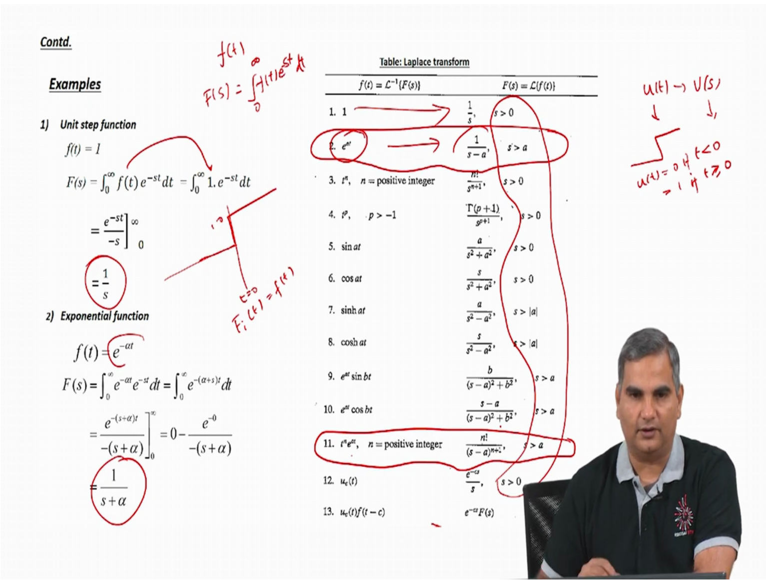
Process Control- Design, Analysis and Assessment
Professor Dr Raghunathan Rengaswamy
Indian Institutes of Technology Madras
Department of Chemical Engineering
Laplace Transforms - Part 2

We will continue with our lecture on Laplace transforms. In the last lecture I explained to you what are Laplace transforms and then I started with state space equation and then I showed you how you convert that into Laplace domain and then I talked about solving Laplace domain equations as algebraic equations and getting the variables of interest related to each other. And then I also said ultimately, if you do not get your answers in time domain it is really not very useful so how do you convert the Laplace domain solutions to time domain is something that we also discussed.

Now in the last lecture I talked about the inverse Laplace transform being a complex integration and we rarely favored actually performed that computation because people have already done that for us. So I said that the usefulness of Laplace transform really comes from the fact that there are these tables which can be used to do Laplace transforms and inverse Laplace transform. So I am going to pick up that idea today and then talk about how this is done and then show you some examples of solution using this idea.

(Refer Slide Time: 1:23)



So typical table is something like this that I have shown here, so let me take some examples first to tell you how these computations are done so that you get comfortable with this, and then we will not do this for any of this, we will assume that these are all correct and then start using these tables for what we need to. So remember again the definition of Laplace transform, so if we have F of T, F of S, it is already on the site but it is good to write this because you will understand and remember this quite easily, so this is the simple definition so which is what you see here. So let us say I want to find a Laplace transform for some input which I am giving.

So remember we talked about suddenly increasing the inlet flow rate in a tank, supposing I have the inlet flow rate let us say I call this time T is equal to 0 so remember whenever we do this time t is equal to 0 it is not something which is wrong or it is not something which reduces the applicability of whatever we are doing, whenever we start the experiment we call that as time T is equal to 0 so that is how I want you to interpret this because this will come back again later when we do control studies and so on. So let us say I am operating a tank at 9 in the morning today, I decided to increase the inlet flow to the tank then that is time T is equal to 0.

So from an input profile if I this is F I T which I am calling as small f t here, from an input point of view supposing at steady-state it was 0, remember we would have defined things in the deviation variable form and then suddenly I increase to 1 unit ok, this is a time function for let us say F I T you which I am calling f of t in this slide. Now what I want to do is I want to convert this time function to F of S because remember in the last lecture you have seen that U of T which is the input should also be converted to U of S ok. So it is not only the output variable which I want to compute is going to convert to Laplace domain, the input also has to be converted to Laplace domain so it is important to see how this is done.

