Process Control - Design, Analysis and Assessment
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MATLAB Tutorial 2 Time domain analysis of first order process

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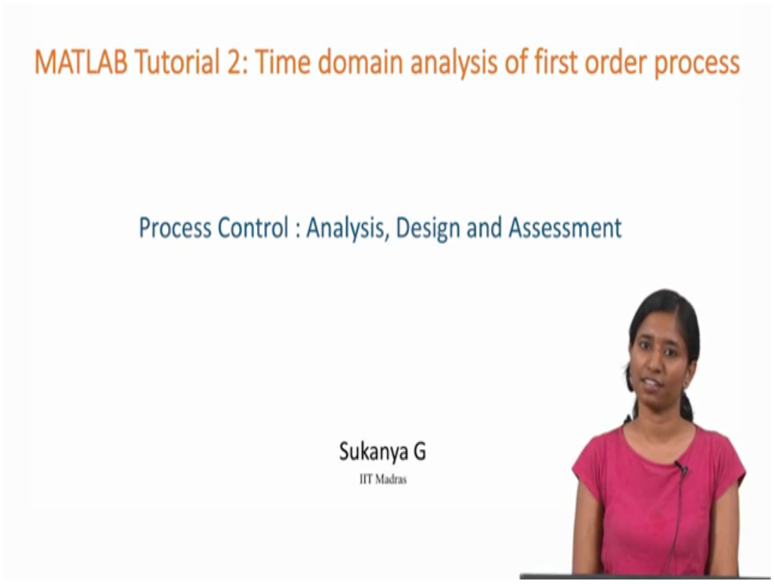

Hello everyone, welcome to this MATLAB tutorial for the course process control analysis, design and assessment. In this tutorial we will be looking at the time domain analysis of a first order process.

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So a quick recap of what we did in tutorial 1, we looked at the modelling of a physical system and we took the example of a liquid level system, we modelled the nonlinear differential equation using the conservation laws, then we found the response of the nonlinear process model to a step input and we linearized the model and we found the state space representation of the model and we evaluated the response of this linearized system to various kinds of inputs and we found that (linear) when we do the linearization around (a state) steady state operating condition the response of the model of the linear as well as a nonlinear model are quite similar.

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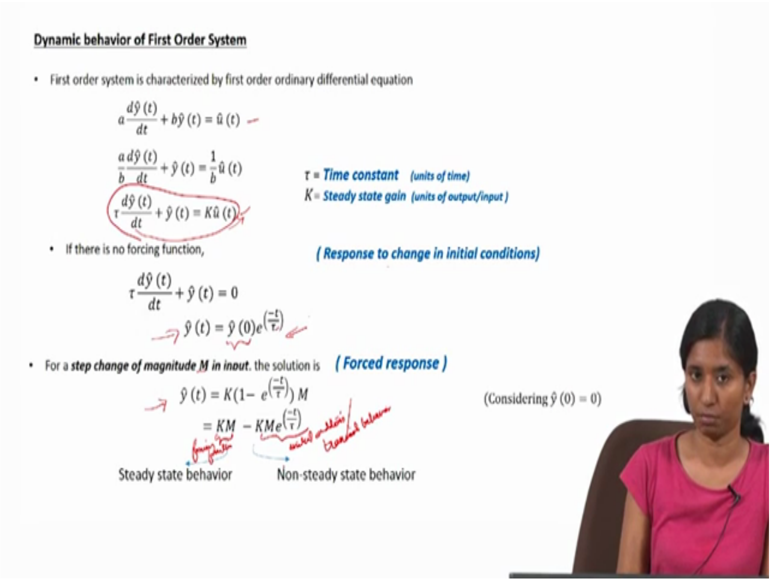

In this tutorial we will focus on the dynamic behaviour of the first order system and what do we mean when we say a first order system, in first order processes the differential equation which describes the behaviour of the process is first order in nature and the order of a differential equation is determined by the degree of the highest order derivative in the differential equation expression.

So the general form of a first order differential equation is as follows it is a into dy hat by dt plus b into y hat of t plus u hat of t, I have written this first order differential equation in terms of the deviation variables, so that is the reason I have used y hat, we saw that in the in the previous tutorial when we linearize the non-linear differential equation and we usually use deviation variable form to write the state space model which is quite useful because then only we will be able to study how the changes in input effects the outputs.

So we write in terms of variables called deviation variables and here this first order differential equation is written in terms of deviation variable. So if we rearrange this first order differential equation into a format like this that is (diff) divide the entire term by a coefficient of y hat of y we will arrive at this format and if we substitute the coefficient of the derivative that is a by b as tau and the coefficient of the forcing function here which is represent as u of t, if the coefficient of the forcing function is denoted by capital K then we will arrive at a form like this, this is the general represent of representation of a first order process that is tau d y hat by dt plus y hat of t is equal to K u hat of t, where tau is called the time constant of the process and K is called the process scheme.

