Process Control - Design, Analysis and Assessment
Professor Raghunathan Rengaswamy
Department of Chemical Engineering
Indian Institute of Technology Madras
State Space Modeling

We will continue our lectures on process control. In the last class we talked about how to model a process. I talked about writing conservation equations, phenomenological models, defining variables, writing these equations for control volumes and so on. In this lecture what I am going to do is I am going to show you the same model in a form that is called state space modeling. Typically, undergrad control classes do not talk about state space modeling.

However, it is a very simple extension to the standard equations that we showed last time and it is a very useful framework when we start doing multivariate process control or multivariable process control. So it is important to understand state space modeling and the way I am going to do these lectures is I showed you the first principles modeling last time. I am going to show you the same thing in state space and I am going to show you how you solve this state space models in time domain, get some results and then show you how you can solve the same problem in frequency domain.

We will introduce Laplace transforms at that time and then I will show you the connection between solving these equations in time domain and frequency domain using Laplace transforms. So ultimately at the end of this series of lectures you would have understood Laplace transform and you would also know that Laplace transform is not some general mathematical abstraction but it is basically representing fundamental physical process.

The only thing as I mentioned in the last lecture is that while most fundamental physical processes are non-linear by nature, the use of Laplace transform is largely for linear systems. So you have to take this additional step of linearization of the model. So you have first principles model which itself is a conceptualization of a process and there will be some errors in the conceptualization. It will turn out that these models are non-linear however we want to work with linear models because it makes the theory and implementation easier.

So we linearize the nonlinear model to linear model so there is a next level of approximation there and once we linearize we get a transfer function model which we will see after we see Laplace transform. And you have to understand that this transfer function model is a representation of the two processes.

I will be with approximation so the quality of the model will depend on how approximate these models are in terms of what is the impact of linearization and what is impact of the errors that we might have made in our conceptualization and so on. So you have to remember that while these models are very useful in control, they are still approximate. Now with this preamble let us go back to the previous lecture that we saw in this course which was simple liquid level system model.

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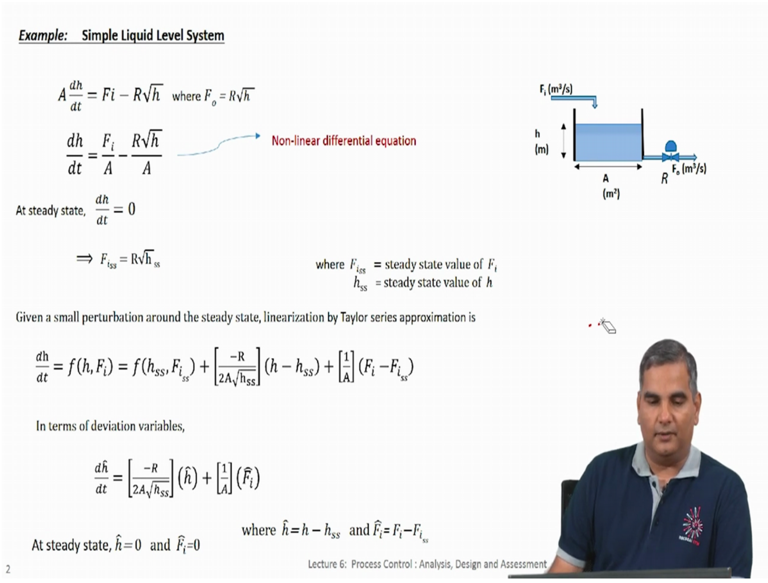

And from last time we did see this. You write this mass conservation equation for the control volume of liquid in the tank and then you use this phenomenological model to come up with consolidated model here. And we said that if you wanted the behaviour of this process at steady state then you have to set d h by d t equal to 0 which will give you this equation, this flow inlet at steady state is R times root of height at steady state.

Then we said because of this term right here if this is a non linear model so we linearize this and we linearize using Taylor series approximation. And if we linearize this root of h we take up to the first term which is what gives you this term here.

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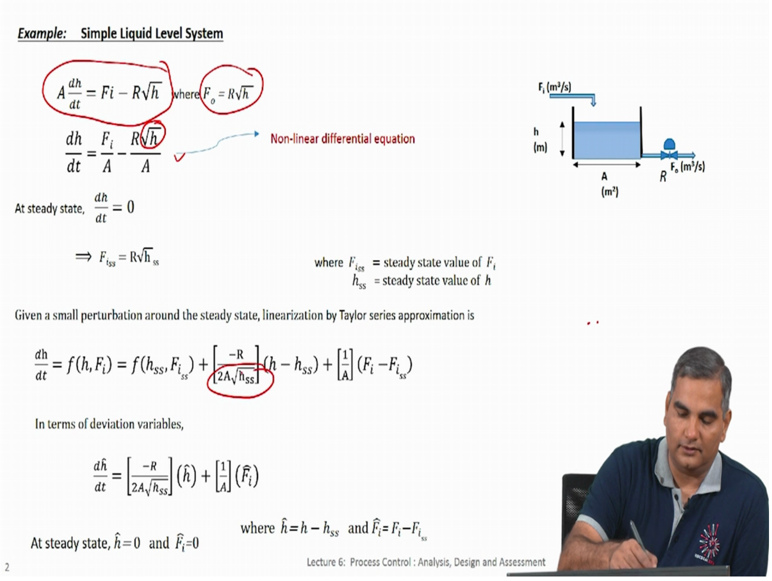

And I also said anytime you linearize you have to evaluate that function at an operating point and the operating point that we chose is this h s s. So that is the operating point on which we are linearizing which you can see here h minus h s s. Remember from your high school whenever you do a Taylor series approximation you will have F of x 0 plus F prime evaluated at x minus x 0, this is the term like that.

