Process Control-Design, Analysis and Assessment
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Model Predictive Control-Mathematical Formulation-Part-1

In the last lecture I talked about the conceptual idea of model predictive control and then talked about the notion of an optimisation problem that we will solve, the notion of control and prediction horizons and basically explain how we are going to look into the future and then ensure that the future predictions are as close to the setpoint value that possible. And to do this we make what is called the move plan, that is what is the control move that I am going to make at the current instant and also into said the number of future instants, so that the mismatch between the predicted output and the setpoint are minimised.

In this lecture we will look at how we formulate this problem in terms of an optimisation problem. Since we are going to talk about an optimisation problem, I will also give a very very quick introduction to how optimisation problems are solved. In fact I am going to just use a couple of slides to quickly show you how these optimisation problems are solved, so that you can understand model predictive control in lot more detail and your understanding about model predictive control can be little more complete, so that is the basic idea.

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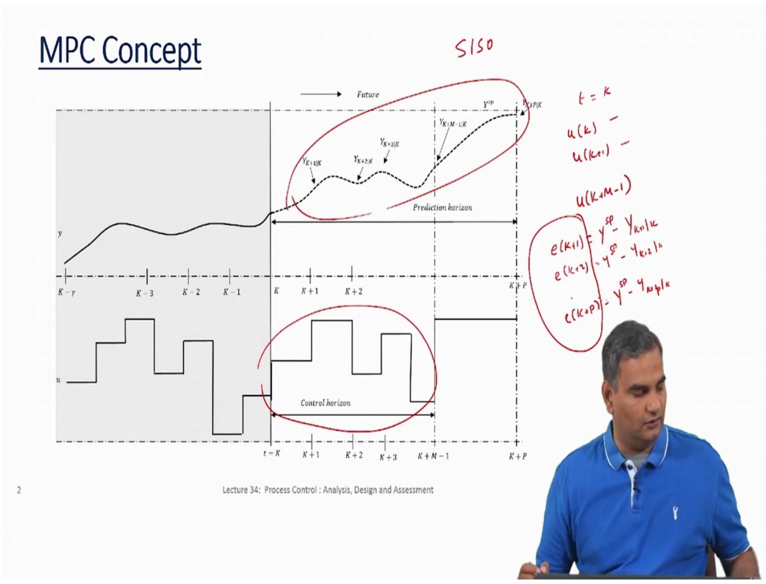

So, just quickly summarising from the last lecture, we talked about Control horizon, which is having a control move plan. So, basically if I am sitting at time T equal to K, if I am making control move plans, since I am still talking about SISO systems here, I have to make decisions about values of U at M steps. And what those M steps are, are UK, UK +1 and all the way up to UK + M -1. Okay. So, if you think about this, this is the 1st decision I have to make, what value that should be input take at time K and the 2nd decision is what value should the input take at time K +1.

And Mth decision is what value should U take at time K + -1. So, these are the M value that I have to choose for this control move plan. And since it is a SISO system, there is only one manipulated variable, so there are M values for this 1 manipulated variable that I need to identify. Okay, so that is how this part of the picture works. And now what criteria will I use to identify these values is where I want to use the criteria where the future predictions are as close to setpoint as possible.

So how would I decide this, how would I come up with the criteria to do this is what we are going to see the next few slides. So, in some sense if you think of the error as error at K +1 is Y setpoint, whatever value I keep mine is Y at K +1 given K and error that K +2 is similarly Y setpoint - Y K +2 given K all the way up to error at K + P is the Y setpoint - Y K + P given K. So, then basically, somehow I have to come up with criterion which jointly or collectively minimises all of these values. So that is a very important idea. So, the optimisation problem should have some way of a jointly or collectively minimising all of these errors is what we are looking at.