So time constant of a process basically determines the speed of response of the process and the process gain relates the input of the system and the output of the system at steady state conditions, we will talk about time constant and the process gain in a bit but let us look at in so finding the solution of this particular differential equation. So if there is no forcing function if there are no input and the system is at a particular operating condition at a particular initial state, let us see how the response will be.

So if there is no forcing function (then the LHS) then the RHS in this equation is zero, so this is a homogeneous differential equation and the equation is tau dy hat by dt plus y hat of t is equal to 0 and if we do the integration of this particular equation we will find the solution as this y hat of t is equal to y hat of 0 into e raise to minus t by tau which depends on the initial condition and the time constant tau. So it determines it is so this solution of this differential equation depends on the initial condition shown here and the time constant tau.

So this if the system was at a particular equilibrium state or particular initial condition and if it was slightly perturbed from this initial condition, how would the system response will be? So that is what is described by this equation. Now suppose there is a forcing function involved and we know that there are different types of forcing function it can be the step, the ramp, the impulse or sinusoidal and so on.

So in this particular case we will consider the forcing function as a step input and for a step input of magnitude M the solution will be consisting of both the response due to the initial condition that is a homogeneous solution as well as the response due to the forcing function. So when we try to solve that is the equation will be of this form for if there is a forcing function and when we try to find a solution with this particular equation you can do the integration and find the solution.

So when you do that the solution that you will get is in the the solution that you will get is this one that is y hat of t equal to K times 1 minus e raised to minus t by tau M this was arrived by considering that the initial condition is 0, since we have defined the variables to be in the deviation variable form so the initial condition is 0. So this is the equation and if we expand it we will see that it is K times M minus K times M into e raised to minus t by tau.

So this is the response due to the forcing function which in this case is a step input of magnitude M and this is the response due to the initial condition or the transient behaviour. So this term here determines the steady state behaviour of the response and this term here defines the transient behaviour of the process to a step input.

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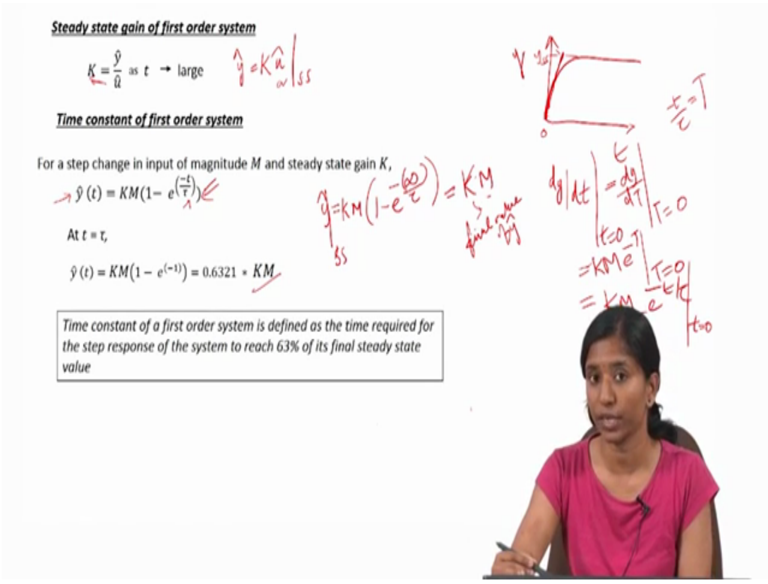

Coming back to the steady state gain, the steady state gain of a first order system as I said before it relates the output of a system to the input of a system at steady state. So this steady state gain at what we mean by steady state is when time tends to infinity what will be the value of the response? So the value of the response is determined by that is y hat is determined by two terms we cross multiply here, so K times u hat at steady state.

So if the value of K that is this K is also known as a process gain so if the value of process gain is very high then for a small change in input the value of the output will be very high that is the change in output will be very high and prominent and the system will be very sensitive in that case but if the process gain is very small then for small changes in input the output change will not be that prominent so only if there is a very large change in input the we can see appreciable change in the output and such systems are quite insensitive to changes in inputs. So this process gain or the steady state gain is basically the ratio of the output as it approaches the steady state to the ratio of the input.

The time constant of a first order system defines the speed of response of a system that is how fast or how slow the system responds to a change in input or change in step input. For a first order system the solution of the differential equation for a step input of magnitude M and the steady state gain K is y hat of t equal to K times 1 minus e raised to minus t by tau. So if time is equal to time constant tau let us substitute and see how the equation will be, so if in this equation for t if we substitute t equal to tau, then we will get as K times M into 1 minus e raise to minus 1 that is equal to 0.6321 times K into M.

So what is the significance of K this term K into M? We have seen in the first slide that K into M determines the steady state behaviour of the system, so and by steady state it means time t tends to infinity. So here we know that y hat is equal to K times M into 1 minus e raised to minus t by tau, so if in a place of t we substitute T equal to infinity let us see how the response will be e raise to minus infinity is 0, so y hat at steady state is equal to K times M, where K is the process gain, M is the magnitude of the step input.