Okay so let us say I give a step input like this and then ask supposing this U was something like this which is step input then what is going to be this function here. So mathematically if you are going to define this U of T so you are going to say U of T time function is 0 if T is less than 0, and equal to 1 if T is greater than equal to 0 ok, so let us say this is the definition we have for U of T, pictorially it looks like this. Then when we try to do this F of S we simply apply the formula, it is 0 to infinity F of T E power minus ST DT, after 0 F of T is always 1 so I replace this by 1 which is what is meant by this as step function then it simply becomes an integral E power minus ST DT.

The integral of E power minus ST DT E power minus ST divided by minus S between infinity and 0, so at infinity this is 0, at 0 T is 0 so E power 0 is 1 so -1 by s but since this is a lower limit it will be minus -1 by S so I will get this 1 by S. So you see this and then you will see here that this unit step function is 1 by S. Now just only once I will explain this, so on this side as we see this table I also have a condition on S, you can largely ignore this and this condition is something called region of convergence. So whenever we are going to talk about things going to 0 at infinity and so on, automatically there are certain conditions on S we are enforcing and all of those are captured by this region of convergence. As far as we are concerned we are not really worried about this, we will simply ignore this and only worry about the two columns here.

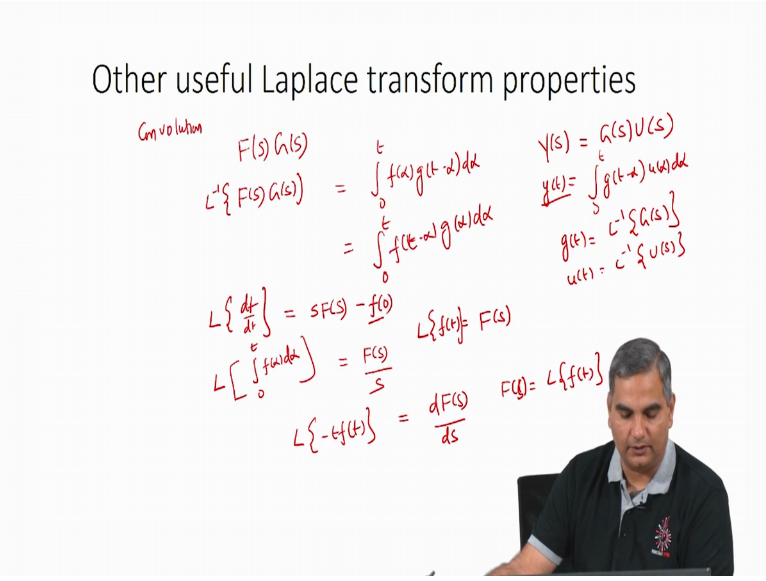
Now let us take another example, so let us say I want to do this Laplace E power A T again I do this E power minus Alpha T, I have a minus here so you will see this is S – A, this will be S + Alpha but in any case so let us say I have E power minus Alpha T then the Laplace of this is 0 to infinity E power minus Alpha T E power minus ST DT which is just the definition of Laplace transform. So this is going to be 0 to infinity E power minus Alpha plus S DT and much like how we did there so this is going to be E power minus S plus Alpha T divided by minus S plus Alpha. This is the integral of this and then if you substitute the limit and then simplify it you will get S + Alpha so here in the table it is for E power AT is 1 by S - A so if you say E power - AT it will be 1 by S minus minus A which will be the same thing here.

So you see that there is nothing very complicated about going in the forward direction and then getting the Laplace transforms from these functions in fact, you can work out all of these yourself if you have the interest and inclination, nonetheless what has been done for us is that this have been computed. Now the forward Laplace transform you go from here to here as I said before but then you want invert then what you do is you look at this column and then find the function that you are interested in inverting and then see what is the equivalent time domain so again as I said before this is a complex computation but we are not going to do this because we simply read off on the backside.

Now couple of things that I want you to notice here is that E power AT this right here is something that we will use quite a bit in inverting so this is of row which is of importance and I will explain this to you. In fact when we use partial fractions, there are only really for us we are interested in, there are only really 2 rows that we look at and you can do almost everything with those 2 rows. The other row is what I showed here and you will see why we are focusing on these 2 as we go through this course. So just look at this, this is if we have E power AT as your time function the Laplace transform is 1 by S minus A or alternatively if 1 by S minus A is a Laplace function then E power AT is the corresponding time domain function.

