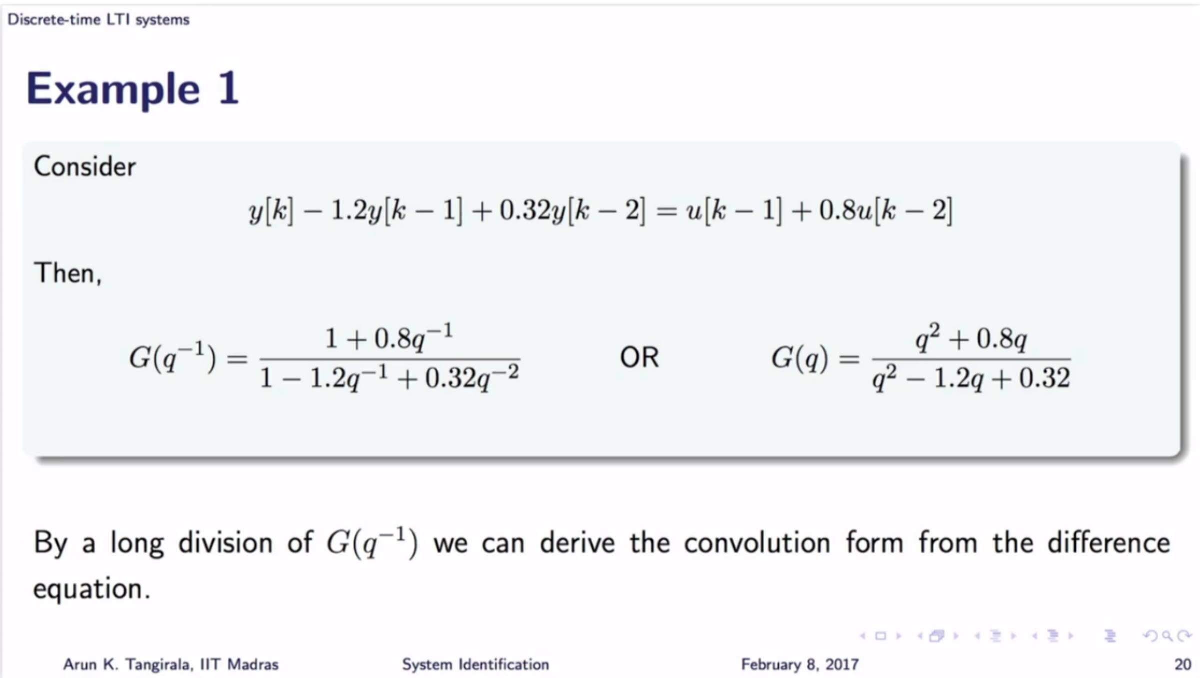

I write G of q inverse as an infinite series. Right. How do I do that? Simple long division.Right. Why would I want to do that, so that ultimately I would be able to write G of q inverse in this infinite series form and then simply read off the coefficients of the polynomial. Those coefficients would be the impulse response coefficients. So the idea behind using this transfer function operator to determine the impulse response coefficients given a difference equation form is to rewrite this G of q inverse in an infinite series form using long division and then read of the coefficients. That way you do not have to solve this difference equation, if you were to solve the difference equation by the way what would be the impulse response coefficients?

