Process Control- Design, Analysis and Assessment
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Laplace Transforms Part 1

We will continue with our 8th lecture, this is the first time in this course I'm going to introduce the notion of Laplace transforms I will explain what Laplace transforms are and why are they useful in control. Laplace transform basically from a control viewpoint can be used to solve the same set base equations that I showed you in the last class. So in the last class we saw how we can solve that set base equation in time domain, the same equation can be solved in what is called Laplace domain and there are certain advantages to working with Laplace domain which I will explain as we go through this lecture.

I also want to point out that I am going to teach Laplace transform purely from the viewpoint of its use in control, so I am going to take certain liberties and definitions and so on so that it is easily understandable for an undergraduate student who is looking at this for the first time. Let us start by defining what a Laplace transform is, when we define this Laplace transform we will see why this definition is useful, so mathematically I will define it and then I will actually motivate why we may use something like this in solving differential equations and as a result use that in control.

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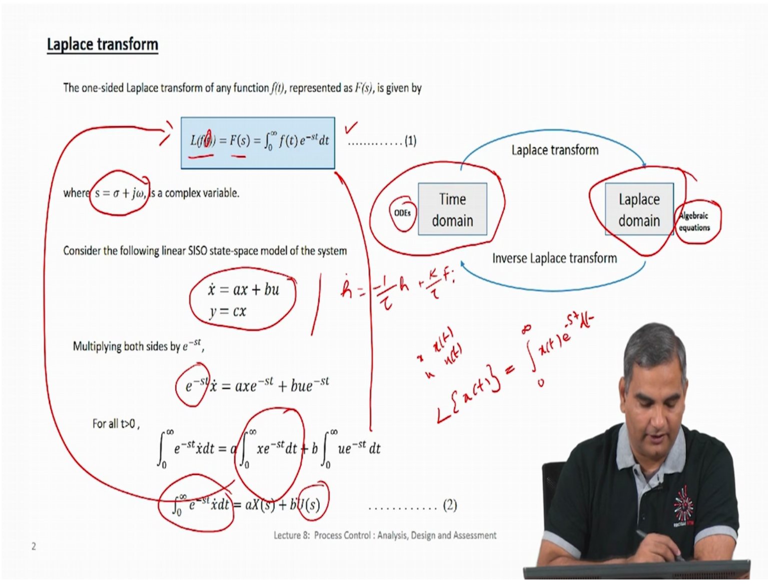

So one side Laplace transform of any function F of T is defined as this Laplace of F of T is F of S, it is integral between 0 to infinity F of T E power minus S T DT as shown here. Now main thing that I want you to notice here is when we take a function which is in time domain and take Laplace transform of that time domain function then it goes into the Laplace domain and it becomes a function of S which is not yet defined for us. So the standard notation is if we define a function in time domain using lowercase alphabet in the Laplace domain, we change the lower case alphabets to an upper alphabets so small f to capital F and this variable is t and here the variable is S.

And let us just make sure that the right-hand side is also in variable S so since we are integrating from 0 to infinity, f of t power minus ST DT, the t will get integrated out and what will remain will just be a function of S. Now what is this S, we just simply say for now that S is a complex variable which is of this form, we will rarely if ever try to worry about what S really is in terms of Sigma and Omega and so on, we will use this complex variable S in terms of its solution what we are going to call as the poles and zeros of a transfer function and then all the judgment we need to make a terms of control can simply be made based on the solutions to this S when we solve an equation in S which we will see later.

So for now all I want you to understand is that you are going to get a function of S and S is a complex variable that is all I am going to say and the complex variable can take many values just like how time can take many values, but we are not going to yet define what values it will take and what are the implications of those and so on. So you might ask why do all of this, so a very simplest explanation from say solving equations and the control viewpoint is the following. We will see that if you have a model in time domain which is a ordinary differential equation which is what a set base model is, I remember we talked about the tank model and then showed how that is going to be simple differential equation ordinary differential equation.

Now we showed that we can actually solve that ODE using integrating factor. Actually Laplace transform also looks very similar to the integrating factor but we can convert that equation which is an ordinary differential equation through the application of Laplace transform into equations in Laplace domain. Now why would someone want to do this? The reason is you will see as we do this that the ordinary differential equations in the time domain become algebraic equations in the Laplace domain so it is much easier to solve large number of algebraic equations than ODE is, so it makes us solve these equations easier. And not only it allows us to solve it easier, it also gives us interpretive ability for the solution, as you get good at this we can bring in notions of frequency, frequency of signal and so on.

