**Process Control: Analysis, Design and Assessment**
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**Frequency Response Analysis – 1**
**Mod05_Lec23**

Welcome to next topic on process control and analysis and assessment, this is Parameswaran, I may doctoral research scholar working with Prof Ragunathan at IIT Madras, so today what we are going to see is like frequency response analysis.

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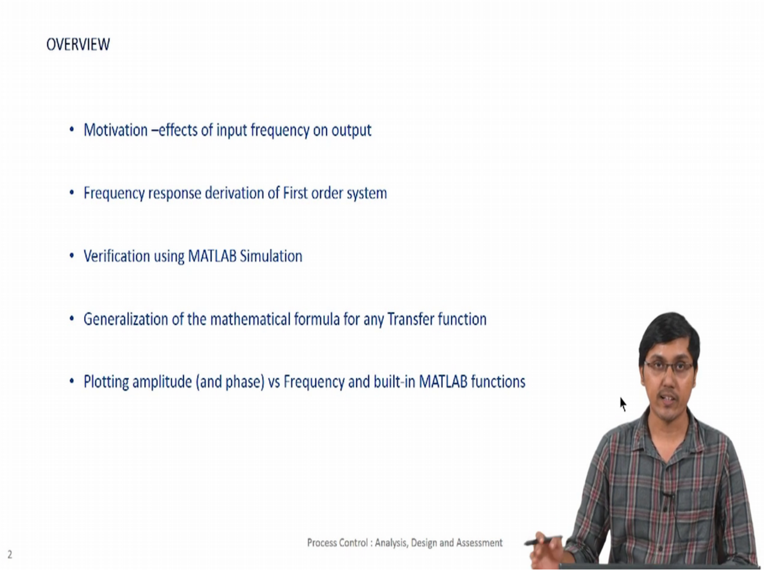


So this is the overview of the presentation that we are going to have today, first thing is like going to see the motivation for the frequency response analysis from application point of view and then we will derive the mathematic, we will drive formula for the frequency response magnitude and phase thing called first-order system and we verify what are we derive using Matlab and then we will proceed to make a generalisation, general formula for response and then will also continue with plotting the magnitude and phase with respect to the frequency in the subsequent lecture.

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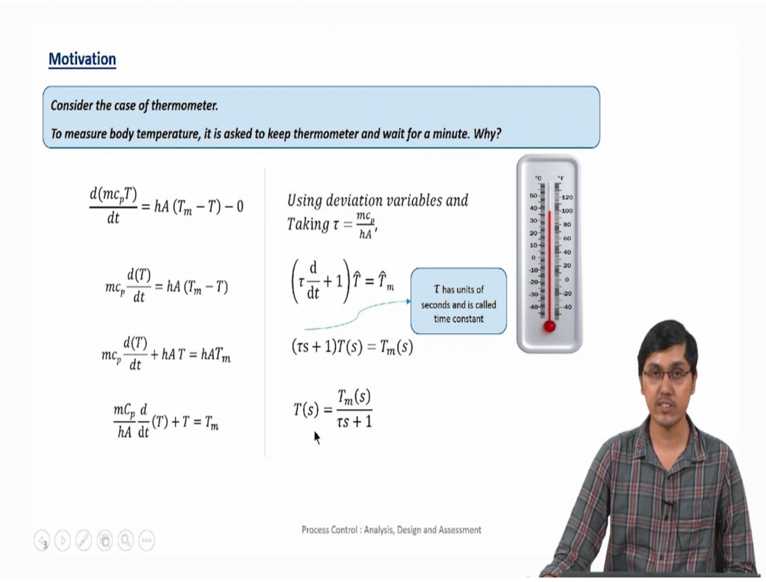


So we will start with the question, let us take the case of thermometer which is used to measure the body temperature, so as usual practice, we are always said to give the thermometer inside the body for a minute or so, do you know why they are asking to keep the thermometer inside for certain minimum time, so we will try to understand at mathematically and we will try to relate why, like what correction it does with to the frequency response, so writing the energy balance equation for thermometer, we can write this particular equation wherein this is the energy accumulation and this is the heat flow into the thermometer.

T is temperature of thermometer and TM is the temperature of the measured entity like the body, so basically while actually going through to some algebraic steps and we finally arrive at this particular equation and taking this particular term entity which is having units in terms of seconds as Tao, we called Tao as a time constant, using deviation variables forms, I am taking Tao as this particular entity, we get this particular equation, taking the Laplace transform this equation we get Tao S +1 TS equal to TMS and this is what we at the file equation.

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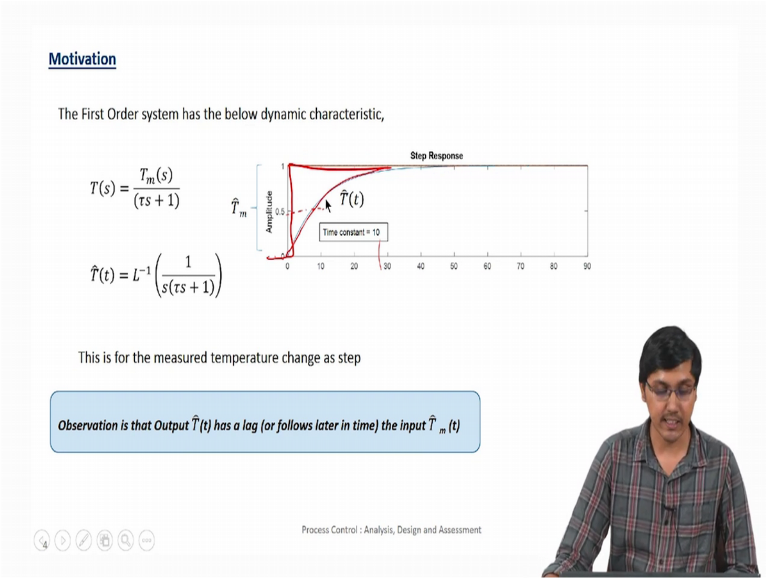