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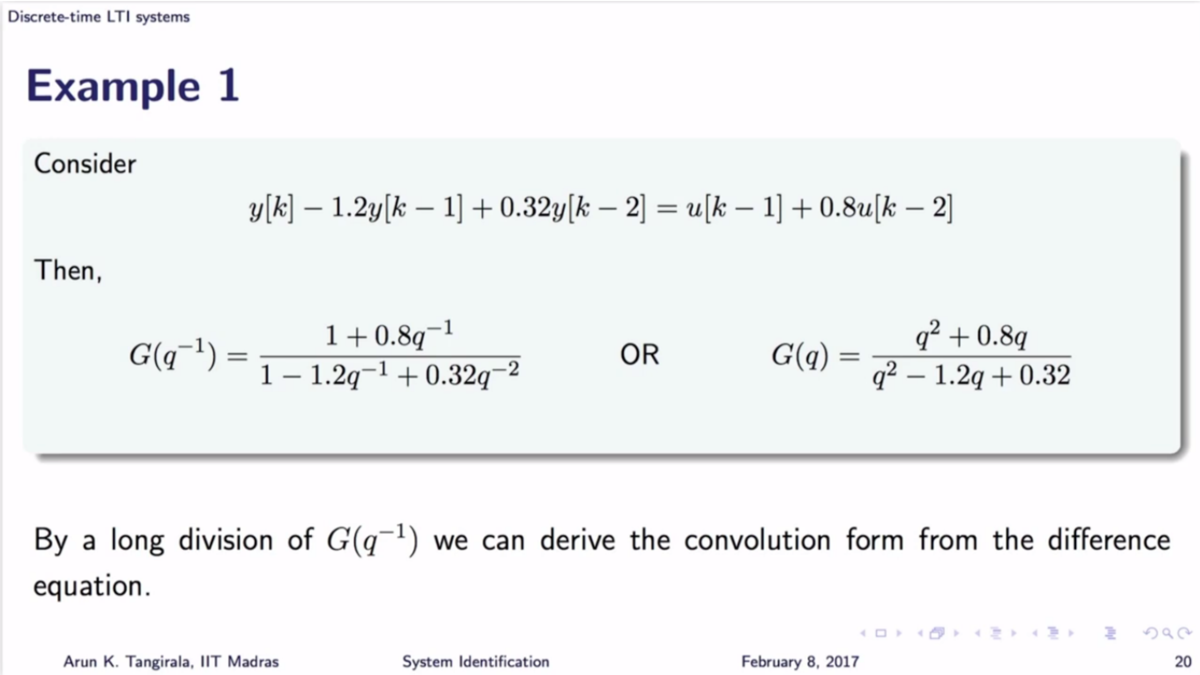

Maybe I have. Let's take this example, there is one more example that simpler than this. So let's take that example on the screen.

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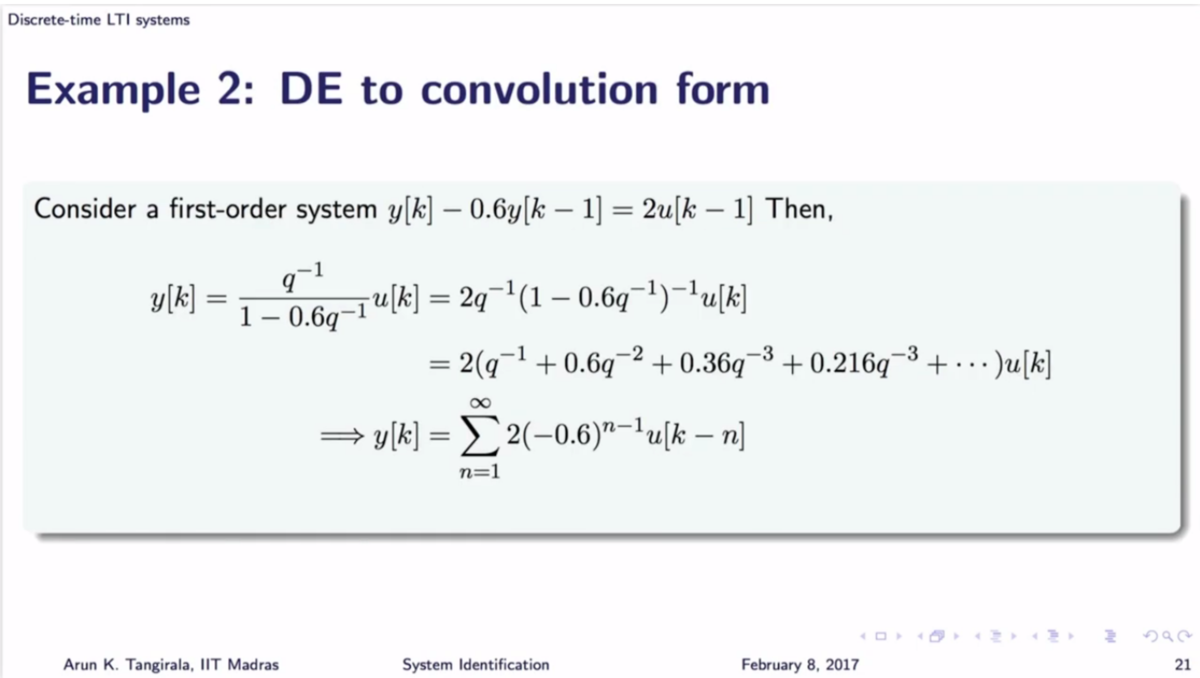

