Estimation Of Frequency Response Functions

Part 2

So let's look at quickly the properties of the ETFEunder certain conditions.

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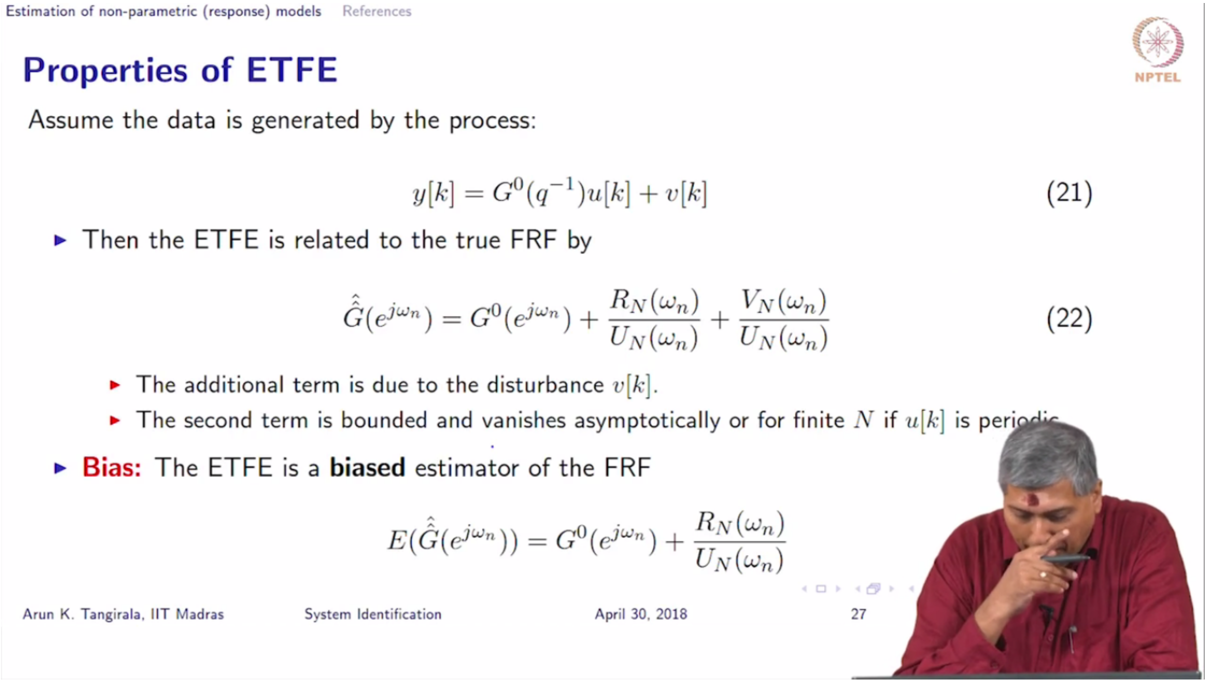

So let us assume that the data generating process is this,y k equals to G(0) not u k plus v k. v k is as usual our stochastic term in y due to the effects of unmeasured disturbances and noise.Then the ETFE now is related to the true one with some error. You can now recognize this two terms. The first time what is it due to. What is a first term due to? I'm sorry.

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Remainder term.Remainder term that we had on the noise free condition. So this term itself is due to an approximation. This has got nothing to do with the noise. The second term has got to do with, what is there at the numerator.

Noise.Right.So this contribution is due to noise. So even if this was not there, this noise term was not there the approximation error would prevail. And that is what we have been saying. Okay. I'm just reiterating this fact many times so that it becomes easy for you to understand and remember. So for noisy data we have an additional term. Now the question is as N goes to infinity. Do these both error terms go to 0.What do we know from our earlier analysis?Does the approximation ever go to 0as N goes to infinity? Yes. So the question is whether noise ratio of the DFT of noise to the DFT of the input. Does that go to 0?