Similarly what this says is, if N factorial divided by S - A to the power N plus 1 is your Laplace function then the corresponding time domain function is T power N E power AT, so these 2 are going to be quite important for the kind of computation that we are going to do so pay special attention to this. If we go from here to here Laplace definition, you can actually derive this but nonetheless at least keep your focus on this because we are going to use this.

(Refer Slide Time: 8:21)



Now these are not the only useful things as far as Laplace transform is concern, there are other properties that are quite useful and that is used again and again in controls, I am going to mention some of those. One of those is called the convolution property, so what this basically says is if I have let us say a product of 2 functions in the Laplace domain F of S G of S ok and I am interested in doing inverse of this, so I am interested in doing Laplace inverse F of S G of S, so what is this going to be is a question that we can answer.

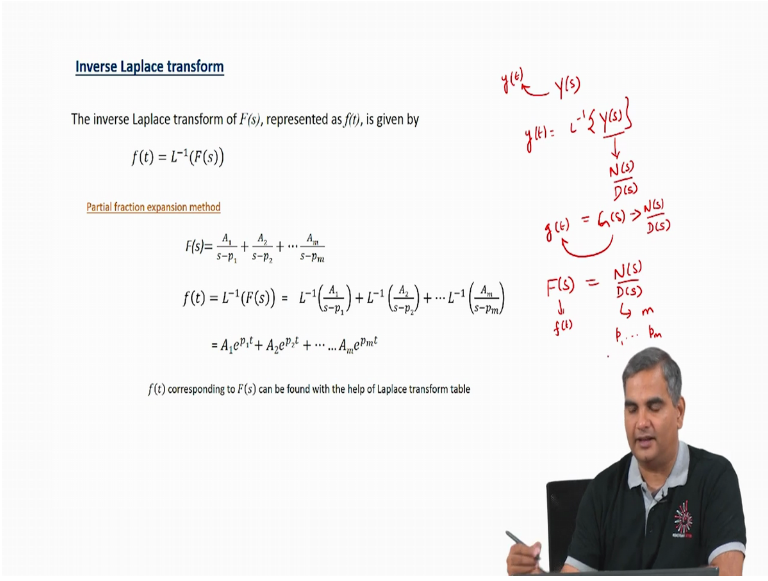
So we will see that this is going to be equal to 0 to T F of Alpha G of T minus Alpha D Alpha so this is what is called the convolution integral. So whenever you do a Laplace inverse of this you would expect to get a time domain solution and the time domain solution is going to be this, 0 to T F of Alpha G of T minus Alpha D Alpha clearly so that we do some sanity checks. You are going to integrate out the Alpha and the right-hand side will be just a function of time. Now you can also quite easily show by changing variables so we can say T prime is T minus Alpha and then this is also completely the same as writing this as F of T minus Alpha G of Alpha, so this also will give you the same result.

So in other words when you do this conversation integral you can write it as F of Alpha G T minus Alpha or you can put T minus Alpha into F and then write G Alpha so both are going to give you the same result. So why is this important from whatever we have seen till now? Remember we got Y of S as G of S times U of S ok so basically I can think about this as let us say product of 2 functions in the Laplace domain and when I want to get my Y of t then this is simply going to be equal to 0 to T G of T minus Alpha U of Alpha D Alpha. And how do I get this G of T, G of T is basically Laplace inverse of G of S and U of T is Laplace inverse of U of S. So that is how I get this G of T and U of T and once I get G of T then I can do the G of T minus Alpha U Alpha D Alpha 0 to T and then I can get my Y function, so this convolution property is an important property that is used quite a bit in Aplus transform.

Now another property that is used is the following, so remember we had the differentiation of a function which if I said I want to take a Laplace of DS DT then we saw that this is going to be S times F of S, so F of S is the ok we will come back to that – F of 0. So if F of T is your function Laplace of F of T is equal to F of S and if you wanted to get the Laplace of differential of F of T then that is going to be F of S which is the Laplace of F – f 0 small f f 0, notice that this is small f 0 so this is evaluating the original time domain function at time T equal to 0 and there will be option of deviation variable generally goes to 0.