So when time t is equal to time constant tau the response is 0.6321 times the final steady state value, so this is the final steady state value. So when time t equal time constant the response is 0.6321 times the final value, so we can found formulate a definition of time constant in this way, time constant of the first order system is the time required for the step response of the system to reach 63 percentage of the final steady state value.

So when time equal to one time constant the response has only reached 63 percentage of the final value and if time t is let us say two time constant, or three time constant and so on the response eventually reaches the final steady state value. Another way to look at the time constant is, so we know that the step response of a first order system looks something like this and this is the steady state value okay this is 0.

So if let us say that if the initial rate of change which is given by dy by dt so the slope of this curve is dy by dt and the initial value of this slope that is at t equal to 0. So let us evaluate this we know that y is equal to this one and let us say let us substitute this minus t by tau equal to some variable capital T, so this will be if we differentiate this with respect t this is also equivalent to differentiating with respect to d capital T at T equal to 0, so if we do this let us say let us differentiate this particular expression here.

So we what will get this KM e raise to minus T at t equal to 0, that is KM by tau e raise to minus t by tau at a small t equal to 0. So if the initial rate of change of the response was maintained as it is then the process will reach its final steady state value at time equal to one time constant so this is how the speed of response can be explained that is if the time constant is high then system will reach the final steady state value very slowly but if the time constant value is very slow then the system will quickly reach the final steady state value.

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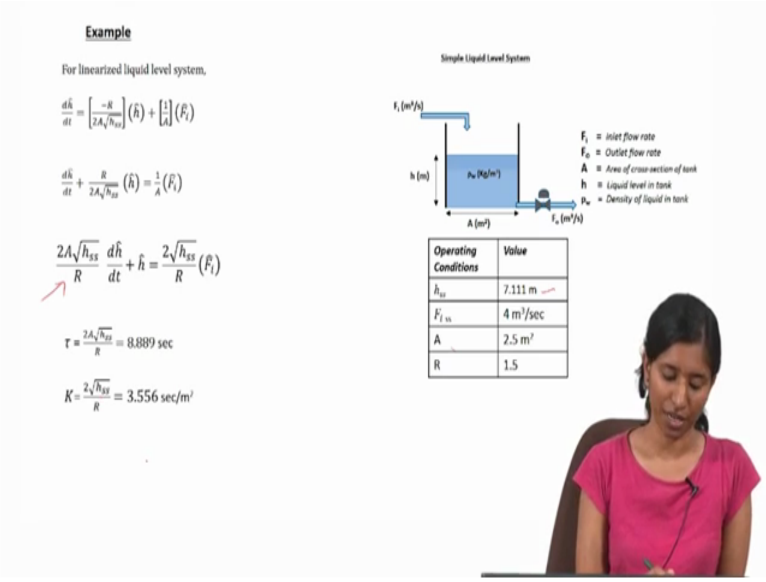

So let us revisit the example that we considered before, it is a simple liquid level system and it has an inflow, it has an outflow, there is holed up within the tank, there is a resistance that is placed at the outlet and these are the operating condition that is shown here in this table, we have the inlet flow initial flow of the inlet and we have the area of the tank, we have the value of the resistance and based on this we have calculated a steady state value of the height which was found to be 7.111 meters.

And for a liquid level system which was say initially it was a only in a differential equation then we linearized it and we wrote in terms of deviation variables and it was in linear differential equation. So here we will consider the linear differential equation and it is it can be written in this form dh hat by dt is equal to minus R by 2 times a root h ss times h hat plus 1 by A times F i hat.

So we know that the first order differential equation in terms of time constant and the process gain has particular format. So if we rearrange this equation in order to get the differential equation in terms of the time constant and the process gain, we will rearrange it and we will rearrange it in this manner and if we substitute the values from here in this table we can find the expression for so we can find the expression for time constant tau has as 2 times A times root h ss divided by R and the expression for process gain as 2 times root h ss by R.

So if we substitute the values for all this if we substitute the values for all this from this table we will find the time constant of this particularly liquid level system as 8.889 seconds and the process gain as 3.556 second per meter square. We can see that the time constant and the process gain is determined by the process operating conditions.

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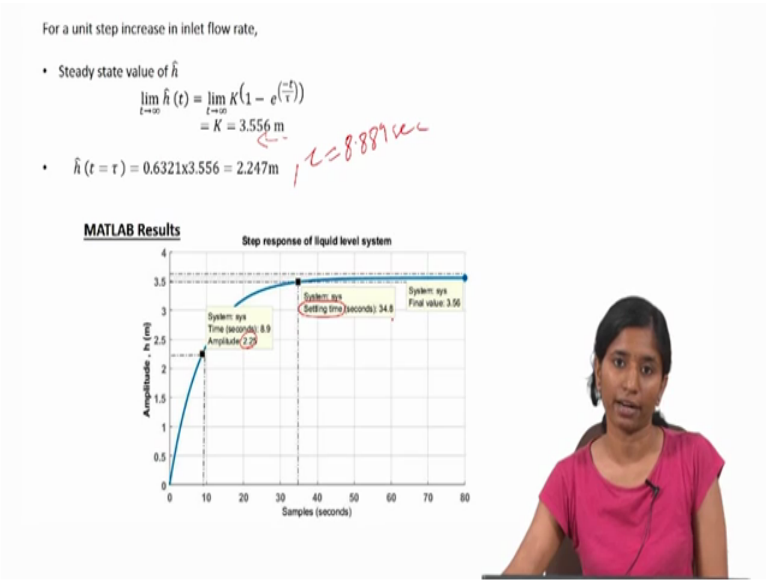