Now we will be give a step input to the system, which is typically what happens when thermometer is placed inside the body like the temperature of thermometer was initially at the same temperature of tester room as environment and then you keep it inside the body, the body is going to be at different temperature, so basically it is a step change and if you can see that the thermometer behaves in this particular fashion wherein thermometer is following this particular line, putting this particular line, so basically we can see that at thermometer take certain time to follow the input, so this recall as a lag or lag seen since like it lags in time basically that is what the concept of lag is.

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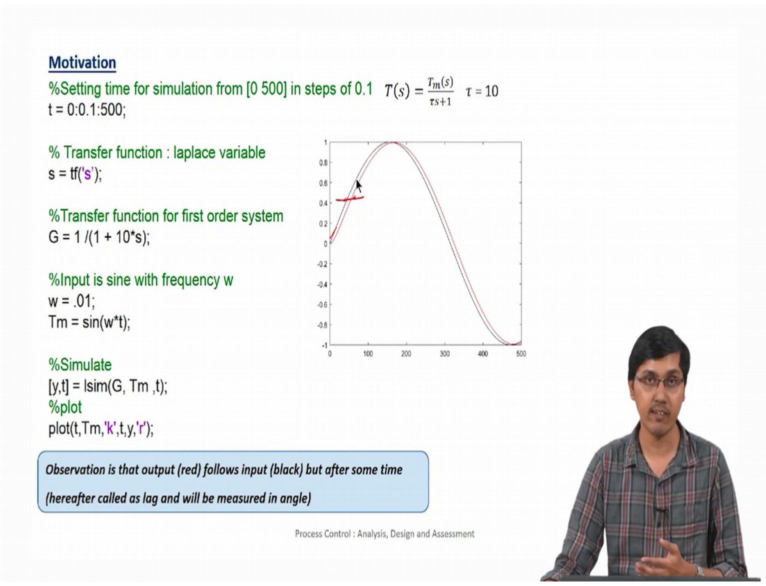

So now we ask another question like okay, first step input this is the, that was the response of first-order system what will happen when we exit the system with the sin order input and we take a sin order input of angular frequency 0.01 and then we see this as a response, so if we could see this response we can understand that the red line is the thermometers temperature, whatever temperature he indicate a bit could a thermometer or the mercuries temperature and the black line is the temperature of the input.

So the input is actually like fluctuating, so basically we can see that the output is following the input but it is following after certain times, so this particular temperature is reached by the actual entity in certain time and the thermometer take certain time to show the same temperature, so this is again like a indicative of lag, it shows the temperature, it records the temperature but it does it with the lag after certain time.

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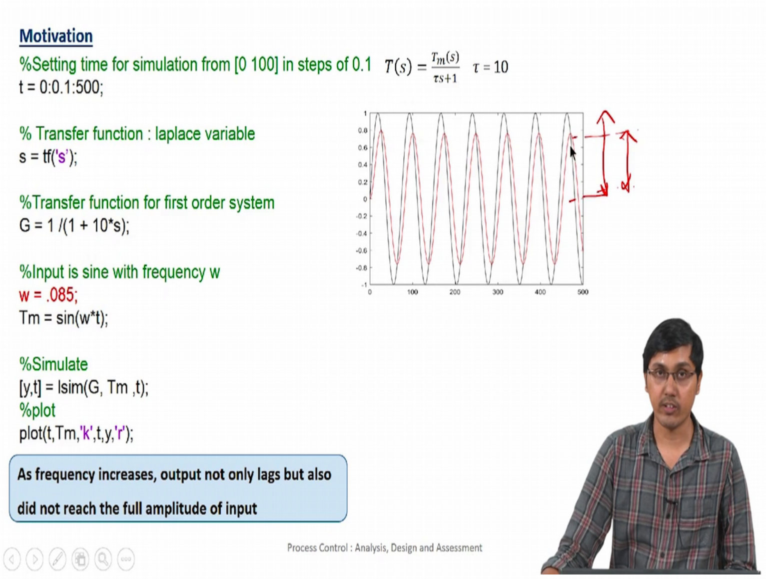