Now you might ask okay so this is the integral, what happen if I want to get the Laplace of for example, 0 to T F Alpha D Alpha so this is equivalent in terms of integration to this differentiation, so here I wanted the Laplace of the differential I want here the Laplace of the integral. So and also notice here I write this as F Alpha D Alpha so when I do this I get this T out so this will turn out to be F of S divided by S. Now there is another interesting Laplace property also which is used which is Laplace of minus T times F of T. This looks similar to F of S G of S but now it is a product of T and a function F of T and you can show that this is actually D F of S differential of F of S by dS where F of S is basically Laplace of F of T. So these are all interesting properties which we can use later when we do little more sophisticated computations with Laplace transforms.

(Refer Slide Time: 13:06)



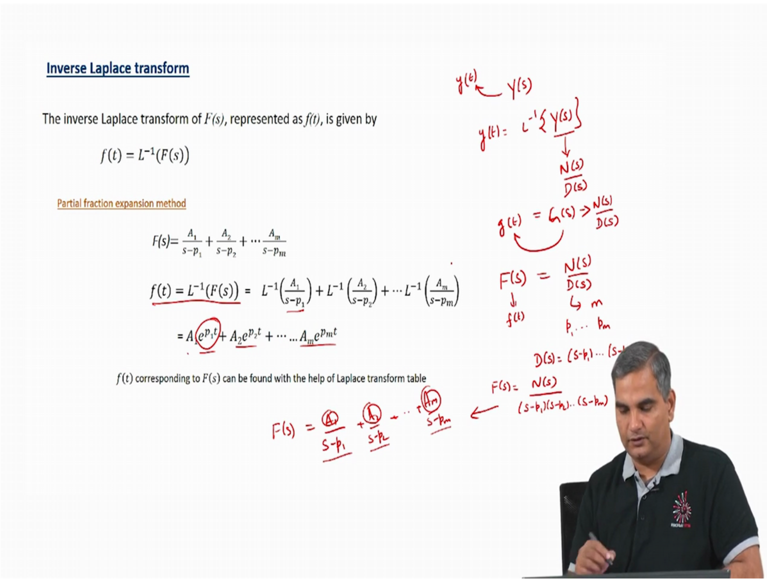
Now let us look at inverse Laplace transforms for general functions ok. So though I am doing this for F of S so what we are basically interested in is supposing I got Y of T that I am interested in, after doing all the Laplace transform and so on now I have Y of S ok, so I have to go from Y of X to Y of T so basically Y of T is Laplace inverse Y of S. And the way I have got this Y of S is through algebraic manipulation after I do the conversion from the ordinary differential equations to the algebraic equations. So in most cases we will see that this Y of S is going to be of the form of N of S by D by S, there is a numerator polynomial and denominator polynomial right.

Now there are also other than Y of S there are general functions Z of T for which I either need to take Laplace transform to get G of S or I have to take an inverse Laplace transform to get to G of T. So these cases also this G of S will have the form of a numerator by a denominator. So the upshot of all of this is most of the things that we are trying to invert from Laplace domain to time domain from a process control case will be ratio of 2 polynomials; numerator polynomial by denominator polynomials. So this whole set of variables Y of S, G of S, U of S, D of S, whatever it is so let us do generic the F of S okay which is going to be numerator polynomial by denominator polynomial.

Now our interest is once we do all the computations in the Laplace domain our interest is in moving this back to the time domain so basically we want to get F of T from this. Now one assumption we are going to make and which you will see will be valid most of the time is that the order of the numerator polynomial is less than the order of the denominator polynomial so that is an assumption we are going to make. And let us assume, to start with that this denominator is of order M ok and from our high school math we know that when I have a polynomial of order M there will be M roots that are associated with the polynomial.

And even though the coefficients of polynomials are real we know that these roots can be complex nonetheless if the coefficients of the polynomial real if 1 root is complex, the other complex conjugate root also has to be a part of the solution, and we also know that the roots can repeat ok so the same route can repeat once, twice and so on. Nonetheless in the 1st step we will keep this quite simple and then say the roots do not repeat and let us say I have M roots P 1 to P M. I am just saying roots do not be repeat however, I am not saying the roots should be real and so on so this P 1 to P M could be complex the only fact you have to remember is if P 1 is complex then there has to be another complex conjugate root also associated with it.

