Process Control – Design, Analysis and Assessment
Professor Raghunathan Rengaswamy
Indian Institute of Technology Madras
Department of Chemical Engineering
Nyquist Stability Criterion – Part 2

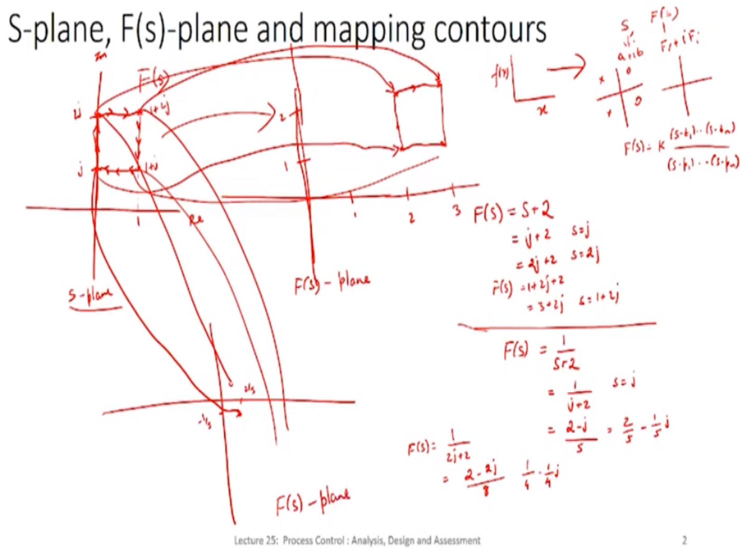
Let us Continue with our twenty-fifth lecture on the process control course we have been talking about NYQUIST stability criterion and how we understand a stability of time-delay systems we started doing this because the things that we have learnt in terms of partial fractions and abode a route stability table they are not useful anymore, so we said we now have a situation where I have transfer function which is a numerator by denominator transfer function and the denominator has this exponential term so I get a polynomial of infinite order so both the partial fraction idea and root stability table are not going to be useful anymore

So we were looking at other ways of doing this and in the last lecture I talked about how you think about this transfer function and the poles and zeros of the transfer function the reason why we are interested in the poles and zeroes of the transfer function is because the poles really tell you the stability of the system, so what we are trying to do is we are trying to find out if there is some way in which I can figure out if there are poles of this transfer function in the right half plane for systems with time delay elements particularly in the denominator polynomial.

So that is the reason why we are looking at F of S and looking at poles and zeros of F of S while all this result are going to be applicable for time delay systems at the beginning to explain the ideas, we said we will still look at numerator by denominator polynomial which can be written as root result forms with zeros in the numerator and the poles in the denominator, so we are going to look at those kinds of systems, we are going to show you the main result of NYQUIST stability and then connect that to control systems particularly closed loop transfer function and then finally we are going to say these result are valid even if the denominator has a exponential term.

So and you will see and that the computations are easily possible even with the denominator having exponential terms but the theory also hold for time delay systems thing that we are not going to prove but we are going to just state that and then that way we kind of complete the whole sequence of ideas.

(Refer Slide Time: 02:37)



So in the last lecture I mentioned this I said if you had a function X and F of X basically if you transited this to S and F of S remember I said this itself is a complex variable and this also is a complex variable, so I cannot just do it in this two axis so I need one separate plane for S and another plane for F of S so this is something we saw in the last call and then we also saw that if F of S some let us write this in this form some K times S minus at one all the way to is that minus Z M divided by S minus P1 all the way up to S minus PN we said that the zeros and poles of this transfer function will have to be plotted in the S plane because what you plot in the F of S plane is actually the value of F of S when S takes some particular argument however if you want to plot this zeros and poles of this transfer function itself you do not plot them in F of S you plot them in S plane.

So you might have zeros here poles here and so on so that is something that you have to remember so there is this distinction you have to remember, now what we are going to do is we are going to look at some interesting ideas when we think about this S and F plane, so let us assume that I have some F of S, so I am not even going to write this in the pole zero form yet but I am going to show you something called contour mapping and then we will pick up on this idea and then see how this is useful from a stability analysis point of view, so let us start with the S plane, so this are the set of values that S can take and here is an F S plane so, let us look at this now what we are going to do is something interesting so what we are going to say is, let us look at this S plane and I will draw something here and explain what I am doing here presently.