So if we give a step input to the process that is if we apply a unit step increase in the inlet flow rate what will be the steady state value of height? So since we have the expression for the solution of the first order differential equation, we can use that to determine the final steady state value. So the final steady state value of a liquid level within the tank can we determine as limit t tends to infinity h hat of t, since the h hat of t has an expression as K times M, where here M is units since it is the input is unit step that is it as a magnitude 1, M equal to 1 so it is K times 1 into 1 minus e raise to minus t by tau.

So if you do this we will find the value of K equal to 3.556 we will find the value of we find the steady state value of height is equal to K and since value of K is 3.556, the value of height at steady state is 3.556 meters. Now let us also see what will be the value of height at time t is equal time constant, that is we know that at t equal to one time constant the response has only reached the 63.2 percentage of its final steady state value, so if we do that apply here, if we apply that logic here we will find the value of height at time t equal to one time constant to be 2.247 meters.

So I simulated the liquid level system in MATLAB and I applied a unit step increase to the inlet flow rate and I have solved the differential equation and found the step response of the liquid level system. So from the step response of liquid system I can verify that when time equal to 8.9 in our case remember the time constant was 8.889 seconds, so when time equal to 8.9 which is an when time is 8.9 the amplitude is 2.25 that is when tau equal to second we found that doing by our hand calculation we found the steady state we found the value of height to be 2.247 so we can verify away from our from the plot that the amplitude is 2.25 and the steady state of the final value which is the steady state value we found to be 3.556 and here we can verify that the final value is 3.56.

But there is a new term here it is the settling time so the settling time is the time taken for the step response to reach and stay within 2 percentage of its final steady state value and for this particularly liquid level system the settling time is 34.8 seconds.

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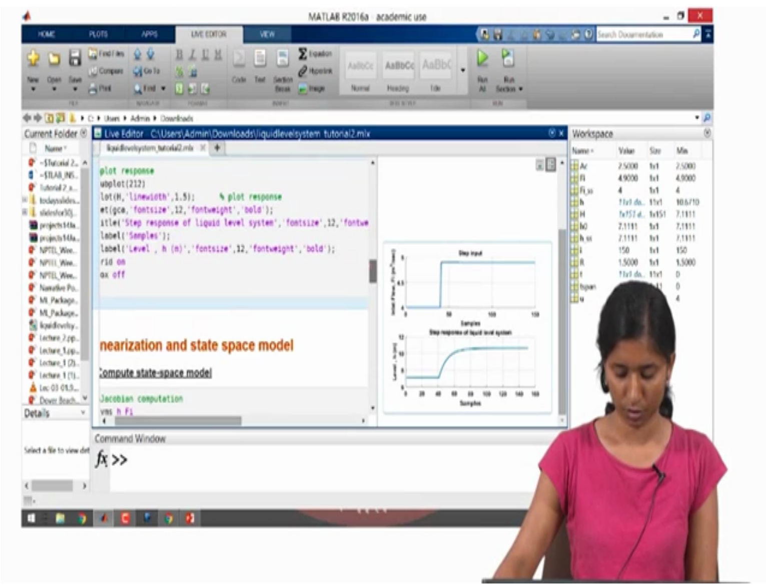