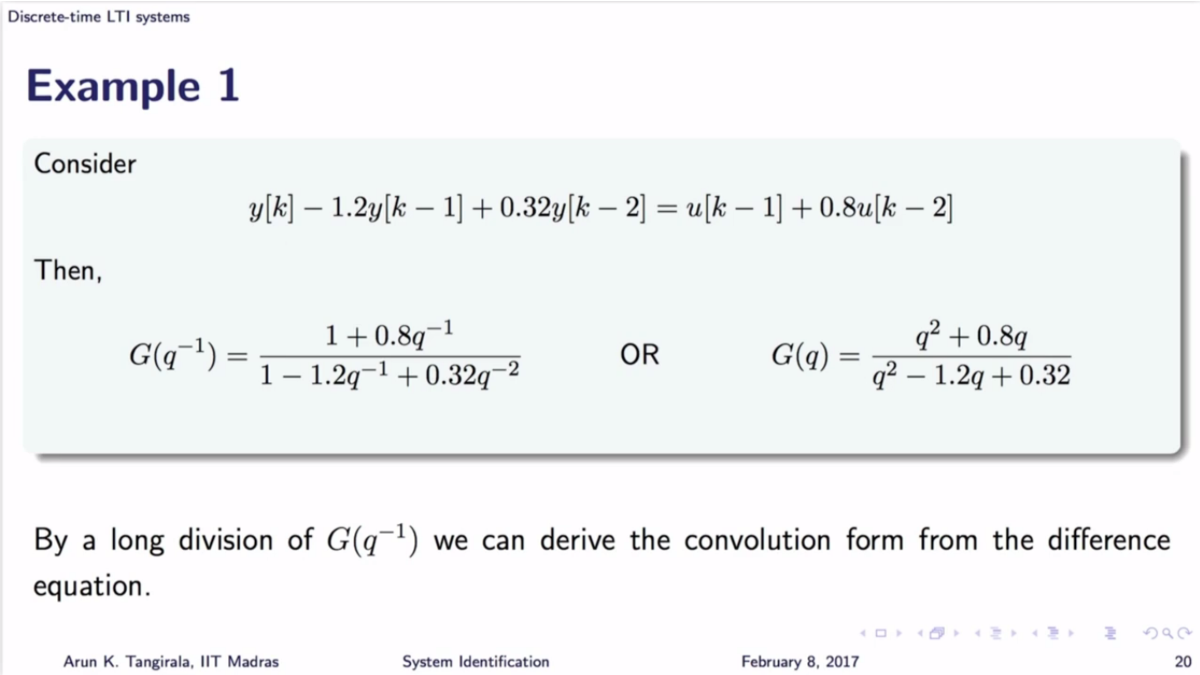

If you were to solve using the traditional way, what would be the impulse response coefficients? What is the difficulty? All you have to do is to inject an impulse into the difference equation. And assume that the system is at zero initial conditions, relaxed. What would be the impulse response coefficients?

Not using the long division approach, using the traditional approach. What would be the g0, g1?g0, you should be able to say straight away by looking at the difference equation. Yeah. There is no need to think because already there is a delay of one sample.

Sog0 is 0.

Good.

g1 would be?

Good.

2, sure? How do you get that?