So let say this is J which is one time say at zero this is let us say J, let us say at this point at one plus J and let us say this point is one plus 2J, so you can see this so this is a real axis, so this is the imaginary axis, so when I go Y axis here then I if you take a single unit that is simply J, two units will be 2J and I do this here this is one, so this point will be one plus J this point would be one plus 2J now what we are going to do is the following and you will see why we do this when you finally bring all of this together right now what we are going to do is let us say I start a contour here and then start I start from this point and then I go along this line and then let us say I go along this line I go along this line and I come back to this line.

So this is what I am going to call as a closed contour which start from here goes around and then comes back to the same place, so this is closed contour and the other thing that I want you to notice about this closed contour is this closed contour is also moving in the clockwise direction so basically I have moving in this clockwise direction so let me pick a contour like this in the S plane and I want to know as I move along this contour for every point on this contour I can get a value for F of S because this is in the S plane so if I take any point here S will take a particular value I can substitute this into this and then see what the value F of S will take, so what I want to do is take a closed contour on the S plane and then see what will happen in terms of the F of S plane.

So when I say I want to see what will happen in terms of F of S plane what I am saying is every point on this contour if I compute an F of S and plot it here how will that look now since the beginning and the starting point are the same I would expect this F of S contour also to be a closed contour but the shape of that will depend on what F of S, so let us do this exercise to quickly understand how this works, so if let us take some F of S and then understand what happens let us say if F of S is actually something like S plus two for example then this point here where S equal to J if I substitute into this then F of S equal to J plus two when S equal to J, so when I take S equal to J I get J plus two.

So that means this is let us say one this is two and this is one and let us say this is two so J plus two will be mapped to this point here so basically you will have this point mapping to this point now if you take this 2J so F of S is going to be equal to 2J plus two when S equal to 2J so that point is going to be basically this, so this point is going to map onto this, now if you take this point S is one plus 2J so F of S will be one plus 2J plus two which will be equal to three plus 2J if S equal to one plus 2J so this point basically will be three plus 2J this point will map on to this point and you can also see that this point will map on to this point, so the four edge point I have shown you how they map on two different points on the F of S plane and any other point here also will be translated likewise.

So basically this will just be a translation by two units in the real axis of this contour, so this S contour will turn out to be the following contour here in F of S plane so this very-very simple idea of how this go, so any point here will also be plus two here so same you moved in relaxes by two, so this is a very simple idea so basically what we have done is we have mapped a contour in the S plane to another contour in the F of S plane the procedure is very simple I take different points in this pane and then compute the corresponding F of S value and simply point plot those in the F of S plane and this is very-very simple function F of S.

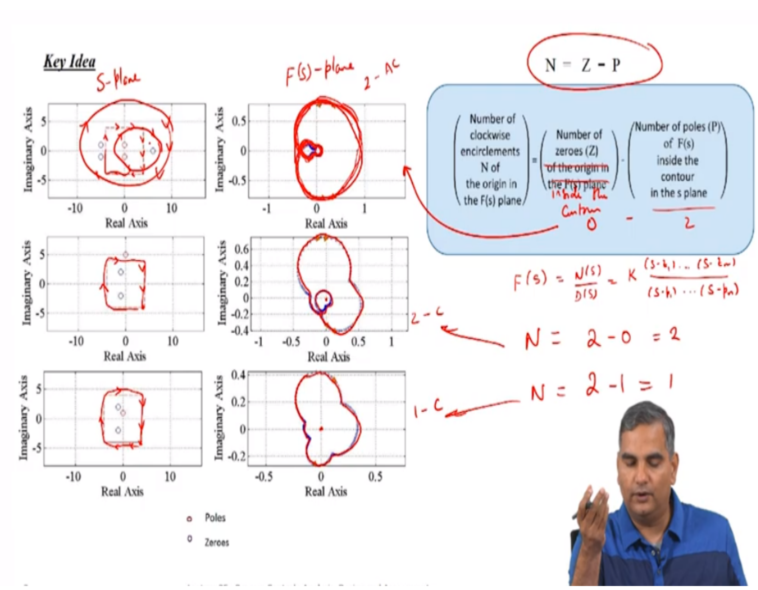
Now let us do slightly more complicated function we will keep this plane to be the same and we will keep the contour in this plane also o be the same but we will change F of S and then see what happens to the contour in F of S, so the contour that you see in F of S really depends on what F of S is so let us just do a simple change, so now let us assume that I have an F of S which is one by S plus two not S plus two, one by S plus two so if you want to think about this is like a zero at minus two this is a pole at minus two, now let us see what will happen to the F of S plane, so now when I put J, so F of S is one by J plus two when S equal to J.