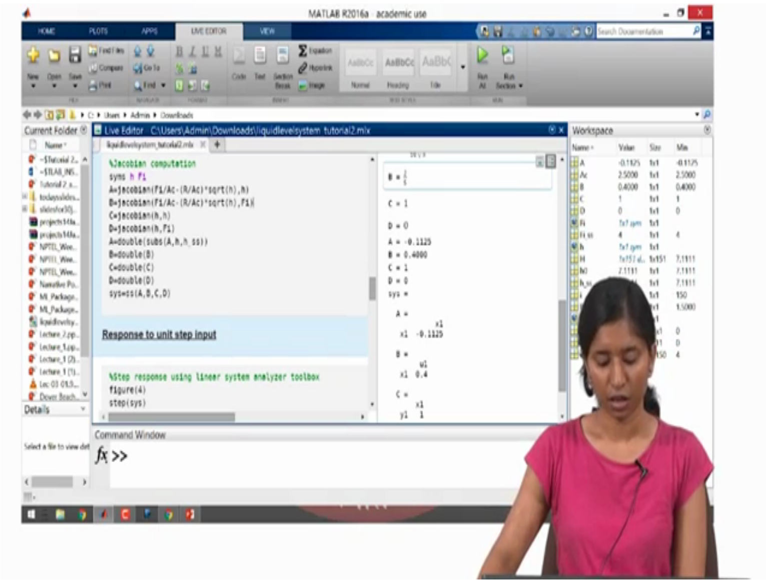

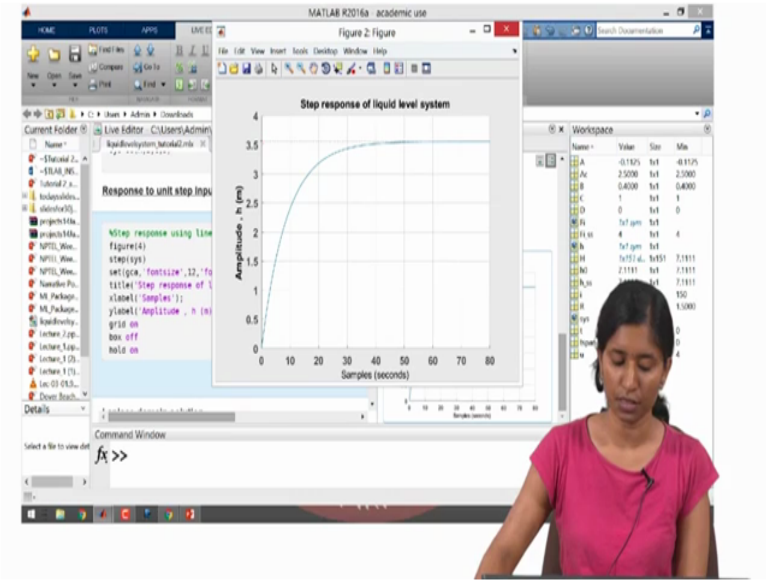

So let us have a look at the MATLAB and see how I simulated it we had covered the modelling aspects and of this liquid level system in the previous tutorial. So I will quickly run through this code and simulate the step response when a unit step was given you can do the simulation on your own and verify the result. So I will not deal deeply into the modelling aspects.

So this was the step response, this was step input given, this was step response, that was my nonlinear model. Now let me do the linearization and determine the state space model which is which consists of linear differential equations, I have calculated the state space model here and it is stored in this variable called ss the (())(20:01) of state space model. Now what I will do is I will simulate the response of simulate the response of the state space model to unit step input and let us verify the result.

So as you can see from this figure the step this is the step response and the time t equal to 0 denotes the time at which we gave the step input, in previously we had given a step input at the 41st time instant but we have considered that time instant to be t equal to 0, so at the 0^{th} instant I gave a step input and this is step response, unit step input was input given and we can see that the steady state value is 3.556.

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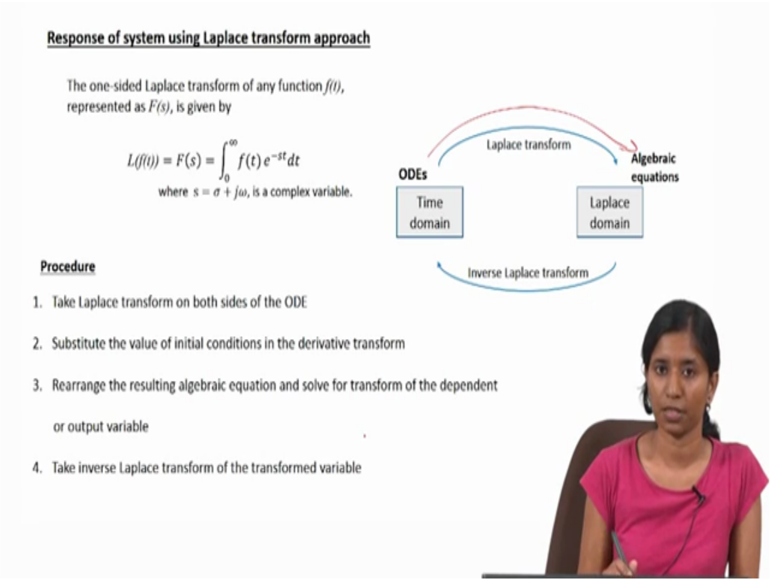

Till now we saw how to solve a nonlinear differential equation in order to get the time response of a process and we saw how to linearize and again solve a linear differential equation to get the response of the system. So in all these cases we had to solve a differential equation but what if we had a simple tool which could make our job of solving differential equations easier.

So one such tool which is very popular in the control engineering field is the Laplace transform tool this Laplace transform tool is used to convert differential equations into simple algebraic equations which are quite easy to study and analyse and manipulate the one sided Laplace transform of a function a time domain function f of t which is represented as F of s is given by L, L is the Laplace transform operator L of f of t equal to F of s equal to integral 0 to infinity f of t e raise to minus st dt, where s is sigma plus j omega a complex variable.