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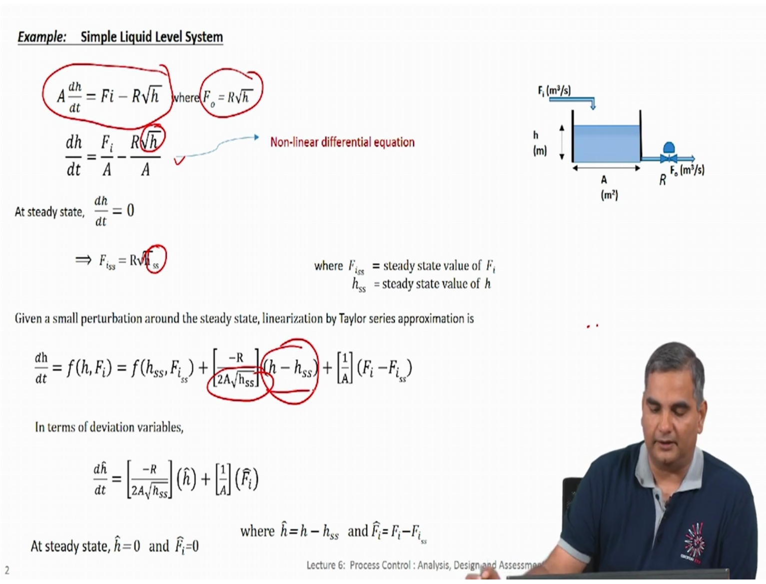

So after all of this algebra is done when you put all of this in terms of deviation variables, this is the form that you get here. The hat basically symbolises that at t equal to 0 this h hat is 0 simply because we start from a steady state. So this equation has been derived assuming we start from a steady state. The actual value of h is h s s and the actual value of F i is F i s s but because we define h hat has h minus h s s at time t is equal to 0, if h were in steady state then this would also be h s s so h hat will be 0.

So that is what we have written here, h hat is 0, F i hat is 0. So look at this equation. This is the form of the equation that we are going to use quite a bit as we go forward.

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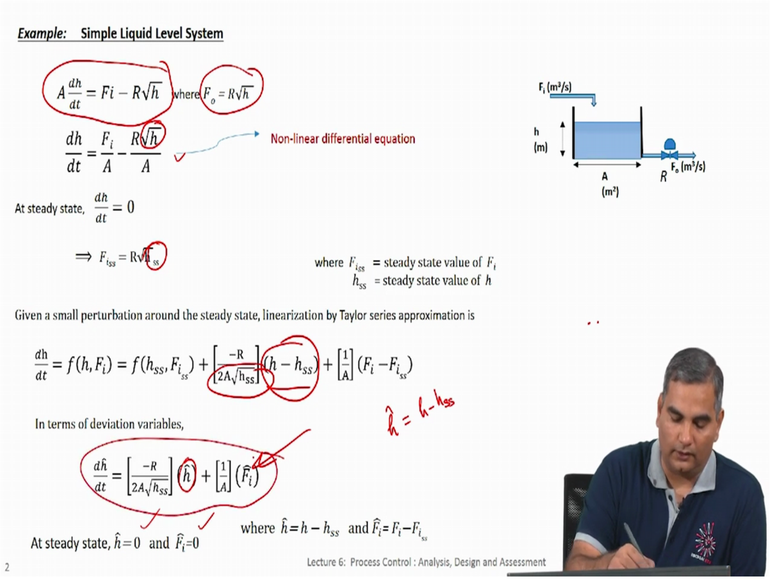

So this is something that we saw in the last lecture and this is a recap of this and I wanted to get to this equation because the notion of our state space model starts from here. So in general a state space representation is written in this form x dot equals A x plus B u, y equals C x plus D u. Here x is what we call as a state variable, y is a measurement or output variable and u is the input or manipulated variable. So in an undergrad course we call this a manipulated variable.

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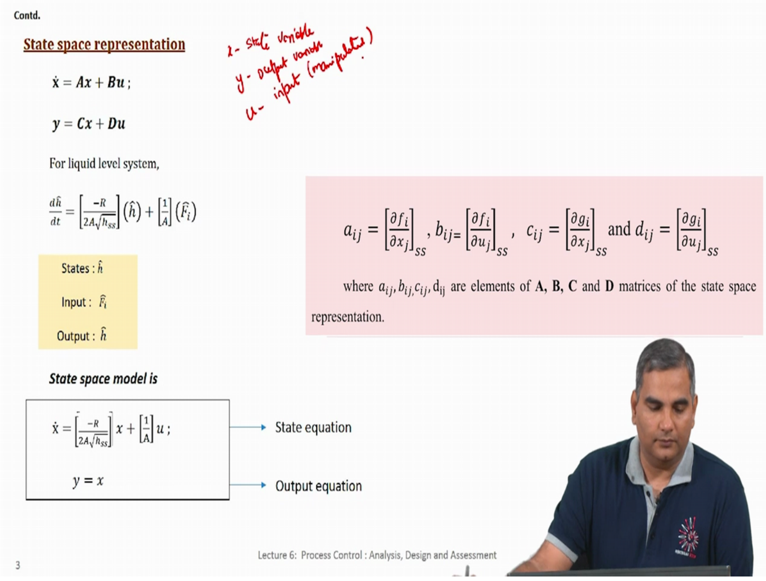

Now typically this is written in the vector form so you could have x with a dimension of n by 1 that means there are n states so x is n by 1. You could say the output variables are let us say p in number so this would be p by 1. And the input variables are let us say m in number so this could be m by 1. So this is the general vector form of the state space representation. We will get back to this when we talk about multivariable control.