Now when you have this J plus two from simple complex variable what you can do is you can multiply this by two minus J in the numerator and denominator so this will be equal to two minus J divided by two plus J into two minus J will be squared minus H squared two squared is four J squared is minus, so two squared minus J squared will be five so you will get two by five minus one by five J, so this point now becomes the following so if I take this as minus one by five and slightly more this is two by five, so this point is this, so this point now you notice changes to this point right here so you can see the difference right and then similarly just let us just do another point 2J so if I put 2J here so F of S will be one by 2J plus two.

Now if I multiply both sides by two minus 2J what I will get is two minus 2J divided by two squared minus of 2J squared two squared is four, 2J squared will be four times J squared, J squared is minus one so this will be four plus four, so this will be eight, so I will get two by eight minus 2J by eight, so one by four minus one by four J, so that is another point that you can put here and so on, so similarly you can compute for this point you can compute for this point and you can compute for any intermediate point and then you will get a closed contour in the F of S plane.

Now notice that this is simple translation just the minus I made this into one or S plus two it is little more complicated in terms of the contour change the contour has change completely in this case however this concept is very-very simple and the same so every point in S you compute a corresponding F of S based on whatever is F of S and you will map a closed contour in the S plane to a closed contour in F of S plane, so this is a first basic idea that we want to understand and so you might ask this is good so we can do this but how is it useful for us which is what we will see in the next few slide.

(Refer Slide Time: 12:56)



So the key idea that is going to be very useful for us is the following so we are still not come yet how we are going to connect this to control we are still looking at polynomials and looking at contours, so the first question that is going to be answer in terms of how is this useful for even polynomial by polynomial we are going to look at this main result here, so let us again do this so let us say I am looking at some F of S and this is a numerator by denominator polynomial, let us assume we have written this in this form, K times S minus all the way up to S minus M divided by S minus P1 all the way up to S minus PN, so there are M zero and N poles for this, so these are just illustrations, so this plot is to give you an idea of the main result that we are going to talked about.

So again and just like we had in the last slide we are going to have two plane so already notice that one graph which we use to understand X and a function of X has become two planes to understand yes and a function of S where S is a complex variable, so this is an S plane this is F of S plane so we have even kind of said what this F of S exactly is but I am going to make such point based on this pictures now take a system here where I had let us say two poles and three zeros the red circle are the poles and the blue circle are the zeros, so for this function obviously I am going to plot the poles and zeros in the S plane so this is something that we have talked several times, so we plot the zero and poles on the S plane and let us say for this function have this two poles and three zero.

Now what I am going to do is I am going to take a contour let us say I start a contour let us say here and basically I keep moving in the clockwise direction in this contour so I start here and I move the clockwise direction let us say and I come back here so that is one encirclement or one full you know contour one full close contour, so I star at the same point and I come back to the same point in a clockwise direction, now let us say just like how I showed you in the last picture I were to look at what happens to the F of S plane contour, so F of S is going to be some function and it is going to have these zeros and poles and I want to see what happens to the F of S plane.