(Refer Slide Time: 16:11)



Now we also know that this DS can be written in what we call as root result form which is S minus P 1 all the way up to S minus P M. If the leading coefficient of D of S has been made to be 1 then we can always write this in this root result form ok, in which case F of S becomes some numerator polynomial, we are not focusing on numerator polynomial and you will see why presently divided by S minus P1 times S minus P2 all the way up to S minus PM. Now an interesting fact is that I can write this numerator by denominator as the following so I can write F of as some A1 divided by S minus P1 plus A2 divided by minus P2 and so on all the way up to AM divided by as minus PM.

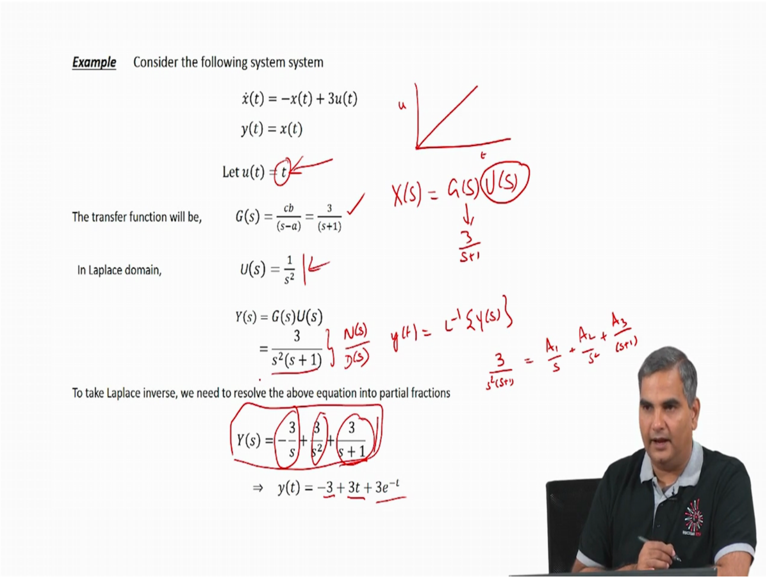
So what we are saying right here is that when you look at this expression, it seems like the numerator has gone somewhere, the action is all in the denominator so the roots of the denominator have given us each one term in this expansion and since they do not repeat the form is always the same and what happened to the numerator What happened to the numerator is, the numerator is defined by these coefficients, so these coefficients will actually tell you what the numerator will be your other words if you want to compute these coefficients then you have to know what the numerator.

So the numerator information has been kind of subsumed or submerged into these coefficients A1 to AM but the real action happen because of the denominator roots and that is the reason why you will see as we go further in control the polls of the transfer function are the ones that attract lots of our attention. Of course the roots of the numerator polynomial also called the zeros of the transfer function which we will talk about later also are important but got out of dynamic is actually dictated by the denominator roots ok. Now like I said before this F of S could be Y of S, G of S, U of S, whatever it is and then once we have done with all of these computations, we are interested in actually computing F of T or equivalently Y of T, U of T, G of T and so on.

So we know F of T is Laplace inverse of F of S and due to the linearity property I can take Laplace inverse of the sum as sum of the Laplace inverses so I have Laplace inverse A1 by S minus P1 and so on. Now you notice that irrespective of what the numerator and denominator is, as longer as the function in the Laplace domain is a ratio of 2 polynomials and with the order of the denominator polynomial being greater than the numerator polynomial we can always do this is the roots do not repeat, and there is a simple extension if the roots repeat which we will see later. And now the whole Laplace inverse which look very complicated till now has been trivially reduced to only one row as of now in the table which is if you go back to the table you will be able to see this which is the Laplace inverse of 1 over S minus P1.

And you will see from the table the Laplace inverse of this is the power P1 T, so F of T will become A1, Laplace inverse of this which is E power P1 T, A2 E power P2 T all the way up to E AM power PM T. So you see how something that looks complicated you know it could be any polynomial in the numerator, any polynomial in the denominator and so on, simply reduces to only one row in the Laplace table and we can actually invert quite easily and get F of T so this is an important idea that you want to remember

(Refer Slide Time: 20:02)



Let us take an example, run through it very quickly. So let us say X dot is – XT + 3 UT, YT is XT ok. Now let us assume that I am going to give an input for this system, UT is T basically what it means is I am going to ramp up so this is UT, at T equal to 0 it is 0 and then it keeps ramping up ok. Now I am going to ask the question saying if I were to give a function time domain function for U like this, what will be X and what will be Y so a quick way to do this a is to do Laplace transform of this and if we assume that X is 0 at T equal to 0 then you know this is S X of S minus X of S will be 3 U of S. So if you take X of S to the other side you will get X of S times X + 1 is 3 U of S so X of S will be 3 by S plus 1 U of S.