But since I am introducing state space representation for the first time I thought I will do the general form and then quickly move on to the simple liquid level tank example that we see and then tell you how all of this figures out in this tank example. Now if x is n by 1, x dot is also n by 1 so this is a derivative of the n state. So, this is n by 1 so this A matrix has to be n by n, only then you can do the matrix multiplication.

And similarly because u is m by 1, there are m inputs. This B matrix has to be m by m so that this matrix multiplication can be done. Now since y is p by 1, the C matrix has to be p by n because x is n by 1 so this multiplication can be done. And similarly this D matrix has to be p by m and u is m by 1 so that you can do the matrix multiplication. So this is the general scheme.

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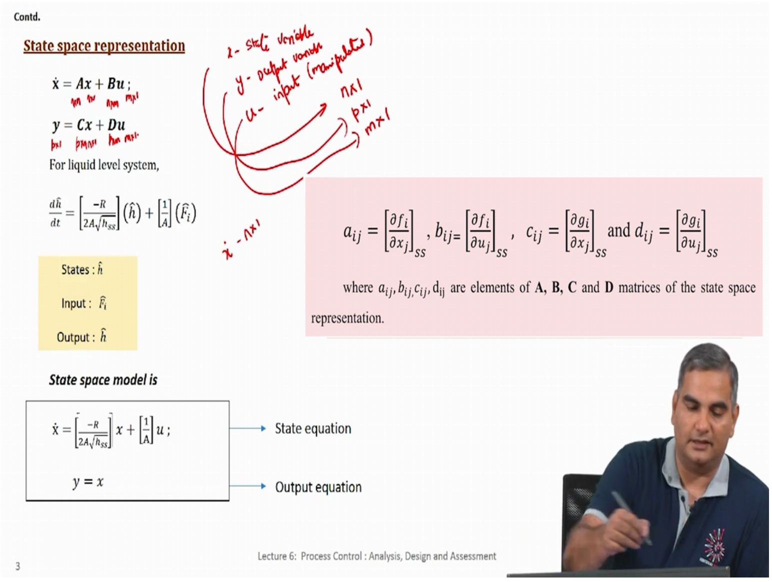

Now if n is 1, p is 1, m is 1 then I have only one state, only one output variable and I have only one input variable. Notice an interesting thing here. When we talk about SISO control we talk about actually one input and one output is what we talk about. So that basically means m is 1 because we said there are m inputs and p equal to 1 because we said there are p outputs. Now n can be anything. So n can be 5, 10, 1, 3, whatever it is.

So these are not dependent on each other, right? So you could have a system where you have five states but only one input and one output. So the key take away from this though it is not relevant right at this point, it will become relevant later, is that you could have a SISO system which is single input single output system, one input one output. However, for the same SISO system in some cases it could have only one state or it could have five states, ten states and so on.

So the notion of SISO really denotes how many inputs and outputs you have and not the number of states you have. Now let us go back and then look at the liquid level system and when we look at the equation from the last line in the last lecture you have equation of this form. And if the notice this equation you will see if I want to write it in this form I will say h hat dot because I am taking a derivative of this is some a h hat where this a for example can be this plus b u, right, where this u is F i hat and this b is this so you can also write this as F i hat, okay.

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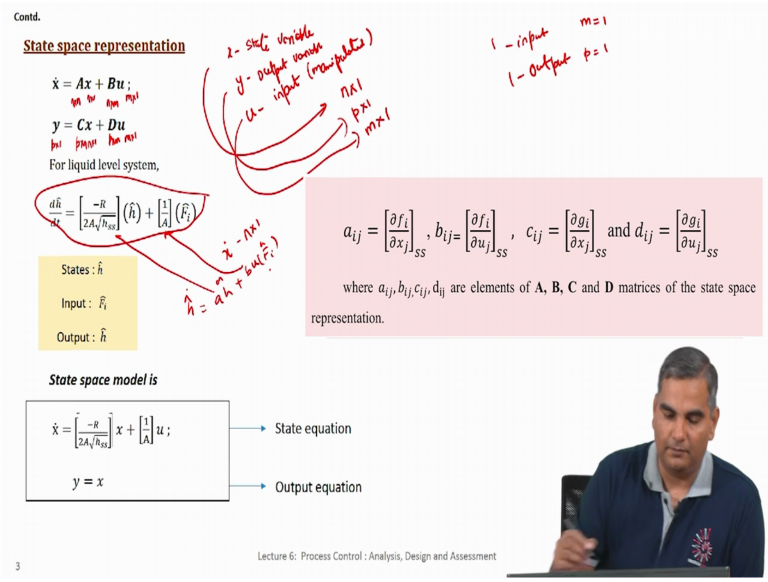

So now you noticed the close similarity (bet) between this equation and this equation here. The only difference is that it is really not a difference. The thing you have to note is that this number of states is one in this case, right? So this matrix equation or a vector equation if you want to think about it that way becomes a scalar equation here. Okay, so there is only one state and there is only one input F i. So if you think about this then you can say the states h hat and the input is F i hat.

Now what about the output? If we were solving a control problem where we are essentially measuring the height of the tank and the output of interest that we want to control is really the height of the tank then the output is also h hat, okay. So that is the mapping from this liquid tank example to state space example. So I just want you to understand really that after a while we will simply be saying x dot equal to A x plus B u but you want to understand that these parameters and this model equations are not abstract.

They come from real physical system and if you were to model a liquid level tank problem as a state space model you would simply write A x plus B u but you have to remember that A is computed using this resistance, it is computed using the area of the tank and also it is computed using the steady state of operation. So these parameters themselves represent or reflect the process that you are trying to model.