So let us say we look at start at some point because it is going to be a closed contour we do not have to worry about where we start, so wherever we start we will come back to his same place, so one place to start might be here so I go here and then I start it is going to be a closed contour so I start like this sorry, so you see this going in this direction I should not so you have to follow the direction, so I follow this direction, so I go like this and for the full contour it is not done yet so I continue like this and then I come back to the same point, so this is what I do so basically what this shows is this is done actually for a particular function the computations we are not showing I am just showing you the results that you get.

So when I start with an S plane and I basically have to complete the whole closed contour and as I complete the closed contour on this plane I have gone through a contour like this so contour something like this, now I will just mark the origin here so the origin is here inside the contour, so the main result is the following and before I say what the main result is if we were to count the number of times we circle around the origin in this case, so if I start here and then I go like this I am so this is the origin so I am circling around the origin and when I come back here I have done one full circle.

So imagine that this is not there the inner contour if you look at this then you will notice that you have done one full circle around the origin however notice the direction of the circling the direction is in the anti-clockwise direction thought in the S plane we went in the clockwise direction in the FS plane we are going in the anti-clockwise direction, so I have done one circle but by the time I have done one circle this closed contour is not finished yet I still am going around in this closed contour in the clockwise direction, so I am here so it still continues now I go around and come back here, now forget the outer circle from here to here I continue to circle and origin is inside you will see I have made another circle again in the anti-clockwise direction around the origin.

So basically when I convert this contour which is the clockwise contour in the S plane to an FS plane in this case I seem to have made two counter clockwise encirclements of the origin , so that is a word that is used so encirclements means if you look at this there is one encirclement here and another encirclement here and I have now two encirclements around this origin, so let us look at the next picture and the third picture and then I will tell you the main result which is very-very interesting that comes out let us say we have another F of S there I have let us say two zeros and one pole, now let us say I start here much like before and the I go one clockwise loop around this and I come back to this same point and then I say how does this one look in this is.

So now again in the FS plane I can start at some point and then when I finish here I should also finish here and see where I end up so let us say o start at this point and then I follow this, now again I am following this contour and then I come here, so this is the first contour still this is not finished so as I go more long there is more to do here that is what this plot show, now if you again look at this now just the outer you will see origin is here you will see I have encircled origin once but now in the clockwise direction I am not done yet I have to then still continue around here and come back and stop here this is when this whole thing has come back to whatever it is in the initial location now if you notice after making one encirclement I come back again and make another encirclement of origin and that is also in clockwise direction.

So if you were to look at all there encirclements and that is where this arrows are given, so this is to encirclements however it is in anti-clockwise direction here I have two encirclements in clockwise direction, let me take a third example and then see this and then we will bring all of this together in terms of the major result, so now it is another F of S this time I have one pole and two zeroes and let us say I do the same clockwise encirclements in the S plane and I want to look at what happens in the F of S plane now difference is that in this plot I had only poles inside the encirclements and when I say inside it is inside the closed contour there are zeroes for the transfer function but they are outside the contour if you look at this picture here.

I have two zeroes inside the closed contour and the pole is outside that is how I chosen the contour you can choose to encircle everything also does not matter but I am just saying what we have done in this case we have encircle in such a way that all the zeros and poles are within the closed contour and keep in mind every time we are doing this movement in clockwise direction, so this movement we are doing in clockwise direction so in this case let me again trace this so maybe I start here and then I go and now you see I am moving in the clockwise direction and then I do this I come back here and I am done.

So when I finished this contour this is what the picture says you are done and now again let us plot the origin the origin is inside now if you notice this starting here I go clockwise and I have made only one encirclement of the origin in a clockwise direction so one clockwise encirclements, so now you might ask why is it sometimes in the F of S plane I have one encirclement and some I have two encirclements sometimes in the clockwise direction sometimes in the anti-clockwise direction while I am always having my contour in the S plane in a clockwise direction, so the result that puts all of this together in a single statement which handles all of this cases is the following.

So this is the single mathematical statement N equal to Z minus P, so let us expand on each of this N is the number of clockwise encirclement of the origin in the F of S plane so that is what we actually find ultimately, so in this case N is minus two because the number of clockwise encirclements in this case would be minus two because I have two encirclements in anti-clockwise direction, so whenever it is an anti-clockwise encirclement we use a minus, and if you take this case this will be two clockwise direction so N will be two in this case if you take this, this is one clockwise encirclements.