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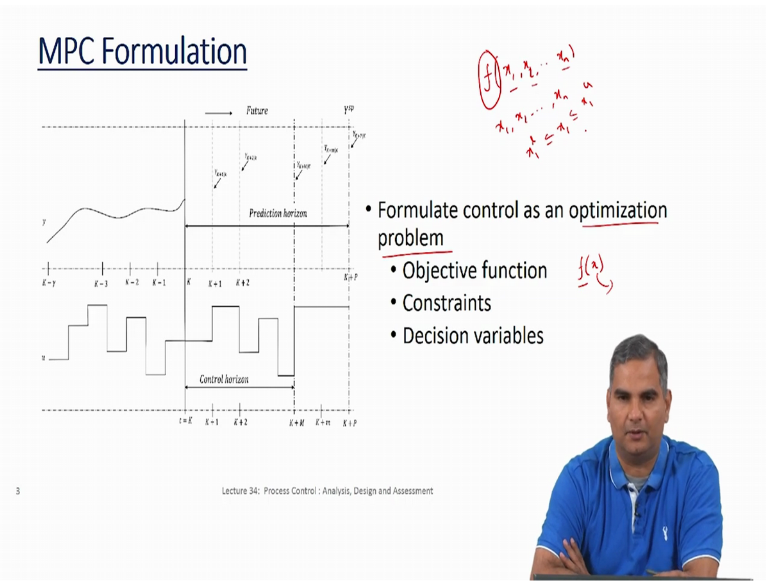

So from an MPC formulation viewpoint, what we are saying is we want to formulate this model predictive control as an optimisation problem, okay. And whenever we talk about an optimisation problem the optimisation problem usually has 3 components. So, if you understand these 3 components, then you know exactly what is happening. So, there is always an objective function f of x which is basically a scalar function in simple optimisation problems. It can also be vector functions when we talk about multi-objective optimisation problems and so on which all far too advanced as far as we are concerned from this course viewpoint.

So as far as this course is concerned, the objective function is some function of variables and it is a scalar function. Now, when I say this is a scalar function, this x can be several variables that you are trying to optimise. So, typically an objective function for the cases that we are looking at will look like this, there is some function of x1, x2, all the way up to xn, so there might be looked n variables, m variables, whatever it is. So, this is an objective function and now we call these variables which are part of this function and we are basically looking for what these variables should take in terms of their values is the things that we are looking for.

So, basically what we are saying is whenever I have an optimisation problem, I have an objective which is a function of several variables and I am going to make decisions about what values of these variables should take. So, these are then called the decision variables x1, x2, all the way up to xn are the decision variables. And basic idea is I am going to make decisions about these variables such that this function is either maximises or minimised, okay. So, in other words what we are saying is what values of these variables will maximise the function or minimise the function, okay. So these are called decision variables.

And a typical optimisation problem also has constraints, the constraints could be these variables have to be between lower bound and an upper bound. So, this is the constraint or some other functional former which involves these variables are between some values and so on. So, you can have several types of constraints here, so the constraints are usually are the decision variables themselves or some functions of decision variables. So, it can be either directly on decision variables or some functions of decision variables, okay. So, that is basically very very compact, very quick, simple explanation of an optimisation problem.

Now there are several types of optimisation problems depending on what type of constraints you have, what type of variables, what type of functions you have and so on. And we are not going to talk about all of those here, so we are going to keep it very simple, very focused towards the type of optimisation problem that we need to solve for solving model predictive control related problems, okay. So, really the summary of this is that an optimisation problem has 3 components, you have to define an objective function, you have to define constraints and you have to define the decision variables.