So that is a very important idea to keep and you should not lose this connection because as we go learning more and more about this we will tend to abstract all of these concepts in terms of models and after a while I do not want it to feel like it is pure math whereas it is really not because this A and B you remember if you have to solve a problem, have to be calculated and those calculations will bring in the specificity in terms of the process that you are trying to model.

So after all of this the state space model would be of this form here because I said h hat is x, x dot is this times x plus this times u and since h hat itself is output, y equal to x. So there is only one output and that turns out to be the one state that we have in this process. And this is a fixed number because resistance is a fixed number, area is a fixed number and we know what steady state we operated and again this is a fixed number.

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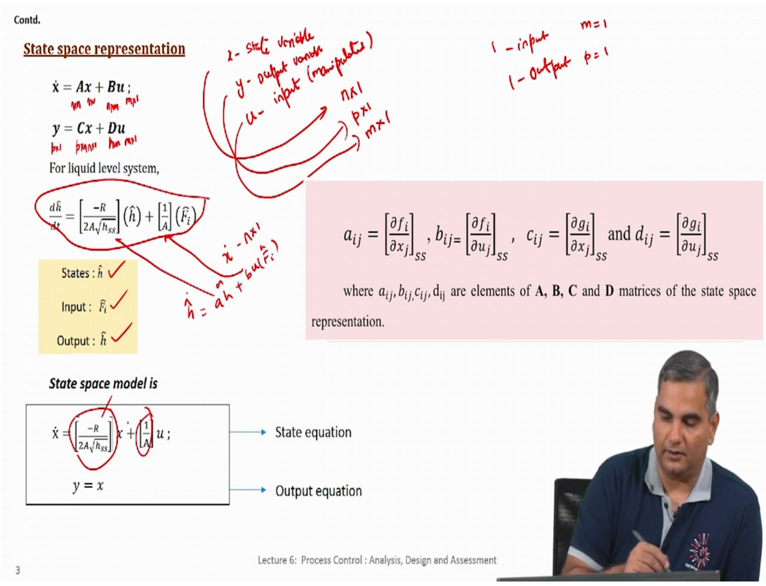

So this is an important thing to remember. Now if you had more than one state and more than one input and output then when we get these fixed numbers we have to do more derivatives for different functions and so on so that we can get a general nonlinear model into this form right here. So if there are multiple functions you have to do the derivative of that function with respect to the multiple states and so on.

So we will get back to that aspect of how do I convert a general multivariable nonlinear model into a linear state space model like this? Right now we have a simple one state one input one output non linear model which we have converted into a state space model. Now let us do one more example to see how this works in a very simple multivariate setting.

To simplify this and not complicate this with the derivatives and Taylor series and so on you will notice that I will use a slightly different phenomenological model for characterizing a flow as a function of height. Again this is an approximation. You can directly use this approximation or you could write the other phenomenological model and linearize that model to get to this approximation. The quality of models will slightly vary, however both are approximates.

So that is something that you want to remember. In most of this modelling there is nothing called absolute true. This is absolutely the true model, right? We will still talk about the true process. That is our conceptualization of the true process. The true true process which is the real process will always be abstracted out a little bit whenever you do modelling. So that is something that is very important to remember. Now just like what we described in the modelling exercise before, let us do that exercise here.

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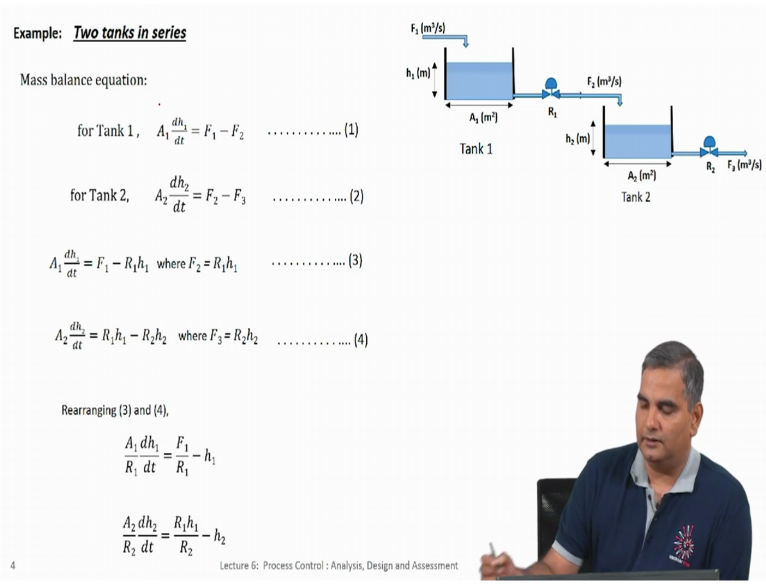

So here there are two tanks and there is flow into one tank and there is flow out of tank which goes into another tank, right? So this is an example which is used quite often in all control textbooks. And this is called the example of non interacting tanks. So there are two tanks which do not interact with each other. Now as before let us start doing the variable definitions. So I know that there is height of the first time so that is the variable that will be of interest to me. I know that flow in I have to define a variable.

I know that for flow out I have to define a variable. So if I define these variables for the first tank it looks like I have enough variables that I need to write the model equations. I do a similar exercise for the second tank and when I come here I do notice that I already have a variable for the inlet so I have to only add two more h 2 and F 3 which is the flow out of the tank. So if you collect all of this as of now I have five variables.