So if I were to write this X of S as G S times U S then this will be 3 by S plus 1 so which is what I have here. Now this U S okay is again the Laplace transform of U of T but since I am interested in finding out how the system is going to behave then I give U as a function of T in a ramp fashion, this is how U behaves with respect to T then basically U of S to get that we have to convert this function to Laplace domain. So if we go back and look at the table you will see that when U of T is T, U of S will be 1 over S square. Now notice that instead of putting this into this equation and doing let us say solving the differential equation what we have really done is actually algebraically set Y of S and X Y T is X T so Y S is X S, Y of S is the output is simply G of S times U of S, and G of S I know is 3 by S plus 1, U of S is 1 over S square ok.

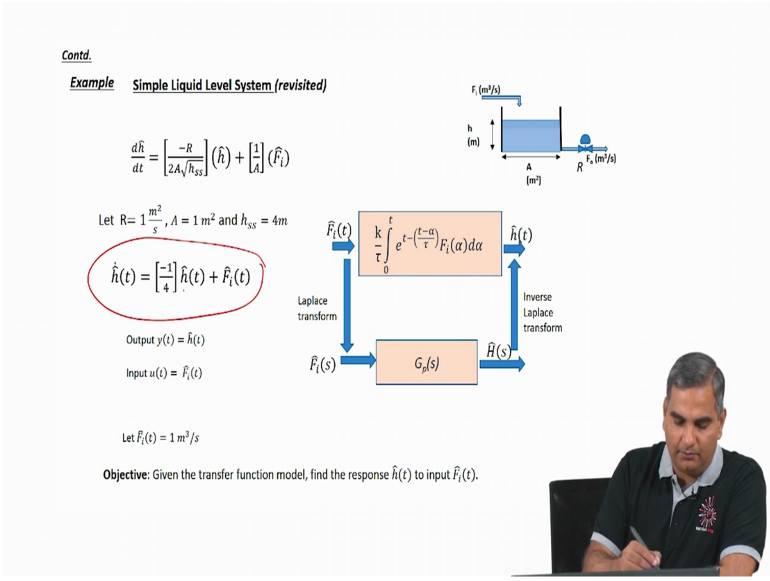
Now when we get to this here, now you notice I have a problem here because if I want to do remember I said Y of T is Laplace inverse Y of S and if I want to do inverse Laplace transform then I have to do partial fraction expansion, before that just notice that I can think of this as a numerator polynomial by denominator polynomial. The denominator polynomial is of order 3 and the numerator polynomial is of order 0 because there are no S terms there clearly satisfying requirement numerator polynomial have order less than the denominator polynomial. However, I have a problem if I look at the denominator polynomial it has how many roots; 3 roots, one root is -1 however 0 is a root that is repeated twice ok.

So we will get back to this in more detail when you do partial fraction with repeated roots what you basically do is you add as many terms as there are repeats so basically the expansion for this the partial fraction expansion for this will be S by S square S plus 1, so it is going to be some constant by S + some constant by S square another term because it is repeated and some constant by the 3rd root. So if it were repeated thrice then you do A1 by S plus A2 by S square + A3 by S cube and so on and you can always do the partial fraction.

Now for this example if we do this partial fraction and then compute A1, A2 and A3, it will turn out to be -3, 3 and 3, you can do this computation and check whether this works out. So I have written Y of S in the last slide however, I did F of S the same way I have these terms I have the 1st term, the 2nd term and the 3rd term. Now if I want Y of T I do a Laplace inverse of this and I go and look up the table, you know the unit function the Laplace inverse was 1 over S, so if it is 1 over S the inverse Laplace will be unit function so this will be just -3. And here you saw if T is your function, 1 over S square is the Laplace transform. So is 1 over S square is what you are looking at then when you invert this you will get the T.