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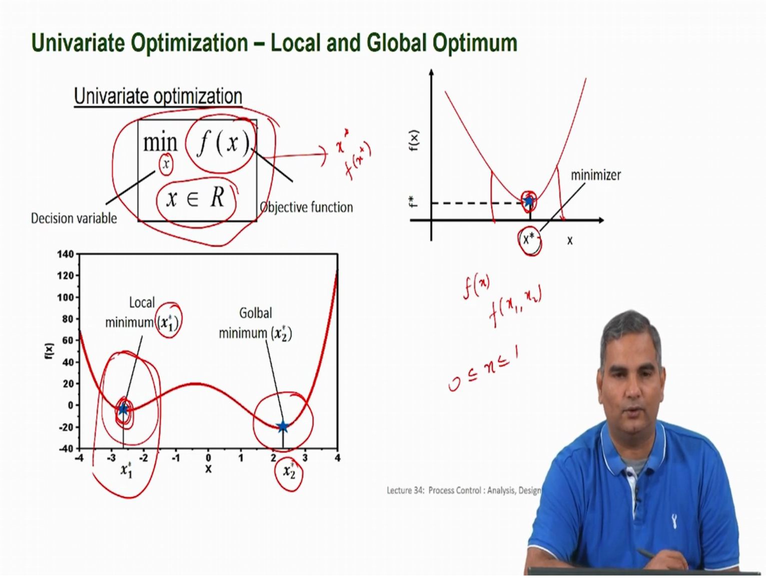

And when you actually define the objective function, the decision variables are also defined because you actually write the objective function in terms of these decision variables themselves. Let us see how all of this fits into this MPC formulation idea. Before we do that, let me explain how you find an optimum solution, so that it becomes easy as we go along, I far as this model predictive control is concerned. So, again I said f is always a scalar function, this is a function that I am either trying to minimise or maximise. Typically people will talk about minimisation problems, most of the times because maximisation problems can also be solved as minimisation problems.

So, you do not have to really worry about whether it is a minimisation or maximisation problem, both of these are completely equivalent. So, we will basically only focus on minimising say an objective function. So, while you write optimisation formulation, I said f is a scalar variable, now within f, the decision variable could be one or many, if that is only one decision variable, it is called univariate optimisation. If there are more than one decision variables, let us say even 2, that this is called the multivariate optimisation problem, okay. So, more than 1 or single univariate. So, let us look at a very simple univariate optimisation formulation here. And these are some typical ideas that you have seen hopefully in some other cause or at your high school level.

So, when we write an optimisation problem, I mentioned in this lecture that you are going to have an objective function, decision variable and constraint. So, the way you write this as this is your objective function and this is your decision variable, so typically you write min and below that you write the decision variable, you write the function here and these are constraints. So, x elements of a real means, now extend take any value here but in some cases you might say x is less than equal to 1, less than equal to 0, okay, so this could be a constraint, okay.

So, you can think of any constraint and as I said before, the constraints are written in terms of the decision variables or functions of decision variables, so, both are possible. Now, you might ask, so what is this optimisation problem solving? So, suppose you were to plot the function f of x as a function of x, what this optimisation problem is trying to solve is it is trying to find the value of x star at which the function takes the smallest value because we are trying to minimise this here, right. So, that point would be this year because any other x if you take f of x, will be larger than this.

So the absolute smallest value that f of x can take for this function will occur at this x star. So, when we solve this, what we are going to get as a solution is basically this x star. And once you get this x star, then you can actually find what the minimum function value is by substituting x star and f. So, from x star you will get f of x star, which is the minimum value of the function. But the key result that any optimisation problem that we are looking for is the solution which is the values of the decision variables should take so that the function is optimised, in our case minimised. Okay.

So you can think of an outcome of solving an optimisation problem as values for the decision variable. So, this is very very important to remember because when I connect this to control, we are going to write the optimisation problem with the decision variables which are the move plans, right, the choice for the input values that we talked about, right. So, I have UK, UK +1, all the way up to UK + M -1. So, those M variables, if I can find values for that, such that some objective is optimised, then I have my move plans. So, that is the connection between control, model predictive control and optimisation.

So this is something that you should keep in mind. So, the outcome of solving in optimisation problem is the solution and the solution is basically telling you the values that the decision variables should take so that the objective is minimise, okay. So, in a simple case, this is called a Uni model function, in simple cases there is only one optimum solution which is this and you can get this x star. But in more complicated cases, you could have functions which are like this, these are not Uni model functions, they have more than one minimum.