But remember since we are dealing with stochastic terms, we may look at on [02:06 inaudible]averages. So the first thing obviously we look at is bias. It turns out that when you work out the bias, so you take the expectation on both sides of this equation 22. The first term is simply a constraint. The second term you have remainder. Because expectation of this first error term is not going to be 0 because that has got nothing to do with noise.It's expectation of that is that term itself. The third time expectation of that is 0.Because we assume that expectation of VK 0 and therefore expectation of V N Omega N is also 0. There's a very simple derivation. You just have to put in the definition of DFT of V N and take the expectation and apply this fact. And you can easily show that on the average the DFT is a 0 value. Okay. So the third time vanishes. Remember the input is assumed to be deterministic. But the second term which is actually the first error term prevails. Clearly saying that for finite N, the estimator has a bias.

But the good news is that we have already learned limit N going to infinity. This term is going to go to 0. Which means that this estimate is asymptotically unbiased. Okay.

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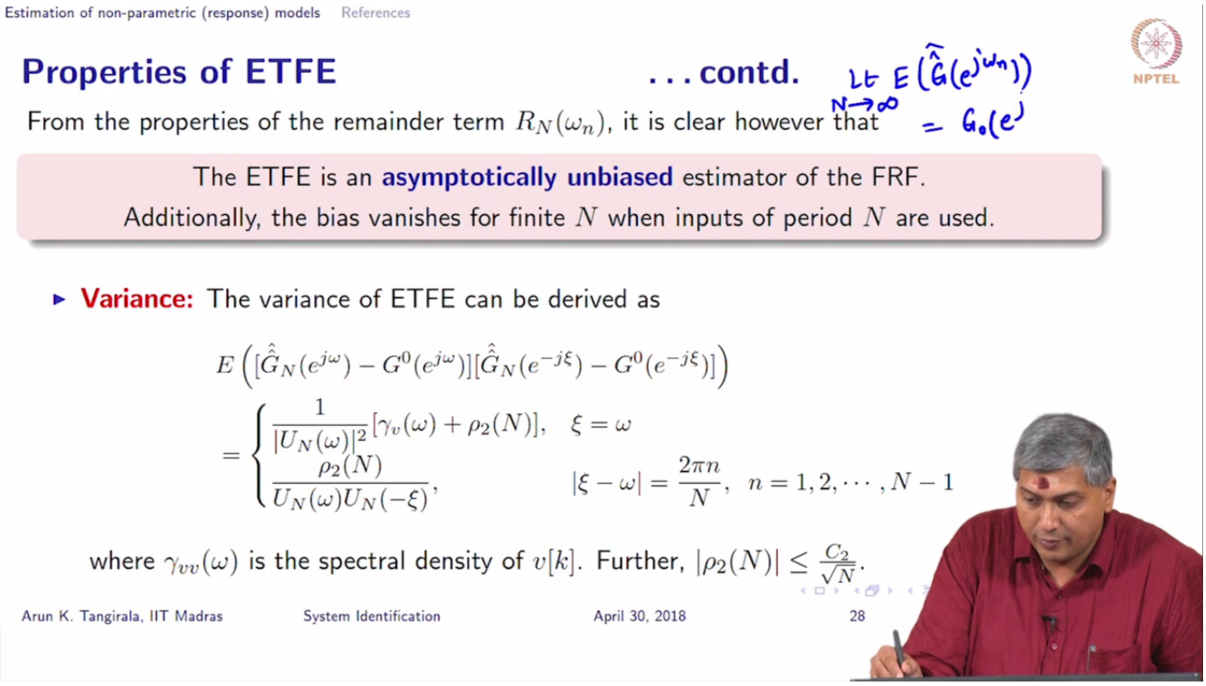

That is limit N going to infinity, in the limit Nas goes to infinity expectation of G double cap is simply G not.So now that is proved beyond any doubt. We are quite convinced.Asymptotically unbiased is acceptable. After all there are quite a few estimators that we use which are biased for a finite sample but asymptotically unbiased. What about variance because that will now determine whether the estimator is consistent or not. Because consistency, typically we're looking atmean square error consistency. So mean square error of a parameter of an estimate is the sum of bias square plus variance, right. So this is bias in theta hat square of this.

And what mean square consistency demands is limit N going to infinity, MSE of theta hat should go to zero.Which means individually, the bias should go to 0 and the variance should go to 0. Bias of course goes to 0. We have already shown. Question is now, variance goes to 0?And the news is not so good. Okay.