So what Laplace transform does is it converts the time domain function into another domain called the s domain or the Laplace domain, so in the time domain we have the differential equations and the Laplace transform converts these differential equations to algebraic equations and these algebraic equations are quite easier to study and this algebraic equations are very easy to manipulate and analyse.

And if you want to convert it back to the time domain itself, we can do the inverse Laplace transform there are many techniques to do this, one of them is the partial fraction expansion, partial fraction expansion so we can do that and get our function back to in the time domain itself. So how Laplace transform tool makes our life easier is that the Laplace transform of so many time domain functions have been already derived and compiled into a tabular form.

So if we want to get the Laplace transform of a particular time domain function, we just have to go and look into the table and write down the corresponding Laplace transform function. So if even if we cannot find any direct Laplace transform of the particular time domain function that time domain function can be written as a combination of various other functions whose Laplace transform have been already derived.

So in one way or the other we will be able to find the Laplace transform of a time domain function from the lookup table itself and that is how we can quickly find the Laplace transform. So the and how can we use this Laplace transform to determine the response of a system? The procedure is very simple first we can simple we know that the system is described by a differential equation a linear differential equation Laplace transform can be only applied to linear differential equations.

So we have a linear differential equation and we can take the Laplace transform on both sides of the linear differential equation and we can substitute the value of initial conditions in that equation and we can just rearrange that we can rearrange it and get it in terms of the Laplace transform of the dependent variable and from the lookup table we can determine the inverse of this particular function and not just the lookup table from the and by using partial fraction expansion and the lookup table of Laplace transforms we can determine the time domain function of the particular Laplace domain function.

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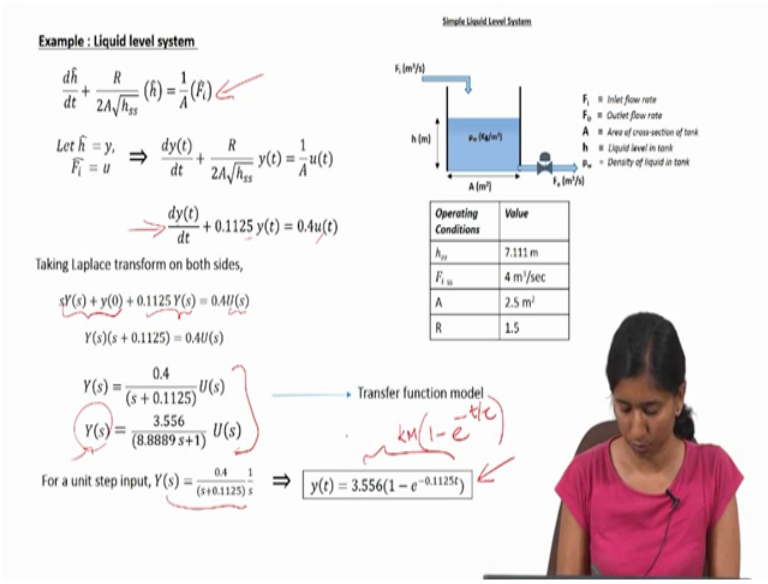

So let us see how we can apply that in for our liquid level system, the liquid level system the differential equation in terms of the deviation variable is this one. So for the sake of explanation I have defined h hat is equal to y and F i hat equal to u so the expression is dy by dt plus R by 2A into square root of h ss into y hat of t equal to 1 by A u of t. So once we substitute the values of all these parameters from the table we will get the equation as dy by dt plus 0.1125 y of t equal to 0.4 times u of t.

So what we can do first? We will take the Laplace transform on both sides and Laplace transform of it this is a derivative term, so we have Laplace transform property which is which is using which we can find the Laplace transform derivative. So if you are not familiar with it, I suggest you to have a look at the Laplace transform functions and Laplace properties.

So if we take Laplace transform of this particular function (how) what we will get is s times y of s plus y 0 plus 0.1125 times Y of s equal to 0.4 times U of s. So this is the Laplace transform of dy by dt (by) and this is the Laplace transform of this term and this is the Laplace transform of u of t. And if we do some algebraic manipulations we will get Y of s equal to 0.4 by s plus 0.1125 times U of S or if we rearrange it we will get is what we will get is 3.556 by 8.889 times s plus 1 into U of s, this particular format is something called transfer function model which gives the which is an algebraic expression relating that time domain behaviour or dynamic behaviour of the input and output in Laplace domain.

So this transfer function is quite popular but we will talk about it in a bit let us go with the flow and derive the time domain expression of the response of the system. So we have this Y of s expression here and for any kind of input we can find the response from this expression using inverse Laplace transform. So for a unit step input Y of s is 0.4 by s plus 0.1125 times 1 by s, 1 by s is the Laplace transform of a unit step, so using partial fraction expansion the inverse of this the inverse Laplace transform of this expression is found to be (y of t equal to 0.35) y of t equal to 3.556 times 1 minus e raise to minus 0.1125 t.