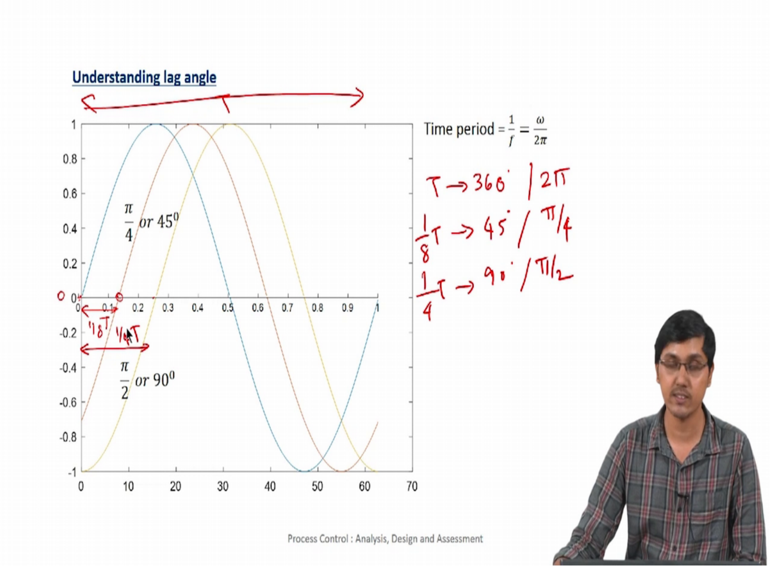


Now we again asked the question like what if we change the frequency, then we see anything interesting, so we increase the frequency to 0.085 to from 0.01 and then we see that the amount the lagging is, the temperature shown by thermometer is not only lagging, but also that it is not reaching the full amplitude of the input signal, so basically amplitude is the, of the input signal is this much and whatever is the thermometer is according is only till this much, so thermometer never reaches, never measures this particular total temperature is change,

So these are the two basic things we observe, one is that when the frequency changes two thing is that, one is that it lags, it can lags, the second thing is that the amplitude is also is limited of certain frequency like it has indeed so the full amplitude after certain frequency, so now coming to understanding what is a lag angle like we keep talking about lag etc right, so it is spent some time on understanding what is lag basically and what is the? And how it is measured etc, so lag is usually measured in terms of angle, so we call it as a phase angle or something, so basically if take a sin wave and let us say this is time period of a sin wave,

So this is the T, total T time period of a sin wave and then let us take the blue line as a thing, so we will compile everything with respect to the low length, if you see the next line this again the sense and solder wave, it is actually following the blue line we can assume it as a following the blue line, but it is starting, when the blue line was that 0 at time 0, it was this came to 0 after this particular time, so this is nothing if you see what is the value of this, this is nothing but like one eighth of the time period of the total blue waveform, so this is nothing but 1 x 8 T and one more thing to notice like all these sign and solder waves are of same frequency and ensure able to compile all this.

So we may not be able to compile like sin waves of like different frequencies, so when you say phase difference, we normally it is general thing like we assume that they are of same frequency and then if you see the like yellow line, it is actually it is also lagging behind the blue line, but now time period of like 1 x 4 T, so one way to measure this is like in terms of angle.

So if you say the total time period T is corresponding to like 360° which is normally that we take was sin wave and or like in terms of radiance 2 pie total time period is corresponding to 360° or 2 pie, then one eight of T will which corresponds to like 45° or it will become as 1 into pie by 4 and one fourth of the time period will see like 90° or like pie by 2, so this is the way we tell for, when we say like wave is lagging behind another wave by pie by 2 it is 90° than what will mean like is basically it is reaching the same value as the initial wave but after a time period of 1 x 4 T and this is okay to compile because the time period of both the wave is our same because the frequency of both these things are the same.

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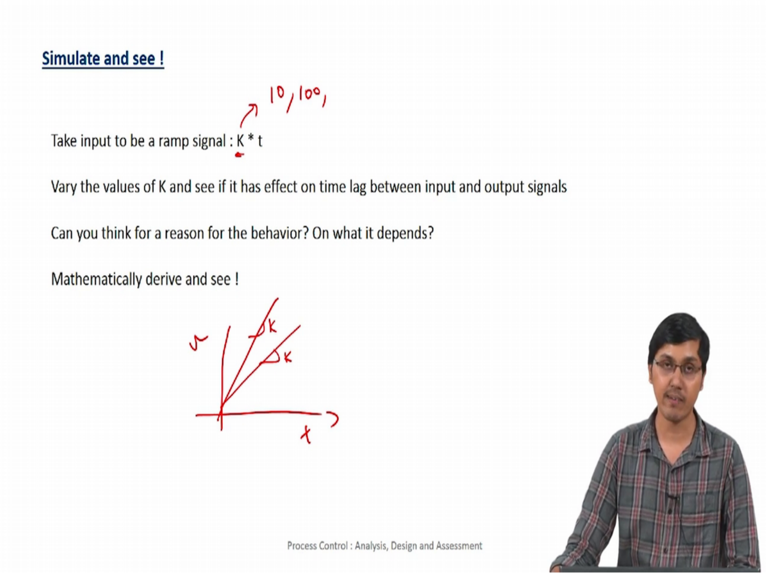


So we are some extensions for you, which you can let stimulate and see, you can use the code that was given in the previous slide to stimulate the same thing and see, so instead of giving the sign input, give a ramp input, ramp input is nothing but some slope into time right, so when time is increasing basically, the signal also will get increasing with respect to this particular slope value, so you change the slope value, say you have 1K here and another K here change this slope value, substitute value for K like 10, 100 or 1000 and then see like how the output is varying and then see whether when the output is following the input, first see if the output is following the input and if it follows the input, see after what time it follows the input and whether this K has some dependent, it has some effect on the time lag between these two signals and also mathematically derive a ramp +1 for first-order system and then check that, whatever you observed is correct based on maths that you have we derived.