So we will get more into the field for what the results are when we think about these aspects in the frequency domain or Laplace domain and we will see this later. So keep looking out for how this ODE becomes an algebraic equation as we go through this Laplace transform. Now once you can avoid the ODE to a Laplace domain of course you got algebraic equations and you could solve these algebraic equations but then the question is the solution will also be in terms of S, but really all the physical interpretation happens in time domain. So I am going to ask let us say for increase in the flow rate inlet to the tank, how is the height going to vary as a function of time. I am not asking how the height is going to vary as a function of later S, which is an abstract concept.

So while it is useful to convert these equations into Laplace domain, get algebraic equations, one then has to do what is called this inverse Laplace transform and then convert the solution back to time domain okay. So the way this thing works is you have the variables differential equations, convert them into Laplace domain and get the solution in Laplace domain and then the solution will convert back to time domain using Laplace transform, so this is the format in which we are going to use Laplace transform. So let us go back, look at the state space model which is exotically AX plus BY is equal to CX, and remember if you are thinking about this for the tank example, the equivalent model was H dot equal to - 1 by Tao, H plus k Tao F I right, this was the model equation that we saw where -1 by Tao will be A, A by Tao will be B and so on.

So this equation again I am just trying to point out though we have done this many times I was just want you to understand that none of this is completely abstract, all of this is rooted in reality in terms of the procedure equations and the equipment and so on. So basically somehow our physical process has been made into some of the equations like this. Now I want to solve this set of equations, so just like how I did my integrating factor when we were solving the time domain equation, just assume multiplying both sides by the term which I am going to use which is e power minus ST with this something I can do since I am multiplying the equations on both sides. So I will have this equation to read as e power minus ST x dot equals AX e power minus ST plus B U e power minus ST.

Keep in mind for not cluttering the slide we have just written X, but really X is a function of time X of t and U is also function of time U of t, so this is actually X is a function of time, U is a function of time. Now nothing to stop us from integrating this equation from 0 to infinity, now I go ahead and integrate this equation 0 to infinity e power minus ST X dot DT equal to

A 0 to infinity X e power minus ST DT plus B 0 to infinity U e power minus ST DT. So what till now we have done is we have taken this equation and simply multiplied it by some term which I am allowed to do and simply integrate it which I am allowed to do also.

Now something interesting happens, if you notice this here you will see that it is exactly the definition of Laplace transform, so Laplace of X of T if I write it like this Laplace of X of T remember I told you that X is a function of time will be equal to integral 0 to infinity X of T is power minus ST DT okay so that is what this is here and similarly, this from the same definition will become U of S. So this equation now turns out to be 0 to infinity e power minus ST X dot DT, we do not know what to do with this yet but on the right-hand side things have simplified quite nicely. I have A times X of S + B times U of S so now take a careful look at what we have on the left-hand side of the slide that I showed just now.

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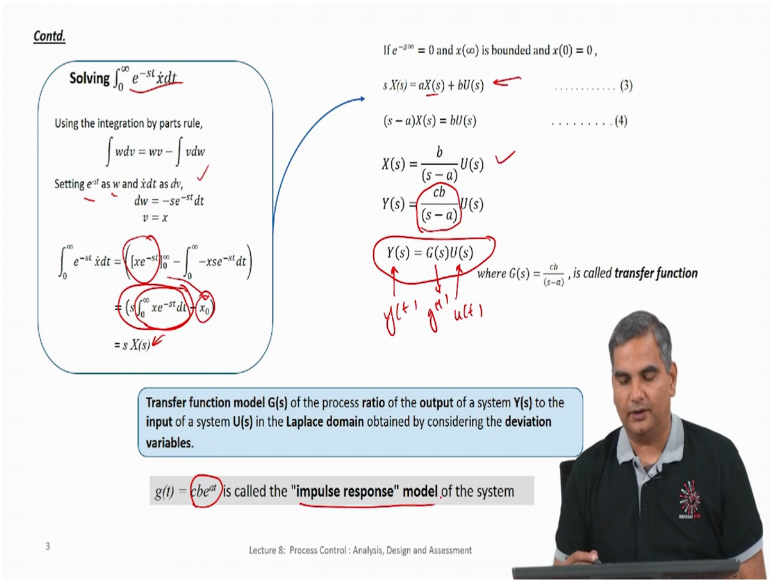

So we had on the right-hand side we had got variables defined in terms of Laplace transform so there is no t term in the right-hand side. But if you look at the left-hand side I still have this 0 to t e power minus ST X dot DT clearly this is going to be only a function of time because X dot of T is also going to be a function of time and once we integrate time out then we will have only function of X but still we do not know what that is. So to simplify this and get it into the Laplace domain with only S variable there what we are going to do is we are going to use room that we are all very comfortable with we have been using this for years starting from high school, which is using the integration by parts rule.