So this expression here, I have already cautioned you earlier. The expression can look a bit intimidating but what is it. It's essentially covariance of the ETFE. So we know that the ETFE is being estimated at two different at many frequencies. So think of theta hat n1 just as double cap e to the J. omega n1, okay. And theta hat n2 likewise or maybe I'll even make it simpler for you. Let me put here and omega, so that it is just an omega. So that it becomes easy for you to relate to the expression and Z at e to the some some frequency Z.

So we are looking at how the estimate at two different frequencies are correlated with each other. After all we are estimating them together, right. So it is natural to look at how the estimate at one frequency is influencing or linearly influencing the estimate at another frequency.

Now when the frequencies are identical you're looking at the variance of the estimate. So that is what is given here and when the frequencies are different then you're given covariance expression. We are particularly interested in the variance of the estimate. So the variance of the estimate expression is given here.

And this we want to see if, if this expression goes to 0 as N goes to infinity. Remember, so this term here has a squared UN of omega to the whole square that term, right. That's a squared DFT coefficient of the input. And in the numerator you have gamma V of omega plus Rho 2 of N. What is gamma V of Omega? It is the power spectral density of the disturbance term.And Rho 2 of N is some function of N, which we say, we can show, I'm avoiding all the proofs. You can show that Rho2 N is bounded above by some constant over root N, which means what can we say now as N goes to infinity, what happens to the variance expression? The Rho 2 N goes to 0. But what about gamma of omega over the squared DFT coefficient, now this DFT is a unitary DFT coefficient. Now, it turns out that as N goes to infinity gamma V of omega is not going to be affected by N because it acts as a spectral density of the disturbance. What about U n of omega to the whole square. Does that go to infinity? It doesn't. It basically settles down to this spectral density of the input.

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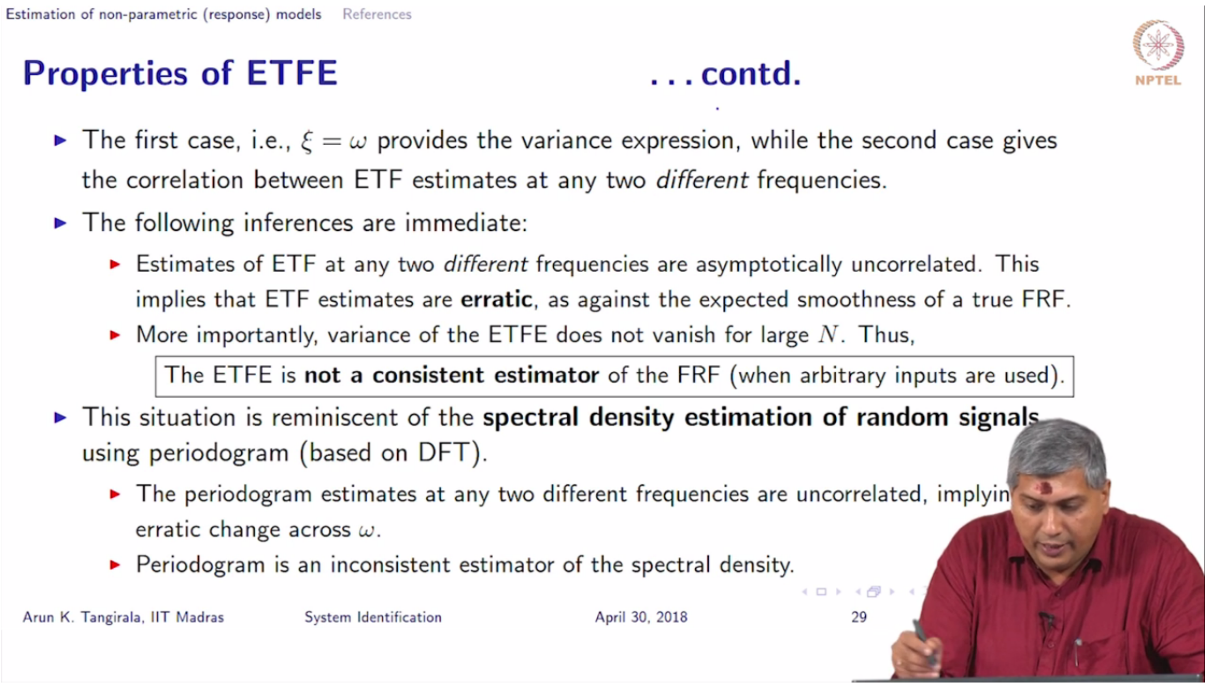