So, if you look at this point here, if you look at in the locality of this point, you will clearly see that this is the lowest value it can, this function can take at this point. So, this is called the local minimum x1 star and if you look at around this region, then the lowest value the function can take is lower than this and the decision variable value at which this happens is x2 star, okay. So, this is called a local minimum, this is called a global minimum. Now, I do not want to get into too much optimisation theory , let me just tell you that it is very difficult once you identify an optimum solution based on certain conditions which I will show you the next slide.

Very difficult to identify whether it is a local or global minimum. You might ask a question saying why is it difficult because if I find this point and this point and then I compute the function value, is clearly this point function value is going to be lower, so this is a global minimum and local minimum. However the difficulty lies in identifying all these points, so in other words supposing you started optimisation problem and then you identify this point based on some conditions, now once you, this point will satisfy all the conditions that are required to be satisfied for a point to be a minimum and you will get a function value.

Once you get this point, you do not know that there is another point that exists like this. So, you have to run an optimisation algorithm again to find this point and then compare, right. Will you find all these points, you can never claim something gives local or global minimum. Right, if I had only this point, I cannot claim, it is either local minimum order global minimum, unless a priori I have proved something which is again not at the level of this course. But the key point being once identify this point, I cannot say it is a local or global minimum, unless I innumerate everything else and then find the function value.

And then see that this is the absolute low value, then it is global, if it is not the absolute value, then it is a local minimum. So, this is another consideration to keep in mind as far as we are concerned, this is not really an issue. So, we will also that whatever point satisfies all the condition for being a minimum, we are going to simply take that and proceed with the control computations. So, that is how we are going to look at when we talk about this optimisation from model predictive control viewpoint.

Nonetheless, I thought this is a very simple picture to explain the idea of local and global minimum without going into details so that you understand as we start understanding model predictive control itself, that at all points we are talking about local minimum. If the control or formulation is in such a way that I can show that the function is going to be of this form, then we do not have to worry about this local minimum at all because there is only one minimum and once you find a minimum, you know that is a solution that you have to live with.

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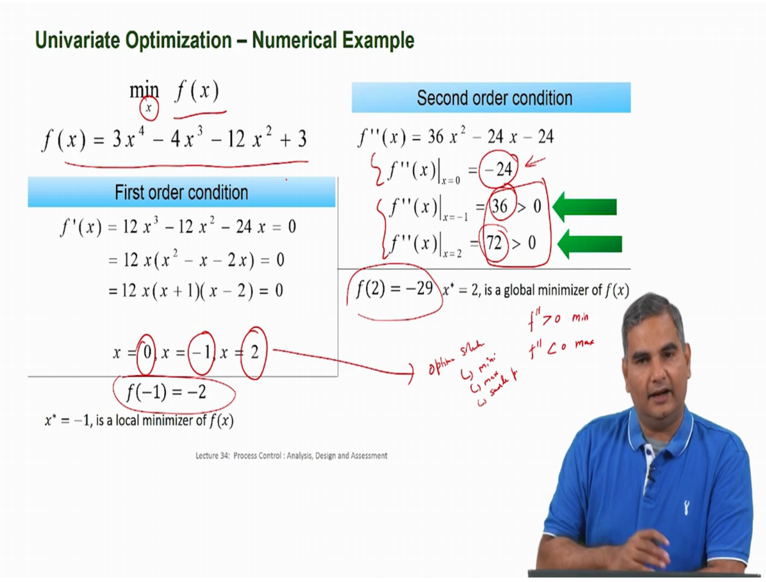

Okay, so let us take a very simple example. This is something that would have been taught to you in high school. So, you have a function f of x, the decision variable x and clearly I write the function in terms of the decision variable. Now I want to find the minimum point x star for this function. Then we know from our high school and probably some courses in your college that if you have to get the minimum point, what you do is you differentiate this function with respect to x.

So, when you differentiate this, you get 12 x cube -12 x square -24x equal to 0. Then you can solve for f Prime x is 0, then you will get these 3 solutions in this case, x equal to 0, x equal to -1 and x equal to 2. So, these are 3 solutions. Now, what this means if these 3 are all optimum solutions, okay. And based on some calculations, we can either find these to be minimum points or maximum points or what are called the saddle points and so on. So the way you do that is the following. You take a 2nd derivative and then if you take a 2nd derivative of this, 12 x cube -12 x square -24x will become 36x square -24x -24.