So N will be one so this result what will happens in the F of S plane will you have two anti-clockwise, two clockwise, one clockwise, one anti-clockwise and so on can be specified actually without plotting this at all that is the beauty of this result, so right now I am going to show you that you could have simply mathematically said how many encirclements will be there and will this encirclements be in the clockwise or anti-clockwise direction by just looking at the contour in the S plane and what is inside the contour in this plane, so that is the important thing so if you take this F of S for example and you went for this contour if you notice within the contour, inside contour there are two poles.

So the number of poles inside the contour in this case is two though the F of S itself has three zeros the way we have chosen the contour we have excluded all this zeros here so the number of zeros in the F of S plane sorry this is number of zeros inside the contour is zero in this case while F of S itself has zeros inside the contour we do not have any zero, so this is zero minus two and you will get this number as minus two so N equal zero minus two, so that is two anti-clockwise encirclements, now if you look at this F of S though this F of S have two zeros and a pole we have excluded the pole and we have done a clockwise encirclements only around zeroes, so in this case for the second case N will be two zeros inside the contour no poles inside the contour, so two minus zero is two, so I have two clockwise encirclements.

Now in the third case if you look at it I am mixing both zeros and poles, now let us see whether we get the correct result in this case N equal to two zeros inside the contour and one pole inside the contour, so this is one, so plus one this is one clockwise encirclements, so this is two clockwise encirclements this is one clockwise encirclement and this minus two case here I have zero minus two will be two anti-clockwise encirclements, so interestingly you could have predicted how many encirclements are around the origin will be there by simply looking at how many poles and zeros are captures within the contour so for example if for the same F of S I had not done this contour but let us say if I had done a contour like this.

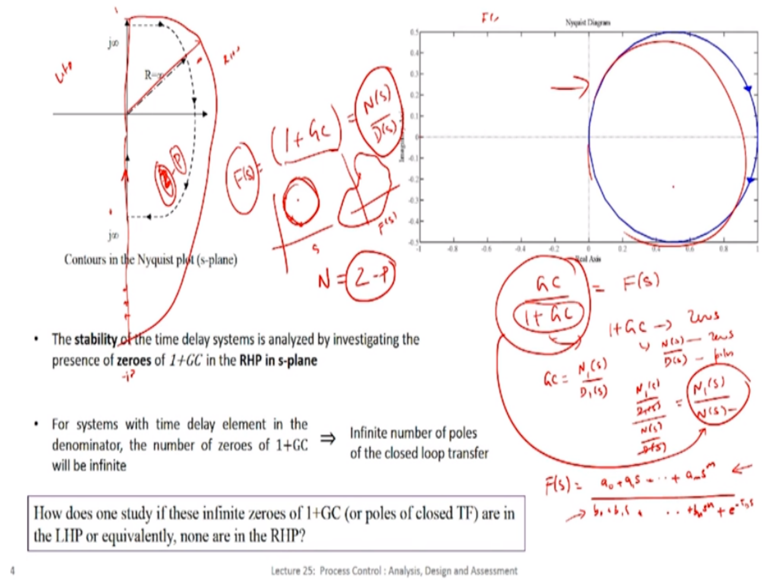
So I go like this which is clockwise for this and then ask how many encirclements will be there in F of S plane you do not have to actually plot this now you can say what it will be, so if you look at this contour and what is inside the contour there are three zeros and two poles inside the contour, so N is equal to three minus two will be one so you will have only one clockwise encirclement here now if let us say my contour was something like this instead then I have two poles and zero inside this contour, so the number of encirclements for this contour when I plot them in F of S plane will be one number of zeros is one minus two number of poles inside the contour, so one minus two so minus one there will be one anti-clockwise encirclement similarly you can make arguments for each of this other F of S and the plot.