So as a result what happens is, that this limit N going to infinity the variance. Let me right here. AlthoughI don't have the expression given here. Let me write. Limit N going to infinity, variants of ETFE is simply gammaV of omega by U of omega to the whole square. That U of omega to the whole square is nothing but the limit. Suppose I define the limit as this. This is what it is. So, this limit whether this value here, whether this is infinity or not depends on whether the input is aperiodic signal or not. If the input is a periodic signal what happens is, the input repeats itself after a few observations and this value blows up because U N of omega to the whole square will blow up and then the variance will go to 0. But for an arbitrary input, that is aperiodic input, this is a non-zero bounded value. As a result the variance doesn't go to 0. So what we conclude is that, firstly ETFE estimates of ETF of ETFE at any two different frequencies are asymptotically uncorrelated.

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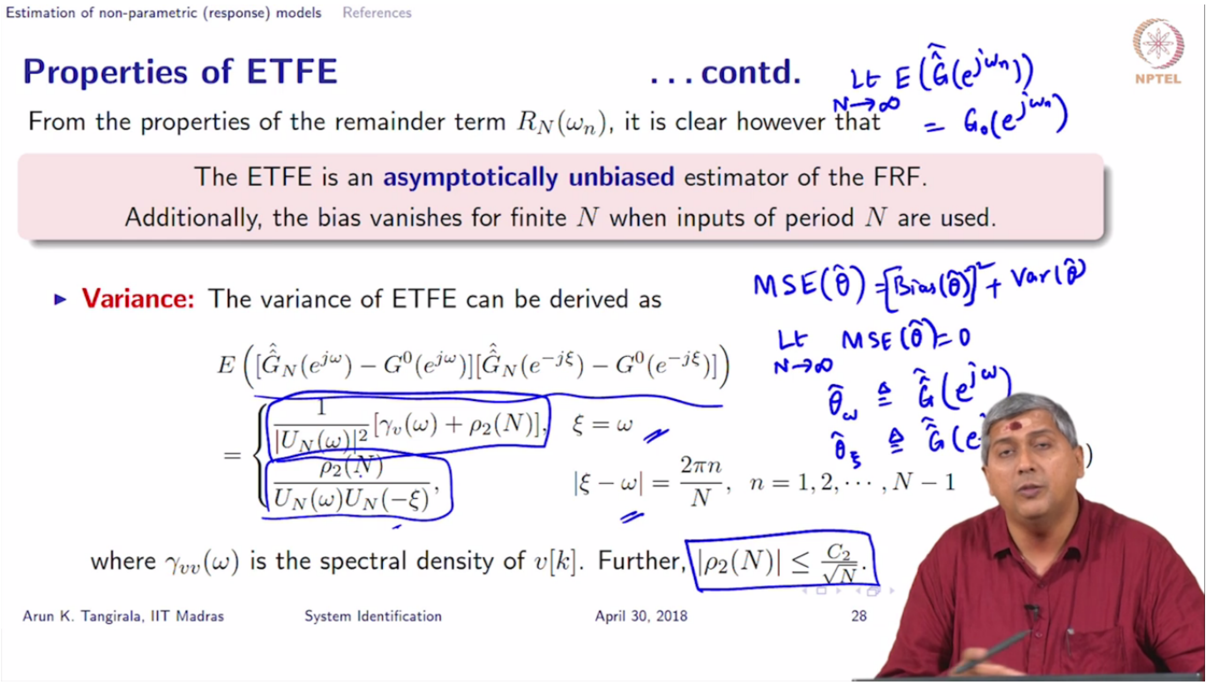

Of course that has got to do with how you look at this. That is by looking at the covariance expression. What happens to the covariance between two estimates at two different frequencies that goes to 0 because Rho 2 of N goes to 0.