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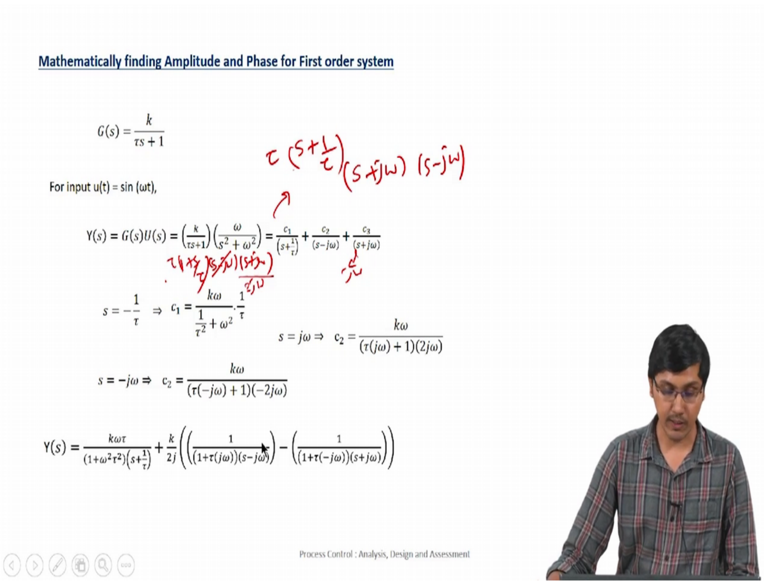


So coming to like mathematically putting everything and like seeing why exactly why to this phase lag and how can we mathematically represent it, etc, that is what we are going to see now, more mathematical thing, so this is the system, this first-order system, so we say generally we say first-order system is K by TS+1 and YS is equal to, so we are giving input as UT is equal to sign omega T and in Laplace domain the output is YS is nothing but the product of trans function into the input in the Laplace domain .

So Y G of S into U of S, so G of S is nothing but we substitute this into your and then we say input sign omega T when it converts, when we take that Laplace and some of it nothing but omega by S square plus omega square, so this is what we can and then we actually dissolve into partial fractions, so we take, so one thing to notice here like we are taken poles like Tao S +1 is taken as Tao into S +1 by Tao and S square plus omega square is taken as S plus J omega into S minus J omega okay.

So we had actually three poles S equal to -1 by Tao and S equal to minus J omega and S equal to plus J omega all that three poles that we have, and then for to find a constant C1, C2, C3 we just substitute the values of S equal to -1 by Tao, S equal to minus J omega and S equal to plus J omega and on substituting we find the constant values C1 equal to, we substitute in this particular question, so we say S equal to -1 by Tao and cancel out this particular pole because we are substituting for that pole, we get K omega and S is -1 by Tao, so 1 by Tao square plus omega square, this is what we get a C1.

And for C2 we substitute S equal to plus J omega and poles that are represents plus Tao S +1 by Tao and S plus J omega, so you can do it yourself and it straightforward and then it comes to be like K omega by Tao of J omega +1 and 2J omega, 2J omega because like this S again becomes J omega, so in this like written as S minus J omega and S plus J omega, so this pole will get cancel when will do the partial correction thing and then whatever remains will be like Tao into 1+ S by Tao and this would have got cancelled when we put S equal to J omega, so we get 2J omega here and Tao into 1+ J omega by Tao so that is what we get here and similarly C2 also we can substitute J equal to minus J omega and then we get the value of C2, so Y of S if we do the algebraic manipulation basically will end up with this particular question, substituting YS of C1, C2, C3.

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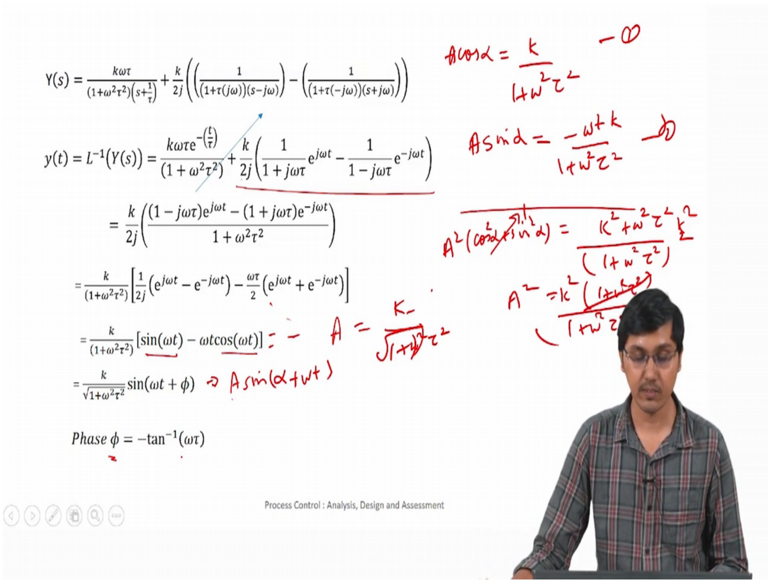


So now we take Y of T, Y of T we take as inverse Laplace transform of Y of S so the one thing we can observe here is like this particular term which is actually E power minus T by Tao will go to 0 when time tends to infinity is because like minus E power minus infinity become 0 this particular term will vanish when times tend to infinity, so what we understand here is like we are actually looking of at the steady-state frequency response of the system, so steady-state means when time tends to infinity.