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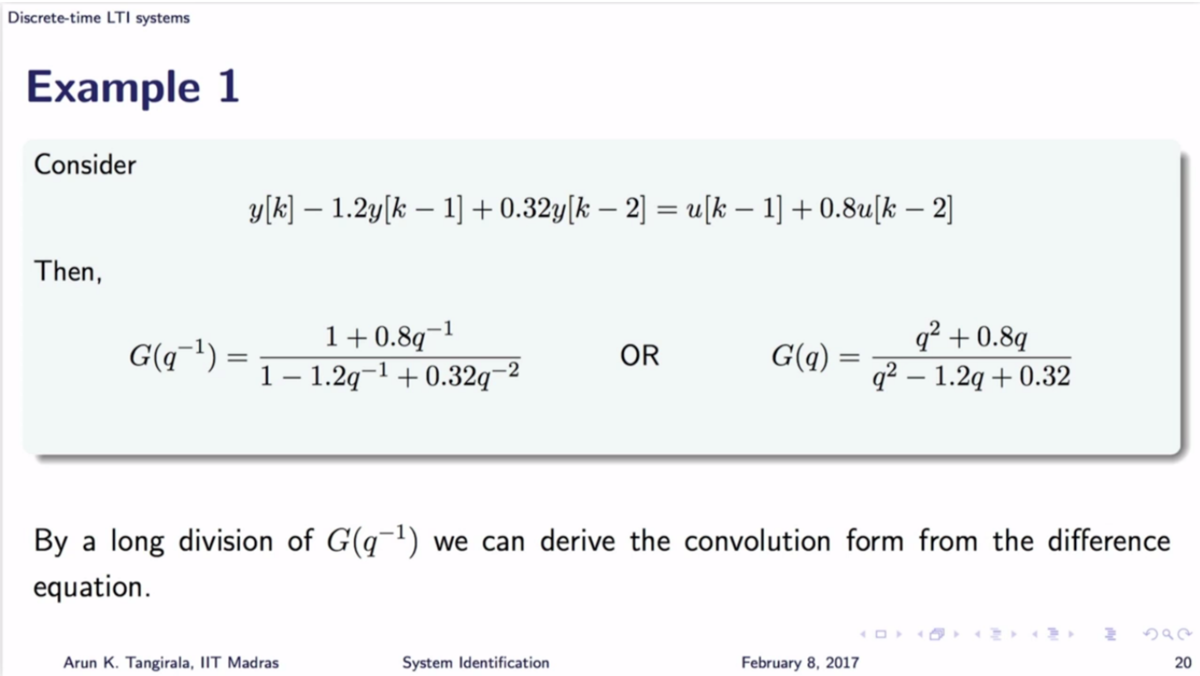

So at k equals to--

[23:40 inaudible]

Correct. 0.8 plus 1.2, that's correct. We'll just write one more and then go to the long division approach. For those students who are just watching, I think you should get into the habit of solving. What about g3? At k equals3, any way the right hand side vanishes. Right. You're left with, so at k equals 3, you have 1.2, y at 2. I'm sorry.

2.08

2.08. Okay. Fine. So we'll stop at g3 now. This is using the traditional approach. Now let's do the long division approach. How many of you are not familiar with the long division approach? Okay. That means you're not. Okay. It's simple division of polynomials, that means you had the denominator polynomial. What is the numerator polynomial of G of q inverse it is 1 plus, what is it? It's on the screen.

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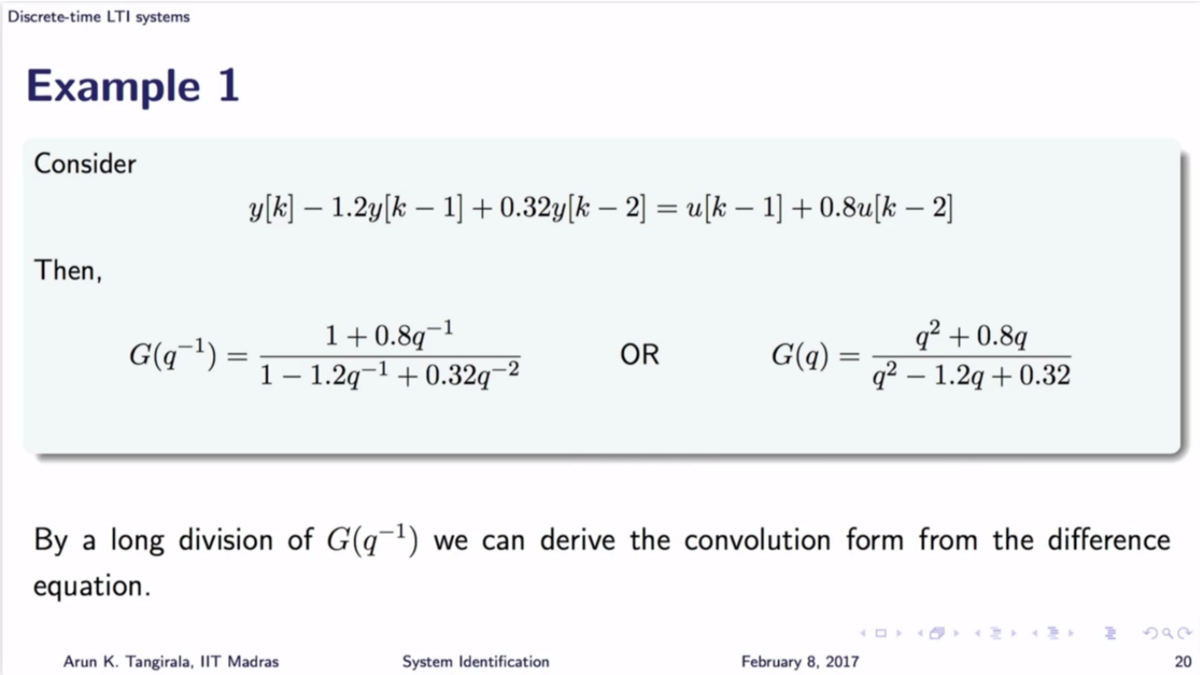