Now when you substitute x equal to 0 is this, you get -24, when you substitute x equal to -1, you get 36 and when you substitute x equal to 2, you get 72. So, now one you get the 2nd derivative, if you want to did defy the minimum points, whenever the 2nd derivative is positive, these are points that are minimum points. So, in other words of these three, -1 and 2 are minimum points and 0 is the maximum because this is negative, 2nd derivative is negative. So, 2nd derivative is negative is maximum and 2nd derivative is positive is a minimum point.

So, even in this case you already see that there are 2 minimum points. Now if you want to find which is the global minimum, then you have to basically substitute these values full so you take these 2 minimum points and substitute into f. If you substitute x star as -1, you will get a value for f of x which is -2. And when you substitute x equal to 2 is f of x, you get -29 for the function value. So, among these 2 minimum points, this x star equal to 2 is the global minimiser of f of x. So, this is the basic idea of univariate optimisation.

So, take a look at the slide, please go through the derivations yourself and basically as long as you are able to come to the same conclusion, you have understood very simple univariate optimisation. The only thing is this f double prime is greater than 0 for a minimum point and f double prime is less than 0 for maximum point, okay. So, these 2 are minimum points, this is the maximum point, okay. So, this is a very simple idea. Now, if the same thing you have to extend it to multivariate case.

So when we talk about, the function is still a scalar function, there is only one f. When we talk about multivariate, now instead of just one decision variable, there will be more decision variables that we have to make a decision about in terms of what values that they need to take. So, how is the multivariate case handled is what I am going to show you in the next slide. And that is a simple logical progression of this. I am not telling you why for a minimum point f Prime has to be 0 and maximum point f Prime has to be 0 and f double prime is negative is the maximum point, f double prime is positive is a minimum point.

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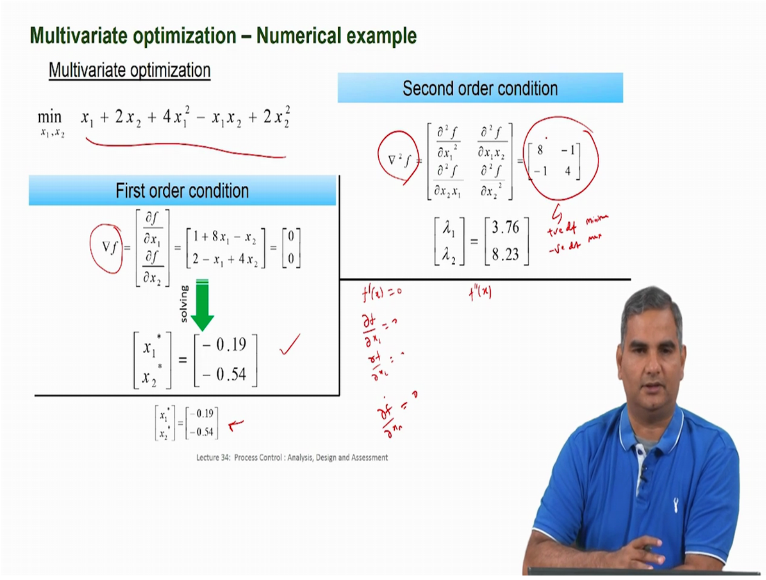

Why that is all is true is something that I am not describing or discussing here. I just want you to understand how these are solved, so that you get a feel for how model productive control problems are solved in terms of solutions to optimisation problem. Now, as I said before, if I have a multivariate optimisation, again that is only one function but instead of the function being you know defined by one variable, in this case the function is defined by both the variables x1 and x2. So, this is to variable problem, so let us assume now f is a function of x1 and x2, how do I find out a value for x1 star and x2 star.