So this term will not go to 0 when time tends to infinity because this is actually will become like a sin wave and but this term is actually E power minus T, so this becomes 0 and if you can see here like this particular term what it remains when time tends to infinity and then we can actually rewrite this particular here and then on simplification this becomes of this particular form and we know that 1 x 2 J E power J omega T minus J omega T is nothing but sin omega T and then 1 x 2 E power J omega T plus C power J omega T becomes cos omega T, so this is what we get at final form and then we can further simplify this particular equation into this form by taking, by equating this to say A cos alpha sin omega T and we sin alpha cos omega T.

So if we can equate this two equations and find values of A and alpha such that this particular equation is the same then, we can actually simplify this as, sorry this is A, so this can be simplified this as A into sin of omega T plus alpha, so that is what we are trying to do here, so basically if you can see A cos alpha is nothing but K by one, plus omega square Tao square and A sin alpha migrating the coefficients we get minus omega T by one, plus omega square Tao square, if we can square this two particular terms and if can add, we get A square of cos square alpha + sin square alpha which is nothing but cos square alpha + sin square alpha is one.

So A square equal to K square plus omega square Tao square and denominator is the same so is get added up omega square Tao square the whole square because we have squared this two terms and added, so now we can see like again these also as a K term here, so K square, now we can take K square outside and 1+ omega square Tao square by 1+ omega square Tao square the whole square, so this get cancel, so A square this particular term and then we get A equal to if we take the square root we basically get the A equal to K by root of 1+ omega square Tao square and if we can divide equation 2 by 1.

Then we get A tan alpha equal to will get A by A cancel so tan alpha equal to minus omega T, so alpha becomes tan inverse minus omega T, tan inverse minus omega T becomes minus tan inverse omega T, so this alpha term, alpha is what is represented by phi here, so basically we can write this K by A into, we got that A into sin of alpha plus omega T, so this is what, instead of alpha we using the term phi here, so A is nothing but K by root of 1+ omega square Tao square and alpha or that phi is nothing but minus tan inverse omega T, so that is the equation that we are going for first-order system, since all this response for first-order system.

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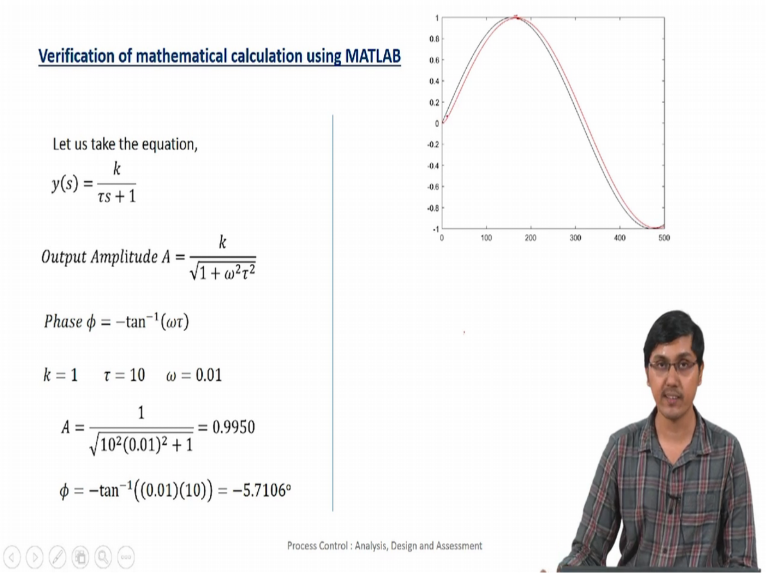


So we have written some mathematical formula for the amplitude and the phase but we can actually verify using Matlab simulation, so that is what we are trying to do here, so we take some arbiter values for K, Tao and omega and then we simulated in Matlab and also be actually find the values of A and phi and then if we see whether we can actually see from the graph the rate is approximately true or not, so A becomes if we have this equalities values into this particular equation is become 0.99 and this red line is actually of about 0.99, we cannot see, actually from the graph, but yes we can see its near close to 1, the maximum amplitude of sin wave represent by the red line and phi is -5° and we can actually draw, try to measure the time period and then we can confirm that this is also -5°.

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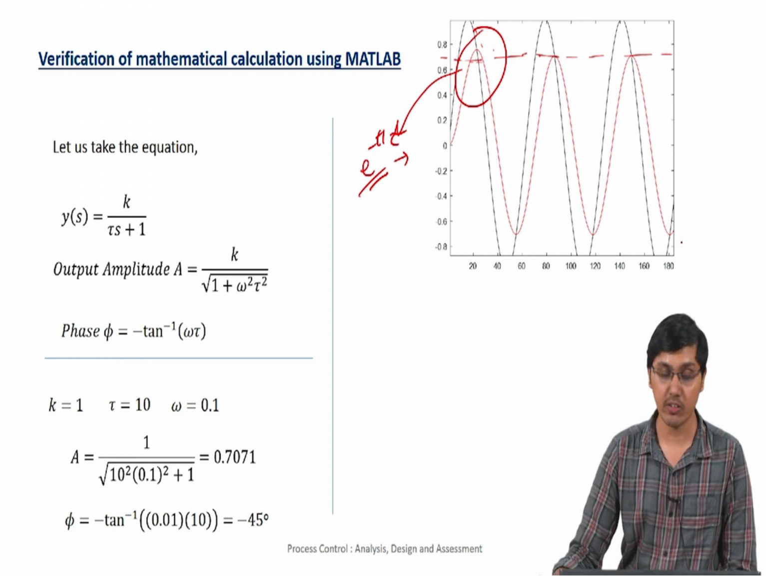