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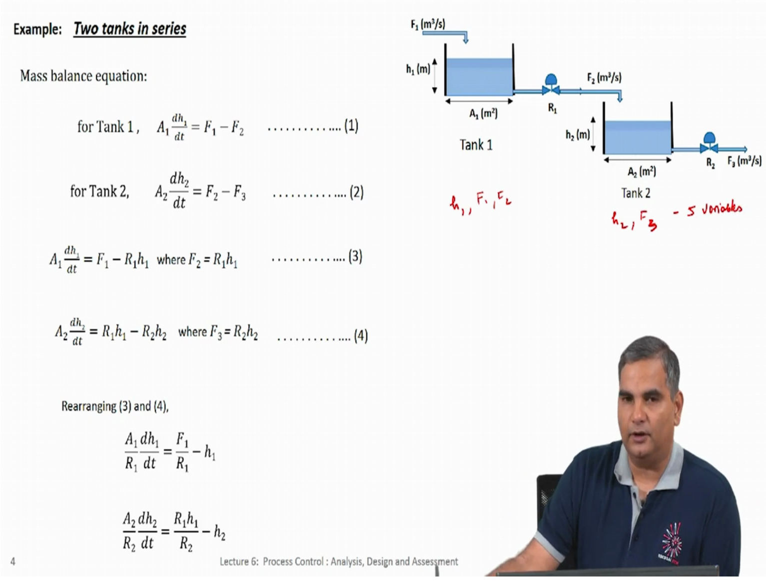

So like before remember, I need to look at enough equations for this variables. Then let us go back to the first tank and then see if I have to define any more parameters. We would define A 1 as the area of this tank. Like last time we could also define a density but we know from the last exercise that the density cancel out if you assume the density is constant.

So to keep the algebra simple I am not going to define a density. However, we know that for this valve we have to define some resistance. So let me define a parameter resistance R 1, okay. So these are two parameters for this tank and similarly for this tank I will have two parameters A 2 and R 2.

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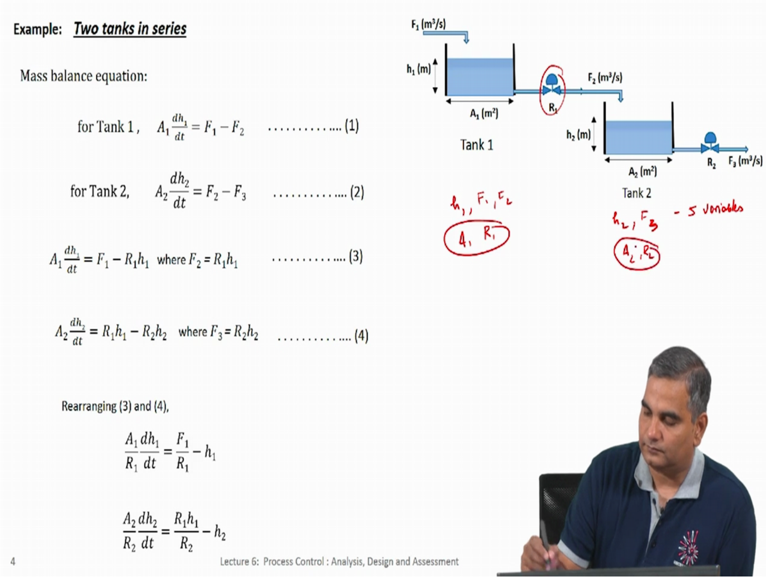

Now I have five variables and I have four parameters. As I said before these parameters are not calculated as part of the model but they are given to the model. Now let us take a close look at the five variables that we have for the process. And then if I draw a battery limit here something like this then I know the only input into this process is F 1. So there is no way in which I can compute F 1 out of a model that I write.

So I am going to call F 1 as an exogenous variable and I am going to say you do not give me the value of F 1, I cannot solve these model equations. So we will have to really look for only the four variables. F 1 comes from outside this system. That leaves us with variables h 1. F 2, h 2 and F 3 that I need to have equations for. And clearly and logically last time we wrote one control volume, this time we do see that there are two control volumes which are independent of each other that I can write a balance for.

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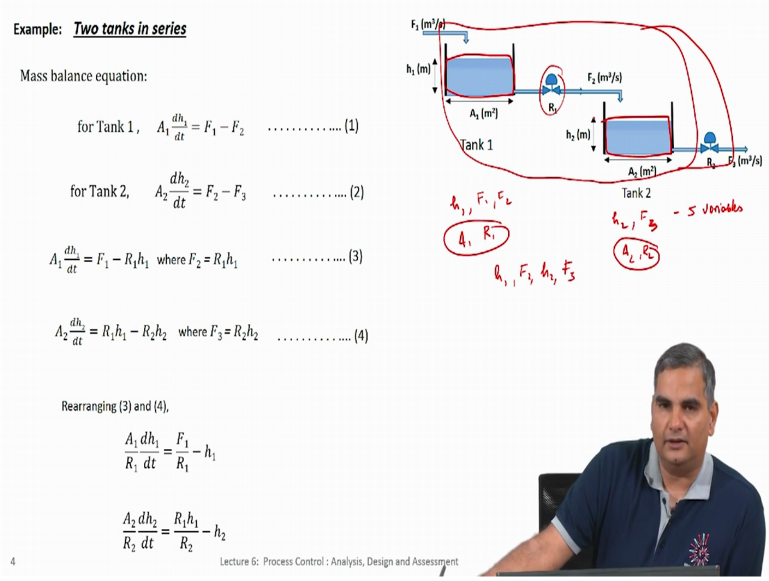

Now I mentioned that you could choose several different control volumes and the model equations though they are the same, in many cases will look slightly different. For example, in this case you could write balance for one tank and an overall balance for both the tanks. So you could write a balance for tank 1 and an overall balance for tank 1 and 2 together. You could write a balance for tank 2 on an overall balance or you could write a balance for tank 1 and tank 2.