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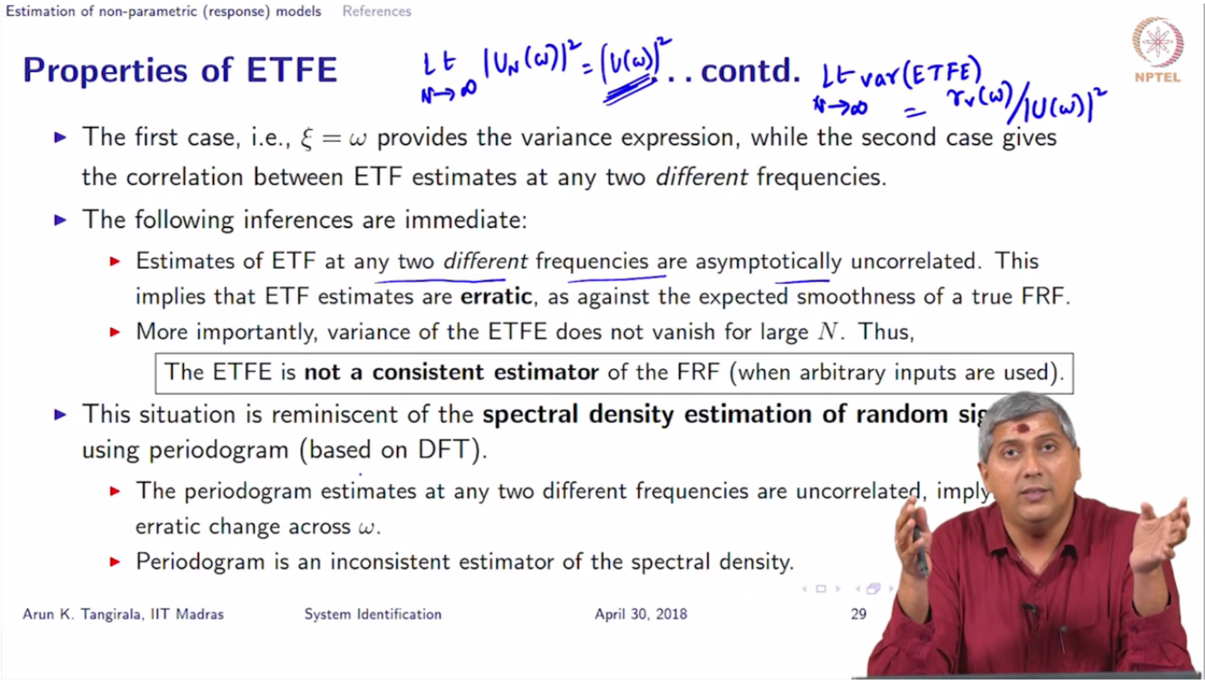

So at two different frequencies the estimates are uncorrelated as N goes to infinity. Now, is that a good news. We'll talk about that very soon. More importantly, variance of the ETFE does not vanish for large N, which is what we've shown in general unless of course, you have a periodic signal. So the conclusion is that ETF is not a consistent estimator of the FRF. Only periodic inputs are used and the number of observations exactly equals a multiple of the period, you will not get a consistent estimator of the FRF.

So if you are using periodic inputs, you can go ahead and use ETFE. That is the news. Okay. So let's come back to this point here. The estimates of ETF, ETFE at two different frequencies are uncorrelated. Is that a great news?Uncorrelated sounds very nice all the time but not really hear. What it says is that the estimate at this frequency is in its own bus. It's an own trip and the estimateat the another frequency is on its own trip. That should not be the case. Why? Because after all what is ETFE, an estimate of the FRF, indirectly it's an estimate of the FRF. So how does a true FRF look like, in general for LTI systems? So let me actually probably take a fresh page for you.

So how does the estimate of the FRF look like. In general let us say I take a first order, right. So if I take an FRF of a first order, typically it looks like this. Correct. So when I'm looking at two different frequencies. So let's say I'm looking at this frequency and let us say at another frequency. So, let's call this as omega 1, omega 2. This is the magnitude of the FRFand this is omega.Okay.