So we will see more better example where we can actually see this value is getting affected like, so here we increase the frequency to 0.1, the same K and Tao values but now we get A something like 0.7 and then we can also see from the graph like it is approximately like 0.7 here, this is 0.7 here and so here you can easily see that the initial response is not something that we actually can capture, this response and this response is the same but, this is not the same, so this is the term that we actually skipped by ignoring minus T by Tao, so initially this had an effect and that is what was contributing to this particular initial change but after sometime this becomes like very small, so that we can neglect it, so that is becomes a constant thing, so when we say frequency response we are not concerned about the initial response how it say it initially, but what we are concerned is about the final steady-state response to the particular frequency.

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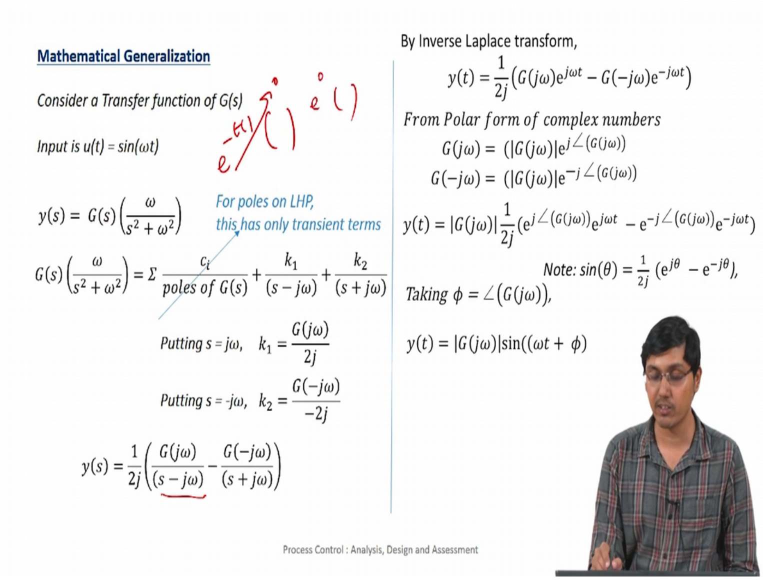


So now we are done it for first-order system, so now we are trying to generalize it in a like for any transfer function, so again, U input is your T sin of omega T, which has a Laplace transform of omega by S square plus omega square and G of S is nothing but again, Y of S is nothing but G of S into omega by S square plus omega square which is nothing but, it is all into partial fraction, we get some terms for each pole of G of S you will get some constant I get pole, so that is what this particular term represents and then this particular term becomes K 1 by X minus G omega and okay 2+, K 2 by S minus J omega.

So this term is actually like the steady state, like at steady-state we can ignore this term because like for all, if we can assume that the of as is having pole only on the left-hand side of S plain than this has all the terms of the form E power minus T something into something, so as time tends to infinity at steady-state what happens this all terms vanish, so provided that G of S has all the poles on left-hand side of the S plain we can actually ignore this particular term.

So we are to very careful about this particular statement. It is on the left-hand side of the S plain because not even the imaginary axis is permitted for now because like even on the imaginary axis what you get basically C power 0 into some term, so that you cannot eliminate when you do steady-state response, so basically this derivation is valid only when you take all the poles to be on G of S on the right-hand side, on the left-hand side of the S plain.

So now again, you should as a putting S is equal to J omega we get K1 value and putting S is equal to minus J omega we can the K2 value and then we substitute those two particular terms in Y of S and then we take the inverse Laplace transform, so now again note that G of J omega now is a complex number because we have substituted S equal to J omega, it is no longer a transfer function, it is a complex numbers, so while taking the inverse Laplace transform we need not bother about what happens to the G of J omega because it just a number, it is not related to the Laplace thing.

So basically X minus J omega becomes C power J omega T and S plus J omega becomes C power minus J omega T and now G of J omega is a complex number, so it can be represented in polar form which is like magnitude of the complex number into the angular of the E power angular of the complex number, so basically we get magnitude of G of J omega into J into angular of G of J omega and then magnitude of minus G omega again the magnitude becomes the same but the angular becomes at the minus so J into minus angular of G of J omega.

So again substituting all this things in the equation we get Y of T equal to E power J angle G of J omega into E power minus J omega T and E power minus J into G of J omega into E power minus J omega T, so basically again we use a formula of sin theta is having 1 x 2 J E power J theta minus E power minus J theta and then we take this angle G of J omega as phi, we get this particular equation, so now we can say that the for any system if we excite the system with sin sort of input then we get the response of the form, Y of T equal to magnitude of geology omega and the phase lag is nothing given by the phase of G of J omega the angle of G of J omega is what the phase given, but when I say any system, it is again like it is nothing but a linear time invariant system, so we are not talking about nonlinear systems here, so for any linear time invariant system, this is LTA system, this particular equation is valid.