So all of those will be similar but nonetheless you cannot get more than two equations. So for example you cannot write the balance for tank 1, tank 2 on an overall balance because overall balance will simply be the sum of the other two balances. So you can only write two equations. We have 4 variables but we know that we can use phenomenological models to get the extra equations. So if I were to write a balance for tank 1 just like last time, this is the accumulation term. There is a rho w remember which is getting cancelled out.

So this is actually mass accumulation, not volume accumulation just so that we say this scientifically correctly. Now that will be input minus output so there will be a rho w F 1 minus rho w F 2 and this simplifies to F 1 minus F 2. So this is the mass balance for tank 1 which gives us our first equation.

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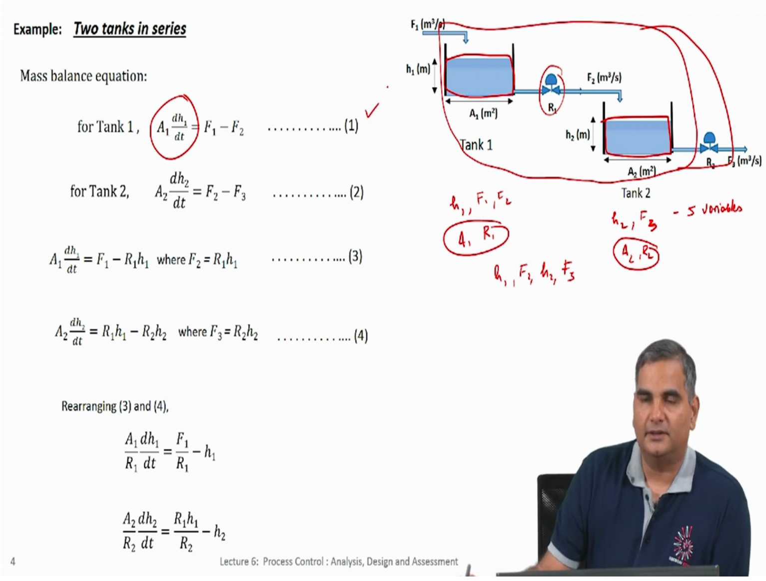

Similarly, we can write a mass balance for tank 2 which gives us our second equation. So if you notice this there are all the five variables that are participating in these two equations h 1, h 2, F 1, F 2, F 3 but clearly this we said has to be given to us for the model to work so we will not worry about that right now. So the remaining four variables I have two equations. So I have to look for two more equations and clearly for each one of these valves you can write this phenomenological model that we have.

Last time we wrote the flow across this as R times root h. We are going to simplify this and then say we are going to use a simpler approximation, a simpler phenomenological model where the flow is related directly linearly. So for example for flow across this valve F 2 I am going to say it is simply R 1 times h 1 which is what is written here and for the flow across this tank valve I am going to say the flow rate F 3 is R 2 h 2.

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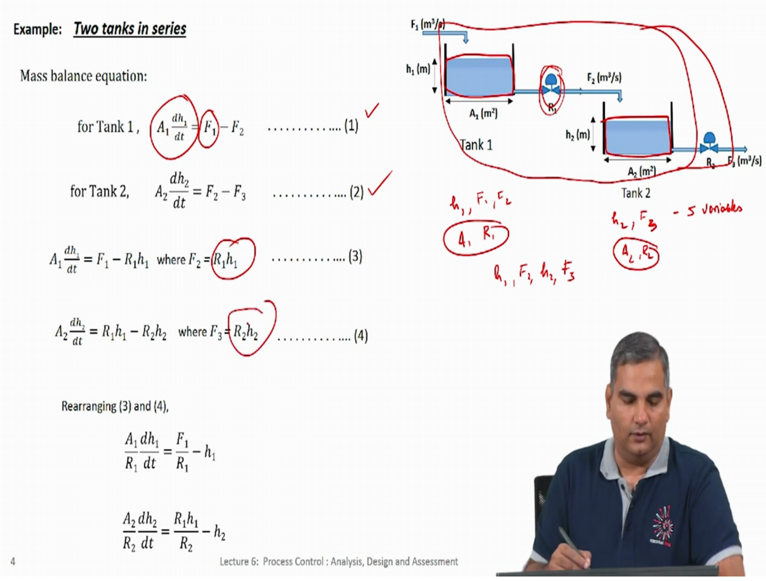

Now if you substitute these back into your equations then you will get A 1 d h 1 by d t is F 1 which is something that we need to be given. F 2 from this phenomenological model is minus R 1 h 1. And for tank 2 you have A 2 d h 2 by d t is F 2 which is R 1 h 1 minus F 3 which is R 2 h 2. So you have this equation. Now what you can do is you can rearrange this equation and simply get some of these terms to the left hand side and you will get this equation. So this is a very simple algebra. In the first equation I just moved R 1 to this side and then the second equation I moved R 2 this side.

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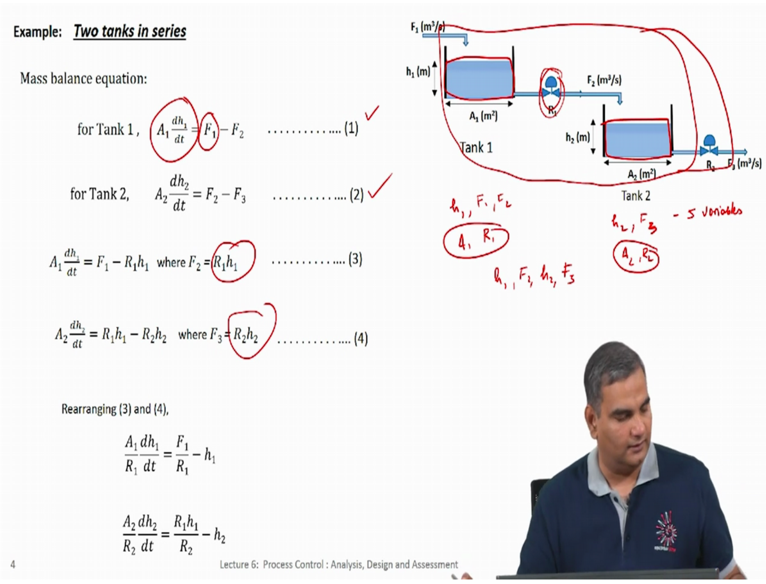

Now just to put this in a particular form what I am going to do is I am going to define some constants and I am going to start using these two constants tau and K which is used quite a bit in control. Tau typically represents what is called a time constant and K represents a gain and we will see this more and more as we do control.