What is it that we observe? The estimates at two different frequencies. Do you think they have no relation between them.Are they on their own or they are tied together. What do you think? Do you think that they have to be tied together orthey are actually on their own.That means what we mean on their own is, FRF at omega 1 can take its own values and FRFat omega 2 can take its own values. Or that if I tell you that FRF at Omega 1 magnitude of FRFat omega 1, if it is that value then magnitude of FRF at omega 2, does it give you an idea? Let me put the question that way.If I give you the magnitude of FRF at omega 1, does it give you an idea of what is a magnitude of FRFat omega 2 orno.Or you willassume some wild value for it from the plot, of course. Definitely if I know the FRF, magnitude of the, amplitude ratio at omega 1. I will know the amplitude ratio at omega 2 more or less or at least, I know that it cannot be at amplitude ratio cannot be anything. There is a certain relation between them and this is a thread that plot that we are drawn is a thread, which means and this is another way of saying and function analysis the curve is smooth. Smoothness in function analysis means correlation in statistics when it comes to estimates. Okay.Which means there is some regularity to your FRF. So the truth is like this. The truth has this characteristic that the amplitude ratios at any two different frequencies cannot be arbitrary. They cannot be on their own. There has to be some relation between them.

Of course, we know the relation. If I know it's a first order then G of q inverse or you can say G of z inverse, with the first order plus time delay is b1 z inverse over 1 plus a1 z inverse, right. So which means that magnitude of. So magnitude of G of minus e to the minus j omega, would be simply be b1 by square root of 1 plus a1 square plus 2 a1 cosine omega, right.So we know that.That is the equation of this curve.That means we're all tied to that. I cannot have any FRF, sorry, any amplitude ratios at two different frequencies taking their own values. Unfortunately, the ETFE, what the way we're estimating, we are not telling the estimator this fact anywhere. Why? Because at each omega, what am I doing? I am computing Y N of omega1 over U N of omega 1. And then taking the magnitude. At omega 2what do I do.I compute DFT and I'm computing the magnitude.

So you may say now, the data should tell me thecorrelation, unfortunately it doesn't because the estimates at omega 1 and omega 2 are not talking to each other. We are actually estimating them individually. Had we somehow tied together the estimates, then it would be great. And that is what is the remedy that is offered to ETFE to improve the consistency properties.So the lack of consistencynow, in ETFE can be interpreted in many different ways. In fact now I would like to draw your attention to the spectral density estimation in time series. We encounter almost a similar situation in estimating the spectral density of stochastic signal.There we use periodogram. The periodogramalso suffers from the same lack of consistency property and the same story the periodogramwhich estimates at two different frequencies are asymptotically uncorrelated.

Whereas the spectral density of a stochastic signal is a smooth curve like the one I drew. So the periodogramalso lacks that tying together. And then three different perspectives are offered in improving the consistency property of the periodogram. Either you do a smoothing in frequency domain, where you artificially bring about a correlation between estimates at two different frequencies or the Blackman-Tukeyapproach, which truncate the ACVF estimates and then take the Fourier-transform. That means it applies a window function to the ACVF or a [17:20 inaudible] method with segments of data into which slices into different blocks, computes a periodogram for each block and then averages it. So for those of you who've taken time series course already, you should know it or anywhere else where you've studied spectral density estimation. You will find an exactly similar looking approach here.