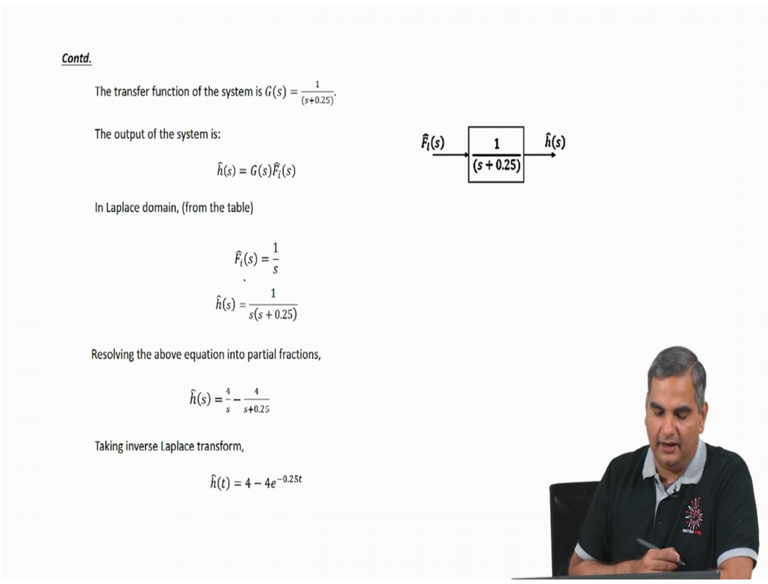
And remember this 1 by S minus P E power PT, here P is -1 so you will get 3 E power minus T. So notice that how I solved this equation very nicely just doing algebraic manipulation and looking at the table to get the solutions.

(Refer Slide Time: 24:38)



We can do the same tank example that we have been talking about. This is something that we have seen several times, this is the equation form now let us put some numbers for this, let us say I have R is 1, A is 1, HS is 4 and so on, then if I substitute these values into this I get this form. So this is the point that I made right, so the coefficients of these equations do not come from somewhere, they come from actual physical values so if these come from what is the steady-state height area , the resistance and all that. Now I can now get this equation and you will also notice that this is basically is equal to AX plus BU.

(Refer Slide Time: 25:20)



I can substitute those equations and then I will get this G of S function this is something that I would like you guys to work out which is 1 over S + 0.25. So basically from a blog diagram viewpoint what we have done is we have abstracted the whole tank to this transfer function and the input is the inlet flow rate and output is the height and the hats basically represents that we have written this in deviation variants. Now the same kind of question that we asked in the last example which is to say if I were to ramp is what we asked last time, instead if I were to step the inlet flow rate what will be the height? Okay.

So if I step then the Laplace transform of that is 1 over S and this equation tells me that H hat is G of S so G of S is 1 by S + 0.25, this S + 0.25 comes from the all the values from the last slide and computing this transfer function. Now this is very easily resolve into some A1 over S + A2 over S + 0.25 and you can quite easily see the value of A1 is 4, A2 is -4 so I can resolve this into this form. And as I told you before now we have actually got a solution for the height here at this point without actually ever solving differential equation, nonetheless I cannot give this as a solution to an engineer saying ok here is your height in Laplace domain.

So what does this really mean? So to do that we have to convert it into the time domain so you know the inverse of this will be just 4 by S is 4 and the inverse Laplace of this will be 4 E power -0.25 T. So this is how you are able to quite easily resolve this equation and one of the things that I want you to remember is that these numbers and these partial fractions and these Laplace transform tables and so on after while it make it sound abstract, it is not really abstract all of these are rooted in the engineering problem that we are solving and the knowledge and the parameters and the variables of the engineering problems have after a while being taking into the constants and coefficients of these functions. And while we are doing this we are actually computing how the height in a tank is going to vary as a punch in of time.

So hopefully I have given you a good idea of how we use Laplace transforms to solve these equations and I have connected it to a physical system and shown you how we go from that physical system equation to a solution. So we will pick up from here in the next lecture and then describe more about how to analyze more complicated cases where we have multiple repeated roots and so on. And I am also going to show you some general ways of computing these coefficients that can be used. So I will see you in the next lecture thank you.