0.8 q inverse. What's the denominator polynomial? You have 1 minus 1.2 q inverse plus 0.32 q inverse square, divide as if you would normally divide, You know, the regular division. It's going to be an infinitely long for the quotient is going to be infinitely long polynomial, right. So what would be the first term in the quotient? 1. Good. That's because you have a 1 here. And then once you do that you have 1 minus 1.2 q inverse plus 0.32 q inverse square and you subtract, your left with 2 q inverse minus 0.32 q inverse square. What would be the next term in the quotient? Good. Right.What would be the next time in the quotient as we proceed this way.

[26:03 inaudible]

Sorry. Be careful. You're going to multiply, so you have to multiply it carefully here. So the first term here would be 2 q inverse, fine. Then you have a minus 1.2q inverse multiplied by--

2.00.

Right.So that would be minus 2.4 q inverse square and then plus 0.64q to the minus 3. It's too detailderivationI'm going through. I call this as the baby steps. But it's okay. We are all babies at some point in time. So what would be the next coefficient here in the quotient, next term?

2.08.

2.08. So do you see something emerging.Right.Maybe the next one would be a minus sign. And so on so, your G of q inverse is simply this quotient. It's a never ending polynomial. And that would be 1 plus 2 q inverse plus 2.08 q inverse square minus and so on.

You should verify. In fact, yeah that's correct. Something wrong. Yes. I'm sorry. The numerator, you should have corrected me. Why you all kept quiet. Don't take my word as a final verdict. Which means this entire thing, I'm I right. So, G of q inverse is q inverse plus 2q inversesquare and so on. Right. Is that we're until this point just compared with this expression here. g is a scalar, so you can actually interchange. You can simply rewrite this as g of n, q to the minus n. That doesn't matter.Whether you have q to the minus n times alpha, I mean operating on alpha or alpha times q to the minus n operating doesn't matter. That's the same. So, now compare terms.

What do you get for the first term.What do you get for g 0? 0. Because there is no constant term, right.Then you have g1 as 1 and then g2 as 2, g3 as 2.08 and then you can verify a couple of more coefficients as a homework. Is this clear, now. How you use the long division approach to derive the impulse response coefficients. The advantage is that, you know can compute the impulse response coefficients in a very elegant way. But you may not be able to see a generic expression, right. You may not be able to see it arrive at a generic expression until unless you're able to visually see. What I mean by genetic expression is. You may not be able to tell me, give me a symbolic expression for gk. How do you arrive at a symbolic expression for gk?

Suppose I ask you derive the general expression for the impulse response for the system. How would you do that? What would be an approach? You have to go back to the theory of difference equations same as the theory of solving differential equations. You would have a homogeneous part and then acomplementary one, right, complementary solution.

The homogeneous part comes from the system the you have a particular solution that comes as a part of the input. So you have to put together both that and get you a generic expression for impulse response.

So when you want a generic expression for gk, this long division approach may not be so amenable. You can still derive but you may have to go through a bit more math to get it. But if you just want the first few impulse response coefficients or you on just some particular ones, you can get it very easily. Okay. So this is a way of going back and forth between convolution and difference equation forms. Clear. So this example I'm gonna skip because this is a much simpler one than what we have taken up.