So this slide is very simple algebra, right? So I am just going to show you the definitions and I am going to show you how this translates into state space model. What I suggest you guys do is simply look at the slide, start with the equation in the previous slide and basically go through this algebra and get to the final form so that you are comfortable with how we do this.

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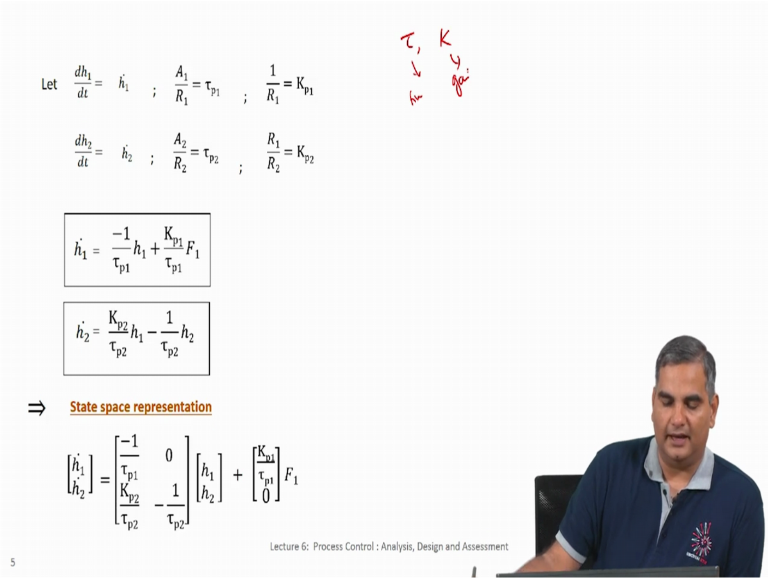

So in this case if you look at this I define d h 1 by d t as h 1 dot, d h 2 by d t as h 2 dot and I define two time constants tau p 1 and tau p 2 and basically I define K p 1 and K p 2. And do some algebra and finally I will get this state space form. So this is an important form. Now notice that if you look at this as x dot equals A x plus B u as we mentioned at the beginning of this lecture.

If you notice there are two states h 1, h 2. So x dot is going to be 2 by 1 vector and that basically meant A has to be a 2 by 2 matrix what we described last time. And you can say that this A is a 2 by 2 matrix and there is only one input u, right? So one input which is basically F 1 or F i. Then basically the B matrix has to be of size 2 by 1 and you can see that this is a 2 by 1 matrix right here.

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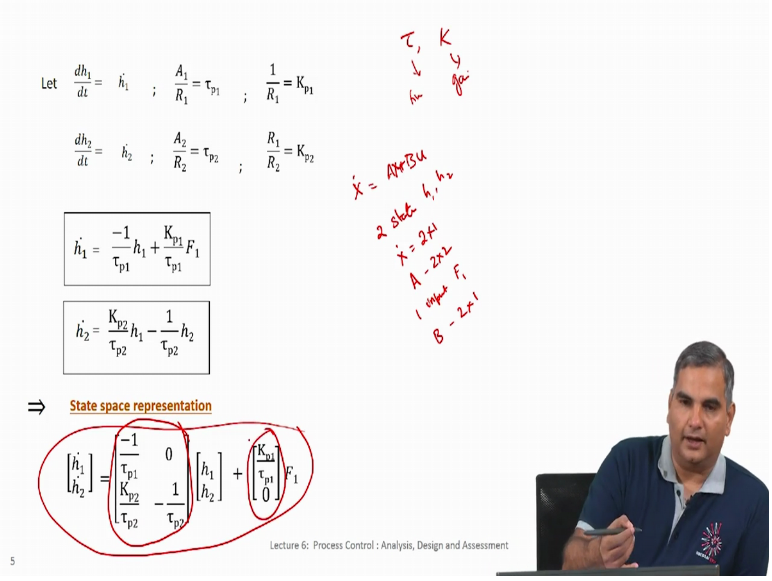

So that is how you have a multivariable state space model that comes about which you see using a very simple example. Now you might ask what happened to the output equation. So the output equation depends on what you consider as output. Supposing you are interested in measuring both h 1 and h 2 and then controlling both h 1 and h 2 then the states are h 1 and h 2 and the output will also become h 1 and h 2. So that will be two outputs.

That means based on our notation p equals 2. That means you should have the C matrix also be of size 2 by 2. In this case if you are output are h 1 and h 2 so basically the equation you will write is h 1 h 2 equal to 1 0 0 1 h 1 h 2. So this basically says h 1 is h 1, h 2 is h 2. So whenever the states and the outputs are the same then you will have an identity matrix for C.