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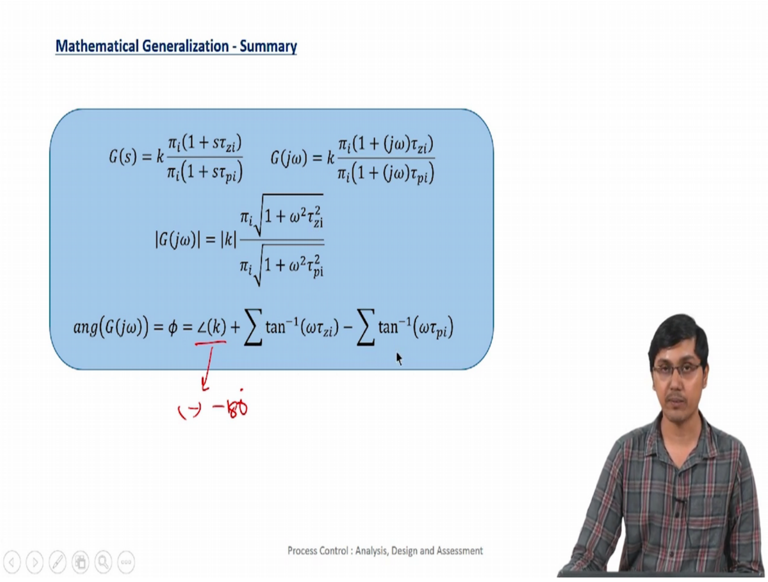


So a summary of what we are to remember basically like if we have a function like G of S is equal to some sort of 0 is and some sort of poles and G of J omega is again, K equal to substitute whenever S there is J omega and then we take the magnitude of it, so this is nothing but product of root of an plus omega square Tao square for all the zeros and this is nothing but divided by a lot of root of one, plus omega square into Tao square for all the poles.

So angle contributed by G of J omega that is a phase thing, but the angle of K, so angle of K has to be considered because this could also, though is a constant it could also have a minus sign which is actually minus is nothing but 180° phase shift, so we all to also consider the angle of K and also I can tan inverse omega Tao is the for all the zeros we have addition term and for all the poles you have the subtraction terms.

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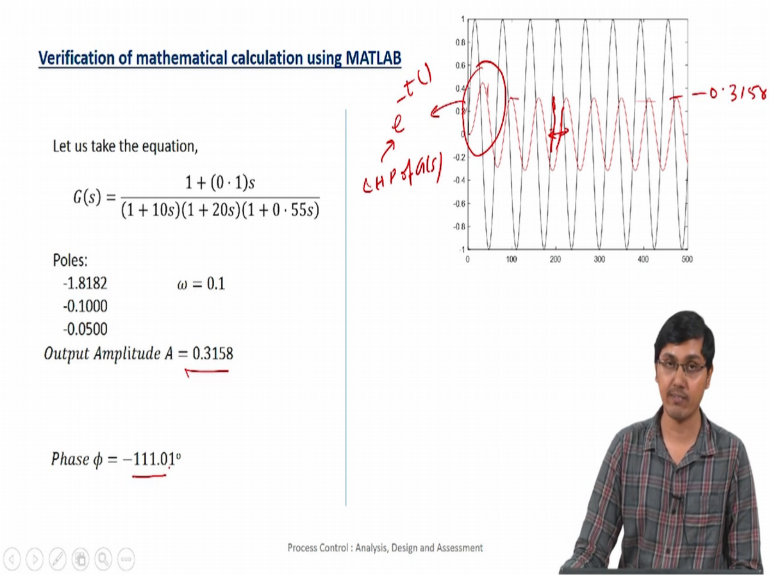


So now let us take general like G of S some random G of as and then verify whether whatever we observed is correct, so whatever derived is correct basically and then we are taking G of as is equal to this and then output amplitude equal to A into, output of amplitude equal to 0 .3 here and phase we get minus 111°, so we can basically see here, here it is very evident see, like the input, the initial portion the transitions is actually showing a very big difference on the other things, so these are all the same, only this one is different.

So this is because of the comes we have neglected as E power minus T into something in the previous derivation wherein these are all cos by the left-hand side of poles of the G of S, so this is what we ignored and then this is what we are interested, so what we are, so what is A will predict is the value of this thing and this is you can see, this is actually 0.3158 and if you see the phase lag, we can all see that this is the input form and this is the output of a system, so the difference between these two the time period between these two will contribute to the phase lag of minus 111°.

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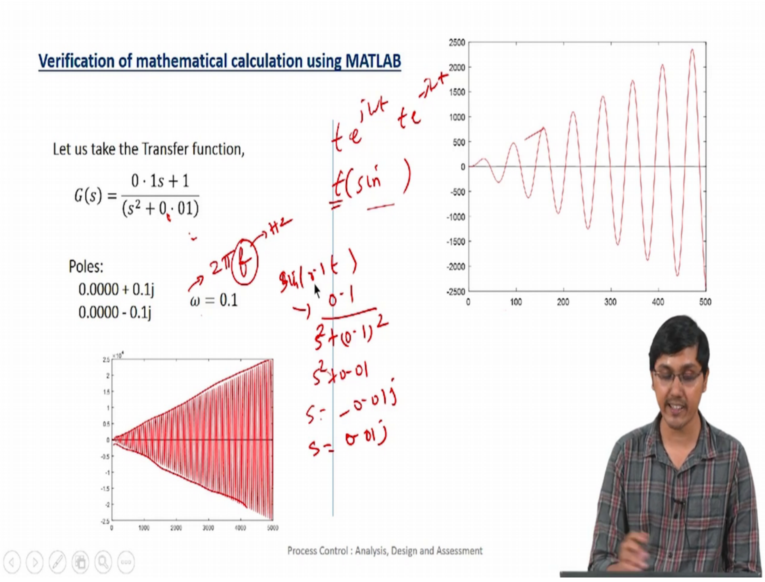


So now something interesting will look into it, so let us say like we have a transfer function 0.1 S +1 by square +0.01 and we excite into the frequency of omega equal to 0.1, you remember omega, when this omega is nothing but the angular frequency which is nothing but a 2 pie F which is the this frequency the actual frequency that we measure in hertz, if you see the response basically we can see that the output is actually like increasing or we can save its kind of like increasing in amplitude and if we can zoom over the image and see like it is increasing in amplitude as a straight line.