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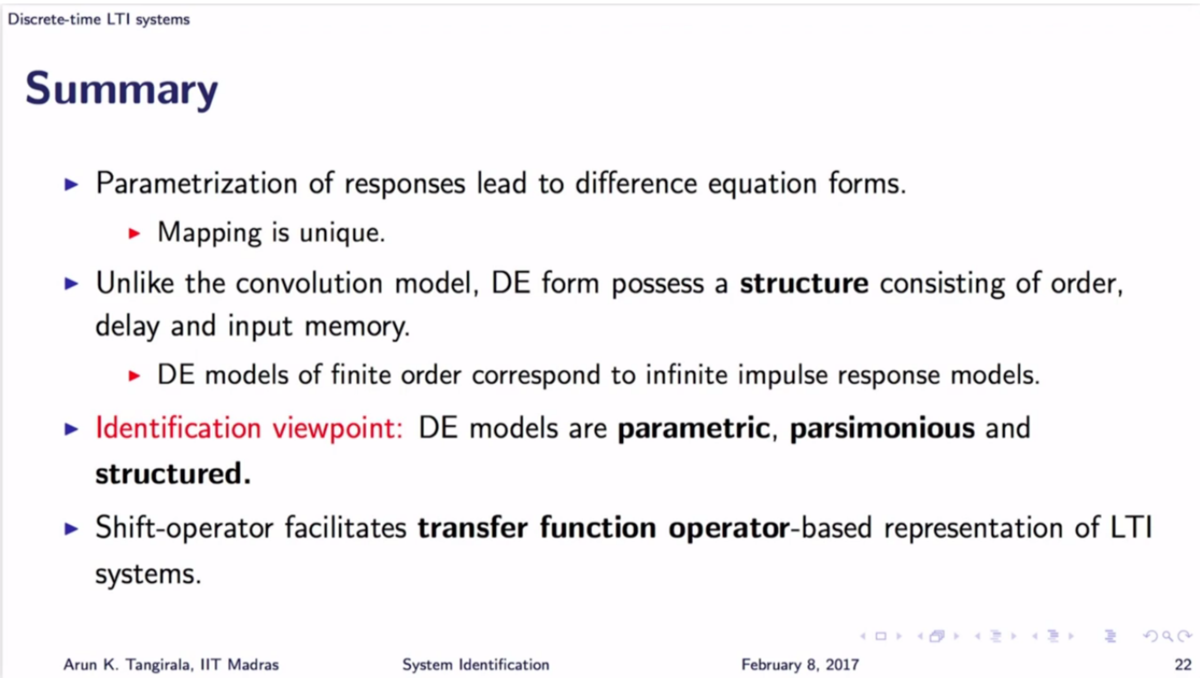

So to summarize now,parameterization of response based descriptions leads us to difference equation forms. You could parameterize,impulse step or frequency response. You can take any of those and do that. And unlike the convolution from the difference equation from possess a structure and from an identificationviewpoint that means the user has to furnish additional information to estimate a difference equation form. And that additional information is in the form of delay order and input memory.And again from an identification viewpoint this difference equation forms are parsimonious for a given system and structure and finally the shift operator notation allows us to write the difference equation forming a compact way to introduce the notion of transfer function operator.

I don't know if you've already thought why that name has come through. If you look at this expression here,G of q inverse in some way quantifies how much of the input is being transferred to the output. But it is not a number, it is not a variable, it's an operator. Therefore it is called a transfer function operator.It's a function of your q inverse. When q inverse being an operator, it's called a transfer function operate. And using this G of q inverse one of the advantages is that we can easily switch to convolution forms from difference equation forms.

There are other advantages of working with transfer function forms which we will see shortly. Now that will enter the world of z domain. Any questions now? So, we are now going to move from time domain to z domain. We have already done one such exercise, we had moved from time domain to frequency domain.And what was the motivation for moving into frequency domain because we wanted to describe the response of the system to sinusoidal inputs. Now you can think of moving to z domain. One of the motivating factors has describing the system, the respond to the system to damped sinusoids.You can think of it that way also. Some textbooks were presented that way, in fact in my textbook also I presented that way. That is one of the motivating factors for looking at the z domain description of LTI systems is that we want to describe the system response to damped sinusoids, not just to pure oscillatory inputs.

But a lot of times the motivating factors, there are other motivating reasons and top one among them is the ease with which we can analyse the LTI system behaviour. The ease with which we can compute the response. So there are so many advantages of moving to a new domain which we call as a transformed domain description. In fact in the literature you should be, if you read the body of literature on system identification, you will see this phrase called transform domain representations or transform domain descriptions.And always one of the reasons that is given is that the mathematics in the transformed domain is a lot easier when it comes to describing LTI systems.