It says if you have integral W dV you can write that as W V - integral V DW, so the key thing here is when I have something like this I have to define what W is and what d B. So if I were to define e power minus ST as W and X dot DT which is remaining has DV and then when I do DW which is I have to differentiate if power minus ST then I will get minus at S e power minus ST DT and when I integrate X dot DT then I will get to V equal to X so this is quite straightforward. So I can write the 0 to infinity e power minus ST X dot DT as W V, so we had W defined as e power minus ST and after integration we got V as X so X e power minus ST 0 to infinity - 0 to infinity V DW, V is X, DW is X the power minus ST DT so we have got this here.

So now if you look at this term, so there is a minus minus so it becomes a plus so I have integral 0 to infinity X as e power minus ST DT. Since the integration is with respect to DT I can bring the S out so I have S 0 to infinity, S e power minus ST DT. And if I come to this term right here when I substitute infinity if X of infinity is bounded and e power minus ST is 0 tends to infinity then the first term goes to 0 and when you substitute 0 here then this is X evaluated at 0 times e power minus S into 0 which is a power 0. E power 0 is 1 so this will simply lead to this, I get a minus here because I first evaluate at the top limit infinity and then I evaluate at the bottom limit.

So this is what happens and if you assume that these equation where actually written in the deviation variable form then you know at initial condition is 0 so we can assume that this is 0 so I am only left with this but now notice what has happened. If you look at this inside this okay, X e power minus ST DT integral 0 to infinity you will notice that is the definition of the Laplace transform for XT ok so this I can replace with X of S. So this with all of these conditions and the initial value being 0 because it is a deviation variable will turn out to be X of S. So if I substitute that into left-hand side of the equation that we saw before then I have equation of the form S X s is equal to A X S + B U S.

Now notice the difference between this equation and the equation that we had in the time domain, time domain we had a differential equation. Now we are solving for X in that as a function of U time and I had to solve the differential equation. But now if I restrict my interest to simply solving for X in the Laplace domain as a function of U then this is a very simple algebraic equation, I can take this term to this side I get X minus A X of S B U of S, then X of S is B by F minus A U of S. So this basically already automatically solves this problem for me and it has given me the solution X of S in terms of U F S.

Now the other equation is Y is equal to C X which is anyway an algebraic equation so if you do Laplace transform on both sides you will get Y of S equal to C X of s so Y of S is C, X of S is B by S minus A U of S so I get this transfer function. Now if you look at only the relationship between the input and output then this is the equation that we are interested in and then we see that the output in the Laplace domain which is Y of s is related to the input in the Laplace domain which is U of S by this equation and I could call this G of S as C B by S minus A. Now this is the first time we are going to introduce this idea so this G of S is what is called the transfer function and the transfer function is modeled in the Laplace domain.

Now though we have come to this equation which is ubiquities in all of control, generally we will sort with this transfer function model. I hope you see the route through which we have come here and there is no magic to this, this comes out of your standard state space equation, and the standard set space equation comes out of the physics of the problem. So I already told you that this B and A for this tank example actually can be calculated in terms of physical variables and parameters in the system for example (())()14:32) of the tank, the resistance value and so on.

So what we have done really is taken a physical system and then modelled it with variables and parameters and those parameters become coefficient in state space model of the form X dot equal to A X plus B U, Y is equal to C X and that state space model can be translated into a transfer function model which is of this form here with the coefficients in the state space model which are actually related to the political coefficients. So these are physical coefficients and that is how they come into transfer function form. So you see that once you work in the Laplace domain you get one definite advantage which is that instead of solving differential equations you are going to solve algebraic equation so when you have multiple equations you want to make judgments about this, it is easier to look at an algebraic solution and get an understanding.

So just like how there is a Y of T which can be converted to Laplace domain and a U of T that can be converted to Laplace domain U of S, clearly for G of S also there should be G of T right because it is a transfer function so every function has you know if it is a Laplace function then there will be equal and time domain function. Now we will see later that if your function is of this form then that would have come from G of T of this form C B e power A T and this C B e power A T is called the impulse response model of the system and we will see why this is called the impulse response model of the system as we will go forward in this course.