So from Laplace derivation we can actually find that out input to this is like sin omega 0.1 T which is nothing but 0.1 by S square +0.1 square which again becomes like 0.01, so as square + 0.01, so this is what we have two poles, now at S equal to -0.01 J and as equal to +0.01 J we have two poles, one pole due to the transfer function and the other pole is due to the input at the at the same frequency, so when we already have a pole on the imaginary axis at a particular frequency and we give the same frequency sine wave as the input than basically, when we take the inverse Laplace transform what we get difficulties like T into E power we get T term extra, so we get basically T into E power J omega T and T into E power minus J omega T, which actually can be simplified into T into sin something.

So basically the sin is still there, the sine wave is still there, but only that the amplitude of this is increasing with respect to time, so when time tends to infinity this also tends to infinity and you can see that this is actually proportional to the time, so the straight-line, so you can see it is a linear thing, so this is called resonance, the concept itself is called a resonance thing.

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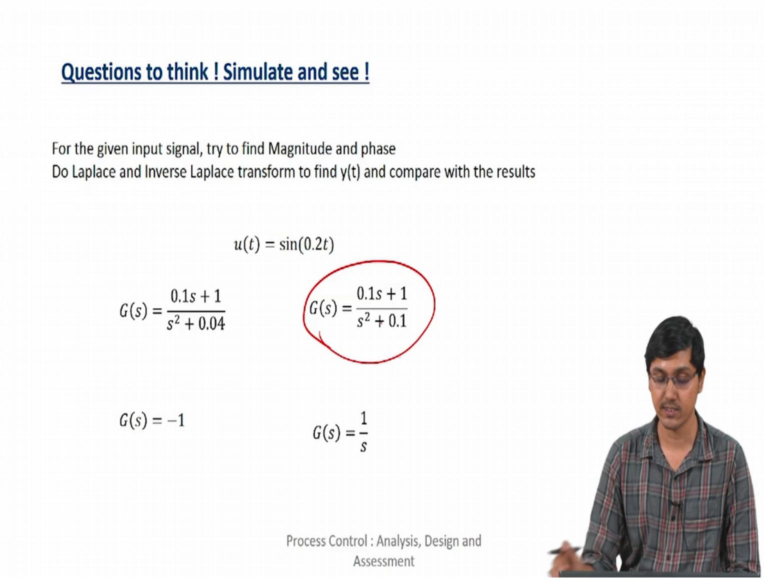


So then now to make yourself like try certain things and then let see think about it, so these are all not like very straightforward thing, so it will be like interesting to try and you will also like is not very tough for us to figure out why it behaves this way, so take the input way from C of T equal to sin 0.2 T and take these four different transfer function and stimulate and see and also like take the inverse Laplace transform and see like if you can find out the equation and compare to with your stimulation and see whether it is, both in the matching or etc.

So we can discuss about this in the next lecture, especially about this particular transfer function and these transaction are better at the same, very much simple, so will focus mostly on understanding what it is this.

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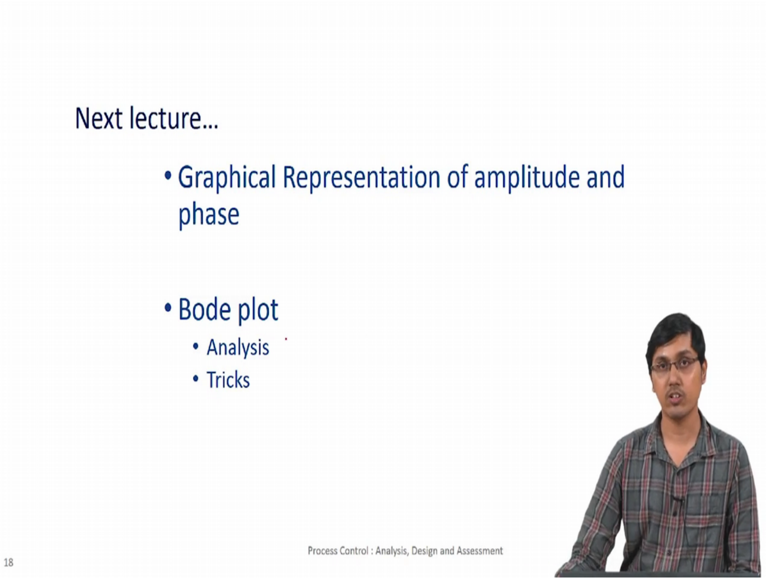


So in the next lecture you will introduce on like graphical representation of amplitude and phase with respect to the frequency and then we will talk about something call about Bode plot which actually the graphical representation and then it will also try to analyse what bode plot and then some tricks that we can use to draw the bode plot without even computing lot of values, so that is what we will try to do, so then bye.