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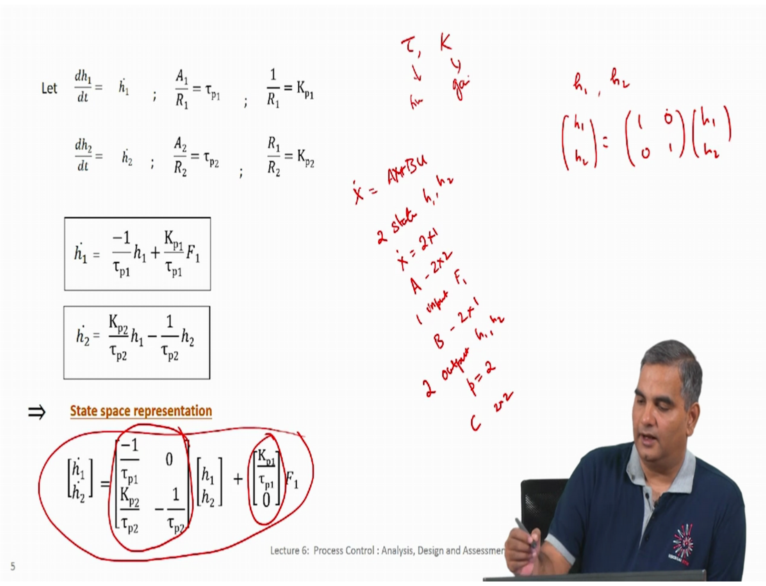

And in the previous slide I also had y equal to C x plus D u. In this case because there is no effect of the flow directly on this, D equal to 0. So that is how the state space model will look like. Now this we cannot call it as a SISO system because though there is only one input which is F 1 or F i, there are two outputs h 1 and h 2. So this is actually two outputs one input system.

Now if it turns out that your interest for whatever reason is only in maintaining the level in tank 1 then h 1 is your only output. So on in this case p equal to 1 so the C matrix will have to be of size 1 by 2, right? So when you write this h 1 you will write this as 1 0 h 1 h 2 again plus D u, since there is no effect of u here, D is 0. So if you notice this, now h 1 is h 1 is the equation and C is 1 by 2.

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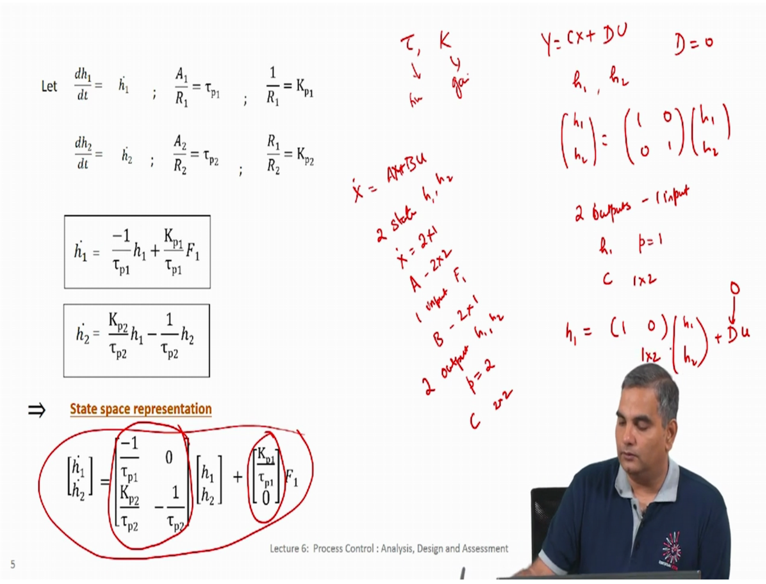

Now for some reason if your interest is in controlling h 2 or the output is h 2 then you will write h 1 equal to 0 1 h 1 h 2 and similarly D u, this is 0, okay. Now you notice that the C matrix differs slightly depending on what your output is. However, the difference between these two cases and the case where both h 1 and h 2 are output is that these two cases are SISO cases, single input single output case because the only output is h 1 here and the input is again F 1 or F i and the only output here is h 2 and the input is F 1, okay.

So basically what this says in this example is I have a SISO system however I have more than one state. In fact in this case I have two states. So this is how we convert a physical model into state space model. In this case by a choice of proper or relevant or phenomenological model that obviates the need for linearization I was able to directly write this in the linear state space form.

However if I had chosen to model the flows as root of h 1 and root of h 2 then I would originally get a nonlinear state space model and then I linearize that state space model and put it into this linear state space form.

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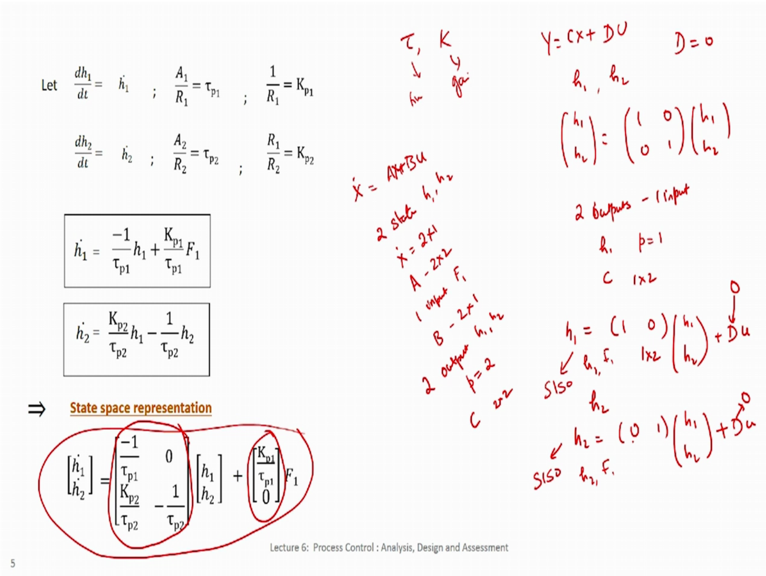

That might be a good exercise for you guys to attempt, to see and understand how all of this works. So with this I will conclude this lecture and I will see you in the next lecture. Thank you.