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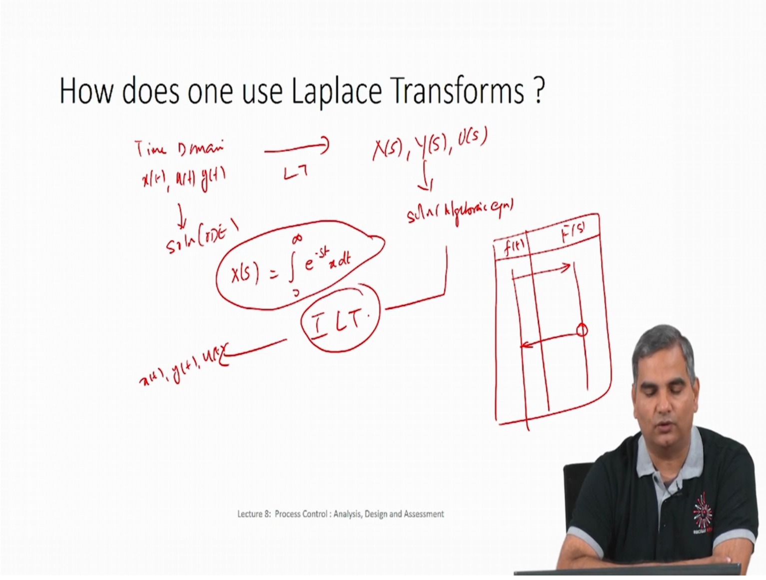

So let us summarize by talking about how one may use Laplace transform. The first thing is I take a time domain set of equation okay, which are going to be in X T, U T, Y T, and then I do a Laplace transform, then I get equations in terms of X as X S, Y S and U S. Once I have this equation so here I have to get time domain solution which is an ODE solving an ODE, here the solution becomes solution to an algebraic equation. Okay so we know for example if I want X of S I know just to recap X of S is 0 to infinity E power minus S T X DT right. So I know how to go from X to X of S and similarly Y of S is 0 to infinity E power minus ST Y DT and U of S is 0 to infinity E power minus ST U DT and so on.

So I know how to go from here to here but I make this point that okay you did all of this great and got the solution and so tell someone Y of S is this so what does it mean? So you tell someone Y of S is this so what does it mean right, I have not yet defined what S is exactly I am looking for an answer to a very specific question which is if I increase my inlet flow rate how would my height varying and if you say height will be varying in the Laplace domain this way it is not very useful. So somehow we have to do an inverse Laplace transform to go back into the actual solutions of X T, Y T and U T otherwise it is not useful. So this part only converts the equations with these variables from ODE to algebraic equations.

And once you write the solution which is explicitly writing Y of S in terms of U of S and so on, I still need to be able to write the solution exclusivity in terms of X T, Y T, U T and that is done using what is called an inverse Laplace transform. Now it is easy to write this Laplace transform, whereas inverse Laplace transform is a more complicated expression which is a complex integral that you have to do, now that is quite difficult to solve compared to the forward transform which is the Laplace transform. So you might ask if it is difficult to solve the universal Laplace transform problem so what is the point in doing all of this?

The point in doing all of this is the following, in almost all cases I am going to show you that we are never going to do this integral itself every time we encounter a problem and as a reason we are not going to do this integral either, so the key idea that you want to remember as to why Laplace transforms are very useful for engineers who do not know complex analysis, residue theorem and so on is the fact that I can simply look at tables and then do all of this without actually doing the integration so all the hard-work in some sense is already being done. So the way we use Laplace transform is the following, so you will have a table which we will see in the next lecture.

So here you have different functions F of T and here you will have the corresponding Laplace transform for those functions already worked out. So if you are looking for Laplace transform what you do is you go from the left side of the table to the right side of the table, first you look at what function you want a Laplace transform for and then look for that function in this column and then you go to the right and then pick the corresponding Laplace transform. Now if you are interested in the inverse Laplace transform, if I have done this right then the inverse Laplace transform is very simple, what I do is I go and look for whatever I want to invert in this and then say okay I find it here so I go in this direction and I get the corresponding time domain function.

So because of this lookup table where I can go from one side to another side and since this has been worked out for a large number of problems and in fact I will show you later through the idea of partial fractions you can really use only one or two of these entries to do wide variety of problems then Laplace transforms become very-very useful. If you have to actually do the integration and then compute the inverse Laplace transform it is much more complicated and it is something we usually do not do ok.

So I hope this lecture has given you good idea about Laplace transform, how the Laplace transform converts ordinary differential equations into the algebraic equations and how once you have these algebraic equations you can solve for the variables of interest which will leave the variables of interest still in the Laplace domain then what you do is you do this inverse

Laplace transform and then get back expressions for these variables in the time domain so that basically completes the whole cycle of solving and interpreting the solution using Laplace transforms. So what I will do in my next lecture is I will take some concrete example of problems and show you how all of this works with numbers and so on so you get more comfortable with Laplace transforms.

And once that is done and I give you general ideas about partial fractions and so on then you will be well equipped to look at these kinds of equations and see how to work with them, I will see you next lecture thanks.