Because when we have more decision variables, we have to be able to say a value for each one of these decision variables, okay, only then we solve this problem. So, the idea is very simple. When I have only one variable, it goes just f Prime x is 0. When I have more than one variable, so basically you do a partial derivative with respect to each of these variables, dow f dow x1 is 0, dow f dow x2 is 0 all the way up to dow f dow xn is 0, okay. Okay, now if I have n variables, I will have n equations, right. Because I will have n derivatives to evaluate.

If I have 2 variables, I have 2 equations. So, in some sense I will have as many questions as variables to solve for, okay. So, if you apply this, we have to say dow f dow x1, dow f dow x2 is 0. So, in the vector form you can write this as grad f, which is basically reading these 2 functions one below the other, that is about it. Or you can look at them as individual equations and solve, really does not matter. So, when we do dow of dow x1, partial of x1 with respect to x1 is 1, partial of x2 with respect to x1 is 0, partial of symbol 4 x1 square with respect to x1 is 8 x1.

Partial of x1 x2 is x2 with respect to x1. So, I have dow f by dow x1 is this. And similarly if I go dow f dow x2, partial of x1 with respect to x2 is 0, partial of 2 x2 with respect to x2 is 2, partial of 4 x1 square with respect to x2 is 0, partial derivative of - x1 x2 with respect to x2 is - x1 and the partial derivative of this with respect to x2 will be 4 x2. So, have 2 equations and both of them have to be 0. Now, have 2 equations, 2 variables, I solve this, there is only one solution I get and this solution could be a maximum or minimum.

That you actually identify by doing what is called a Hessian matrix. And this is far too above what we want to teach in this course. But just quickly I will explain this to you. What you do is you to this 2nd derivative matrix, in the one variable case we look that f double prime x, in this case we look at a matrix a which is dow square f dow x1 square, Dow square f dow x1 x2, Dow square f dow x2 x1, square f dow x2 square. So, once you compute this matrix in terms of x1 and x2, you substitute the solution that you get x1 and x2 into this matrix, you will get a matrix like this.

Now simply if this matrix is what we call as positive definite, so if this matrix is positive definite, then this point is a minima and if this point is negative definite, that this point is a maximum. Right, so I am not defining positive definite, negative definite and so on. But this is how you figure out whether this point is the maximum or minimum point. As far as we are concerned, we know that we can find a solution to as many variables as we want, right. So, this function could be a function of multiple variables, we will never run out of equations.

So, if there are 3 variables, then I will have 3 partial derivatives to set to 0, if there are n variables, I will have n partial derivatives to set to 0. So, one thing that I want you to remember is once I write this objective function in terms of multiple variables, I can always find a solution. Once I find a solution, whether it is a maximum or minimum point, have to figure out by doing a Hessian and then finding out whether it is positive definite or negative definite. You do not have to worry about this, you simply have to understand that once I have a solution, there is a way to find out whether it is a maximum or minimum.

And any software program that you use to solve these optimisation problems will be able to do this computation, basically tell you that this point is a maximum point, minimum point and so on. So, this is just to understand that it can be done and the procedure by which software will do this. But as far as you are concerned, as far as this model predictive control portion of this course is concerned, all we need to know is that we can solve this objective function when it has multiple variables because this is an important idea.

And let me make the connection again. So, the reason why we are interested in multiple variables is because when I am making a move plan, have variable UK, UK +1, all the way up to UK + -1. Though they all represent the same control input, these are values the control input will take at different points. So, I am going to think of them as separate variables and then I am going to formulate an optimisation problem based on these variables. And I have told you that once I formulate an optimisation problem, I can always solve and get a solution to all of this UK all the way up to UK + -1.

Once I get a solution, whether it is a minimum or maximum point, you do not have to worry about that. Whatever programming platform you are using to solve these problems will be able to tell you what that is and in fact we will work that into the solution process itself. So, now that you have understood basic optimisation, how we approach optimisation problems, the next step is to really pose the model predictive control problem in terms of the horizon and so on as an optimisation problem and show you how we can solve this optimisation problem so that we can actually get a model predictive controller going. I will see you in the next lecture, thanks.