So even for the same F of S when I look at the contour in the F of S plane that depends on what is the contour in this plane and all you need to do is you have to look inside this contour and then see how many zeros and poles of F of S are inside the contour and then you plot this close contour in F of S plane then the number of clockwise encirclements is simply going to be Z minus P, so this is the key result that we have, so with this idea of contour mapping S plane and F of S plane and so on hopefully you have an idea of this picture why this happens is something that I have not said and I am not going to say it because that would require lot more complex variable analysis.

Which is not really required at this time for an undergrad control all you need to know is how you interpret this get the basic ideas as to why there is an S plane and an F of S plane and how you can actually guess how many encirclements will be there in there F of S plane based on the contour that we choose in the S plane, so this are important ideas that you have to understand basically if you understand this picture and you are able to basically understand clockwise, anti-clockwise encirclements inside outside contour zeros and poles why do we plot zeros and poles in the S plane and look at contour in F of S plane and so on that is good enough, so now that we have seen all of this the next question that naturally arises is that this is all good how is it useful for control.

So this is what we are going to see and once show you how it is useful for control then of course the next question how are we going to use this for time delay systems.

(Refer Slide Time: 28:51)



So when it comes to the control system the way we are going to use this is the following so we typically, let us say have a closed loop control transfer function and we are looking at the stability of the closed loop control transfer function, so basically we are looking at let us say GC by one plus GC this is something that you have seen many-many times, so this is going to be some numerator by denominator form and if I have time delay elements in my controller which basically gets introduces because of time delay in my system then what is going to happen is this denominator polynomial is going to have infinite number of pole because if I conceptually expand a power minus tau DS into an infinite polynomial I get lot more poles.

So that was our problem, so how do you guarantee that none of the poles chili clip into right half-plane is what we are interested in so basically if you think about an S plane for this so this is basically what we are going to say is F of S and this is itself is a transformation with S, so if I want to know if this system is stable then what I can do is I can just take this one plus GC, why I am taking one plus GC because when I write this in a numerator by denominator form the zeros of one plus GC will become the poles of this, so if I for example write this one plus DC itself as some numerator by denominator form, now GC itself you can write let us say some let me call this some numerator by denominator form.

So when I do GC by one plus GC I will get numerator by denominator divided by numerator by denominator but you will notice the way this is written the denominator of one plus GC will be the same as the of GC, so this and this will get canceled, so you will have some numerator of GC by this numerator of one plus GC is what will be the final expression for GC by one plus GC, so again let me explain if I write this as NS by DS and GC as known as by D1 S because we are going to take this denominator polynomial multiplied by this and then divide by the same denominator polynomial the denominator polynomial of GC and one plus GC are going to be the same, so they are going to get cancelled.

So this whole transfer function is going to be the numerator of GC divided by the numerator of one plus GC, so if you look at this as this then the stability of this GC by one plus GC depends on the numerator polynomial of one plus GC and in fact when you think about this numerator polynomial of one plus GC, we are looking at the zeros for one plus GC, because if this is numerator polynomial of one plus GC this will be the zeros of plus GC and this will give you the poles of one plus GC, so basically what we are saying is I want to find out if this numerator polynomial of one plus GC has any zeros in the right half plane right if the numerator polynomial has any zero in the right half plane then the closed loop system is unstable.

So this is an important idea please work this out for yourself and convince yourself that this is the way to look at this because this is going to be something that will keep looking at again and again, so basically what we can do is ultimately because we are looking at numerator of one plus GC we are looking at one plus GC itself, we are forgetting GB because we are now established that the numerator of one plus GC is important or in other words this one plus GC will be some numerator by denominator polynomial and if I say this is F of S then basically I want to know if this F of S has any zeros in the right half plane, so the whole question of stability of closed loop has boiled down to this where I am saying F of S is one plus GC which I can write as a numerator by denominator polynomial.

Now from this I know if the zeros of this F of S which is one plus GC which is the roots of the numerator polynomial are in the right of plane then I am going to have my closed loop system becoming unstable, so the question is how do I connect what I want to solve here with whatever I have seen till now, so till now what I have seen is in this plane if I take a contour right if I take a contour in this plane without actually computing the poles and zeros inside the current I can always find out how many poles and zeros are inside the contour not exactly the number of poles and zeros but a difference.