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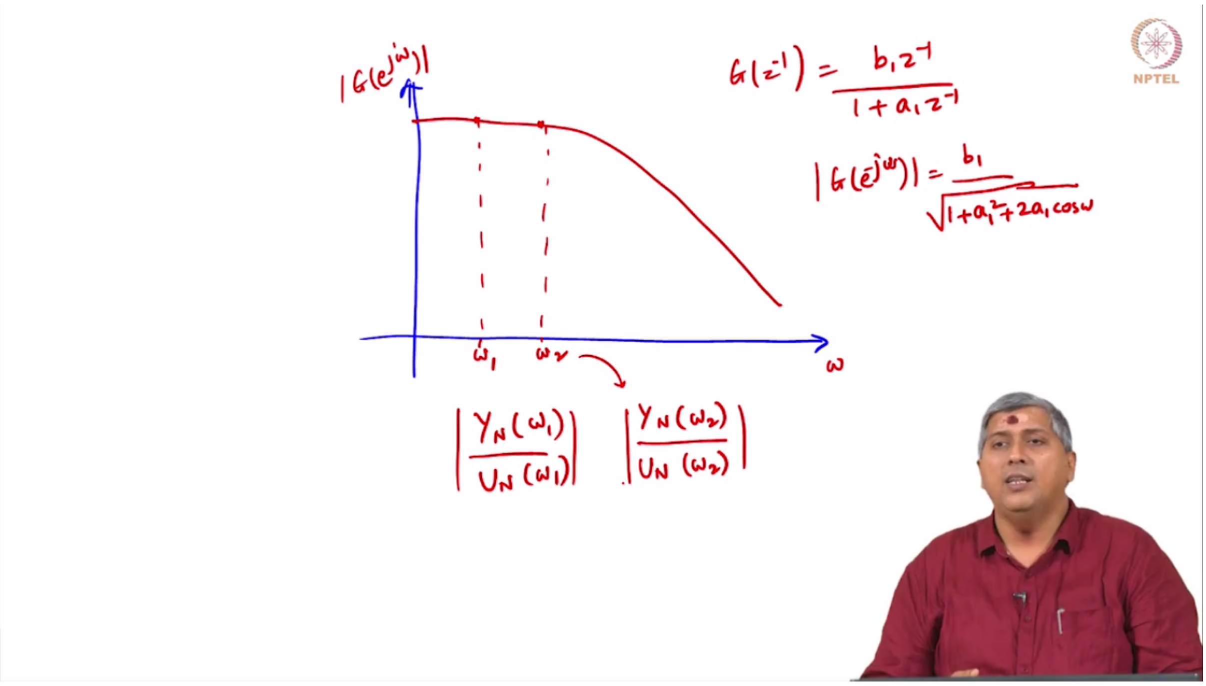

So let me first show you an example here.

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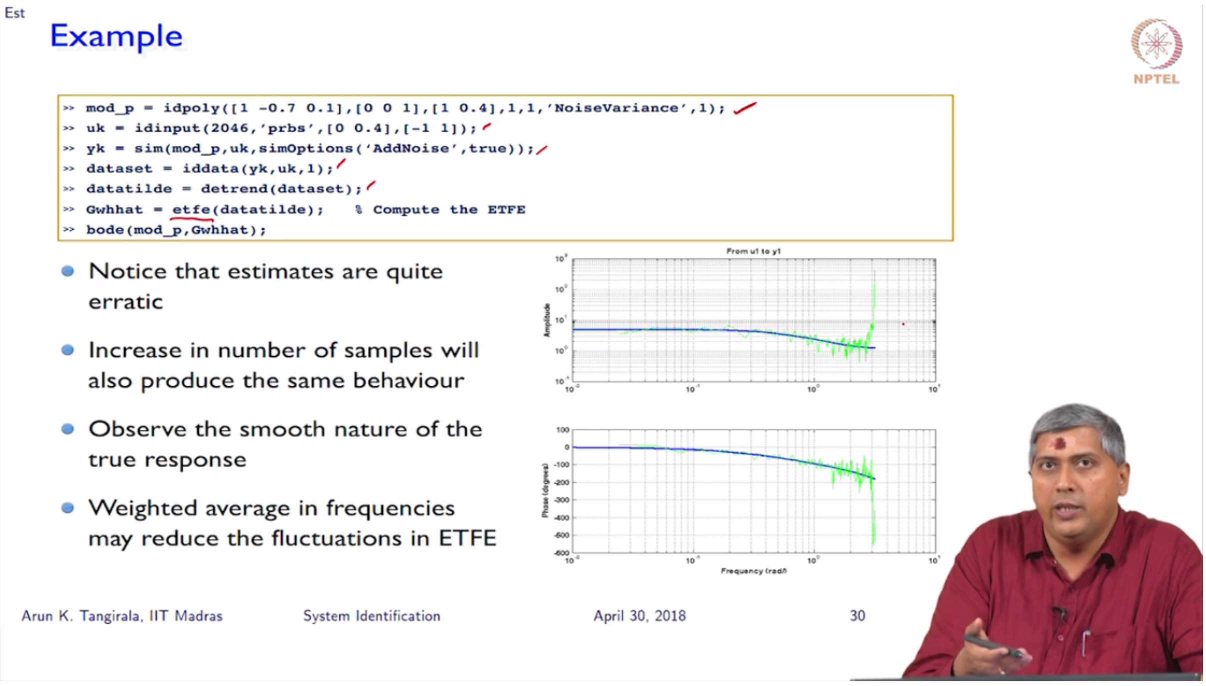

A lot of discussion theories, symbol,and so on. To break the monotony, I'm just simulating here. You can figure out what process this is. Whether it in ARMAX, OE or whatever. And then I use PRBS input, a band limited one and then I simulate add noise and this is the dataset. And I remove the trend and then ETFEis a routine in MATLAB that does the estimate for you. And I'm showing you here the estimates in green and then the true one is in blue.