So this expression can be used to find the response of the system at various time instants so what we did was (we did) we found the Laplace transform of the differential equation, we rearranged it so that we get in terms of the Laplace transform on the dependent variable and then being for a particular value of input we found the inverse Laplace transform that is the time domain function of the output as you can see this is in the format km into 1 minus e raise to minus t by tau.

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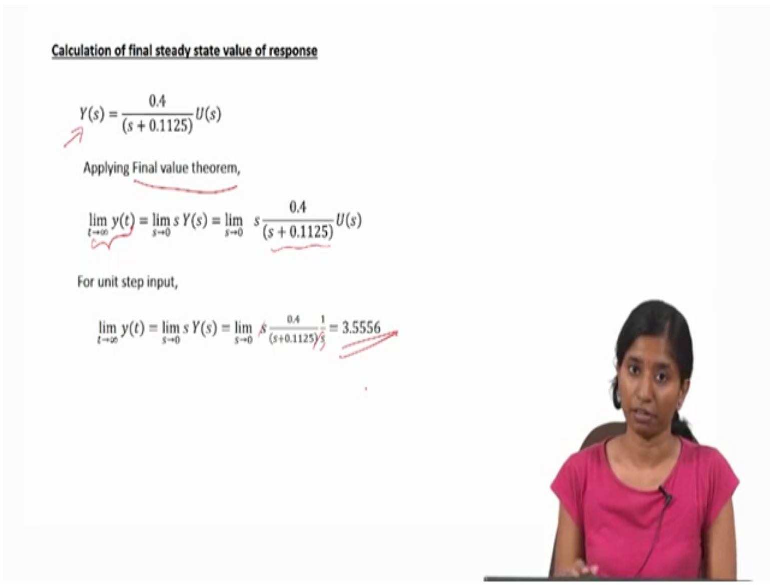

So if we have a function in the transform domain that is the Laplace domain, we can still find the final value final steady state value of the response by using something called the final value theorem. So we know that the expression for the response of the system in the Laplace domain is Y of s equal to 0.4 by s plus 0.1125 times U of s and if U of s is the step input using applying and if U of s is the step input applying the final value theorem we can find what the steady state value output will be.

So let us see how to do that according to the final value theorem the final value of the response as t time tends to infinity is equal to limit s tends to 0 s into Y of s so if we substitute the value of Y of s here and if we give if unit step so that the Laplace transform unit step is 1 by s this and this will get cancelled and s equal to 0, so what will get this 0.4 by 0.1125 which will give us 3.5556. So this is the same steady state value that we had obtained earlier and this confirms that if we solve the differential equation by integrating or if we solve it by this Laplace transform approach, the steady state value is that big obtain is the same.

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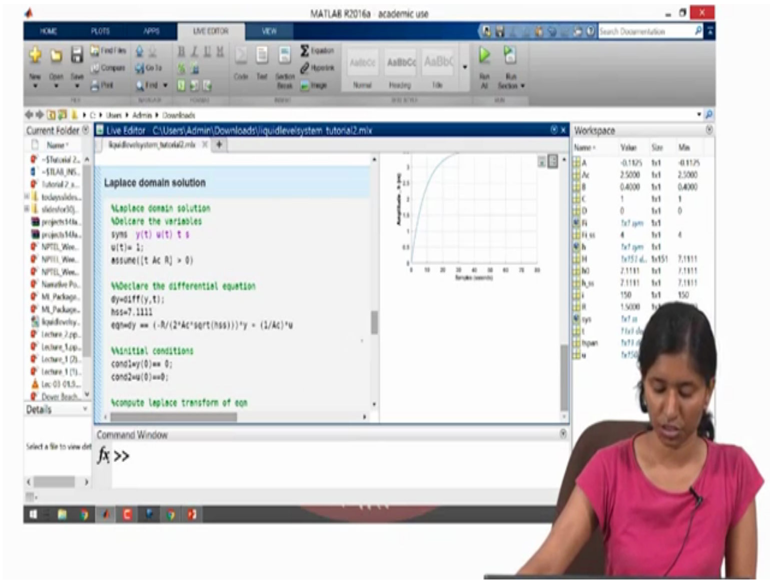