So remember the last time we had N equal to Z minus P, so in the S plane if I take any contour and I know the F of S form one plus GC if I were to just take each point here in S plane and then I generate a corresponding F of S whatever contour that is, so I go from S to F of S remember this you do not need to know the zeros and pole because it does not even have to be in the root result form because I have a polynomial numerator, polynomial I have a denominator polynomial conceptually what I am doing is I am taking every point here and then I am finding a corresponding F of S and simply substituting I, so once I find the corresponding F of S I will get contour here.

So it can be just a polynomial form not even root result form, now without resolving for rules roots without finding the roots what I can tell you is within contour for this F of S what number of poles and zeros are there in terms of a difference so I can tell you this Z minus P so how many zeros minus poles are there inside the contours, so this I can do without resolving this root result form without actually computing the poles and zeros because once I get this form every point here can be translated to a point here by simply substituting this S value into this F of S, so that is the key idea, so this is something that you should understand, now essentially what is the count, now I should take an S plane to look at the stability of this GC by one plus DC.

So ultimately what I want to show is the whole right of plane has no zeros right, so if for one plus DC if the whole of plane has ne zeros for one plus DC then there are no right of plane poles for GC by one plus GC, so basically the contour one have to take in this plane is the whole right half plane and the way you conceptually do it is you start at minus J infinity here and then you keep going, this is in the clockwise direction, so this is minus J infinity somewhere here and you keep going in the clockwise direction up to J infinity and then you take a big are around here and come back here, so this is what is plotted here.

So if you make this radius infinity then basically what you are saying is I am going to cover the whole right half plane, so this is right half plane and this is the left plane, so basically I start from minus J infinity go all the way up to J infinity and take a very-very large in finite circle and come back to minus J infinity, now if I take this whole contour this whole contour basically covers all of HP, now if I plot a corresponding like this plot or I plot this in the FS plane as I move here what happens to F of S then depending on how many encirclements of the origin by C in the F of S plane I will be able to say, how many Z minus P's are there within this contour and remember because I am looking at one plus GC, I am interested in the number of sets because the zeros of one plus DC will become the poles of the closed-loop transfer function, I cannot separately get Z than P.

So we will solve that problem again so we will go one by one so first from the NYQUIST diagram I just want to explain how we convert to a control concept, so we will still see how to get rid of this Z minus P problem but let assume that I can find out how many zeros and poles are there within this contour at least we have got the quantity of interest the number of zeros of one plus DC into this picture already so if I can plot this then by looking at just that plot whatever it might look like this or something else I can actually tell you how many is N minus P what is the value for Z minus P that is just simply given by looking at this encirclements of origin.

So if it encircle origin once in clockwise direction then we will say Z minus P is one if it ends circle origin once in anti-clockwise direction we will says that minus P is minus one and so on, so that is how this notion of NYQUIST diagram like this plot is converted into a control idea so that is a conversion from whatever we saw in the previous slide to a control idea and the key inside is we know, how to characterize how many zeros and pole are there in term of a difference that minus P inside a closed contour by looking at number of encirclements around the origin in the F of S contour and this contour computation is very simple whatever be there for F of S, so F of S can be some A not plus A one S all the way up to AM S power M divided by B not plus B one S all the way up to Band S power N.

So I do not even have to have this in root result form it could have for example just to make sure that we understand this it could have some other time delay term here also it does not matter, so as long as I have this F of S for every point on this contour I can substitute the S value into this remember and simply compute an F of S it need not I do not know do not know the roots of the polynomial clearly there are infinite number of roots for this polynomial all I am going to do is I am going to take an S value and simply plug this in to the surface and I will get a corresponding points here.

So conceptually if I am able to take all of these points and then compute this and plot this F of S diagram here then I am done, now I can actually look at the encirclements of this around origin and then find out how many zeros and poles are there in terms of a difference, so this is the key idea that translates whatever we have learnt till now into a control concept now we will see how to take this forward in terms of clear result on stability for closed loop control systems in the next lecture, thank you.