Just now you saw when I drew the ETFE, theoretically true ETFEit is smooth. That is what exactly you see here in the blue line. The green line, by the way I'm plotting and Bode plot. So it's a log,log scale. You can see that it's quite erratic. You may not see, so visibly here because the spacing is so far on log scale there. But as you approach the high frequencies the erratic behaviour, erratic meaning the arbitrary jump in the estimate from one frequency to the other is clearly visible here. And that is what is not good. Okay. So now what do we do? What is a basic idea?

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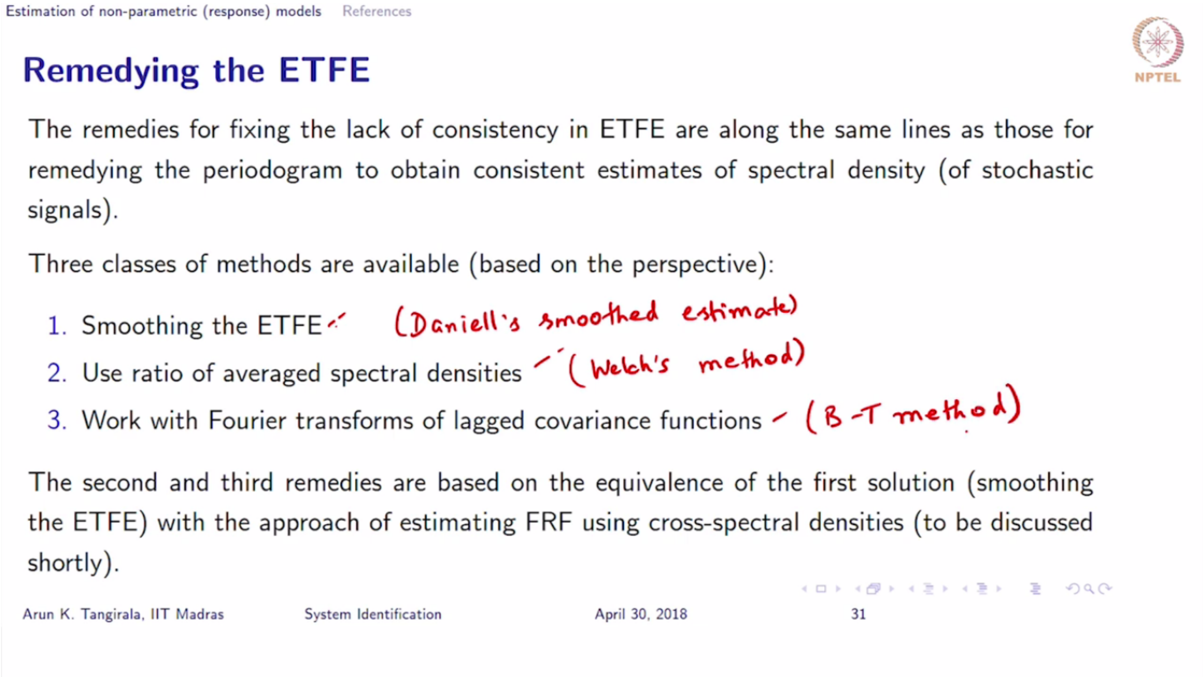

So there are three, again as I've told you. There are three approaches and improving the properties of periodogram. Here also you have three approaches. So this is the equivalent of the first approach in periodogramis a Daniell's smoother, just for you to strike parallels Daniell'ssmooth estimate in the spectral density estimation literature. And this is basically Welch's method, gives segment and then give. So you segment the data here. Take the ratio of the spectral density average or the Fourier transforms you can say. And then this is the Blackman-Tukey method. Okay.