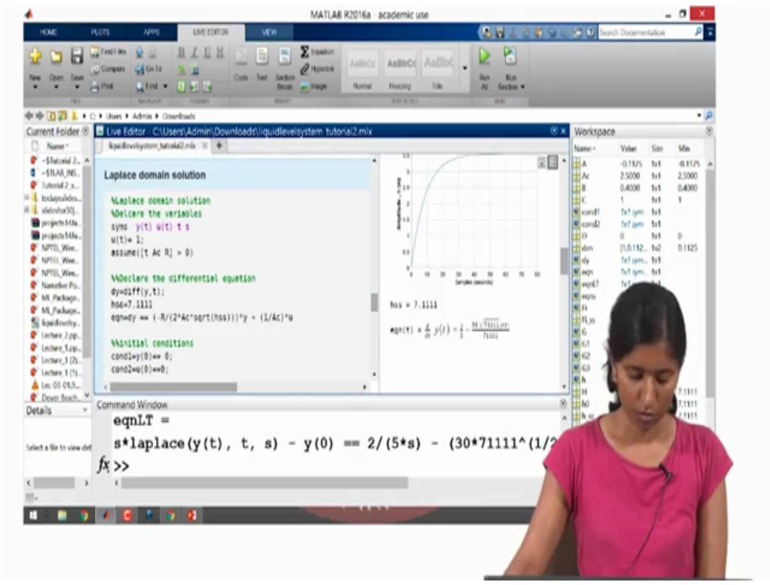

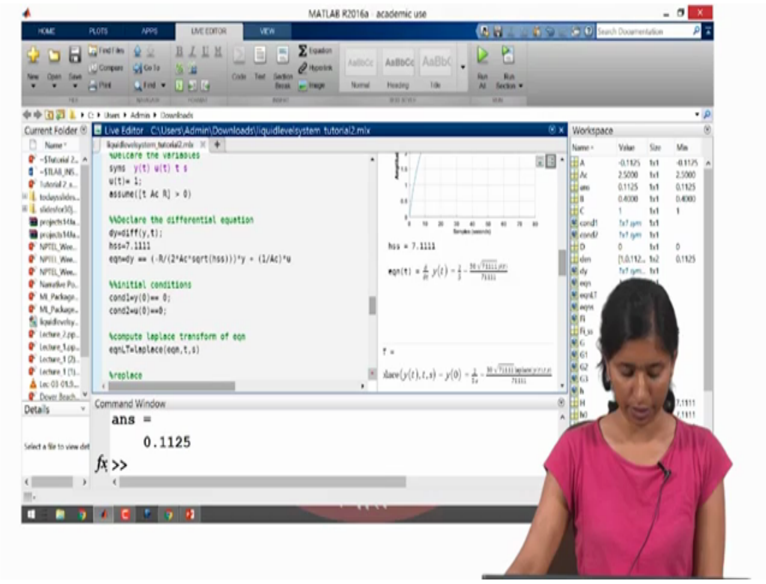

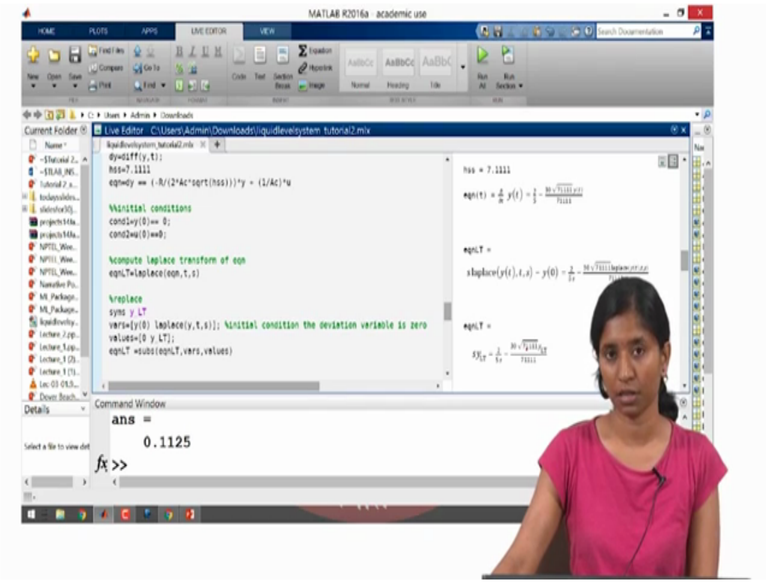

So let us verify this using MATLAB to find the Laplace domain solution I will use the symbolic tool bag toolbox in MATLAB, which will give me the Laplace domain solutions in expression form. So I have to first define what my symbolic variables here are they are y of t u of t, this is the time domain variable t and this is the Laplace domain variable s, so these are declared as symbolic variables, my input u of t is 1 since I have given its unit step input and first I have to declare the differential equation the differential equation is defined declared as dy equal to diff of y comma t, (y is the independent) y is the dependent variable, t is the independent variable, then I have my expression I have the differential equation which I name eqn equal to dy equal to minus R by 2 times (a) Ac times square root of h ss times y plus 1 by Ac times u, Ac is the area of cross section, R is the resistance, h ss is the steady state value which I calculated to be 7.11, then I have to define that what my initial conditions are since this this equation is written in terms of deviation variables the initial condition is 0.

So the expression the the command for finding the Laplace transform in MATLAB is Laplace of equation which I have defined my which is the function whose Laplace transform needs to be calculated, t is the time domain variable and s is the transformation variable. So eq and LT is the name given to the Laplace transform of the equation. So let me run till this this part and show you what the so I will run this particular I will run till this line and show you the result.

So I have run this section and this is my equation this is equation that I defined. So this is equation that is the Laplace transform of this particular term here, so this is the derivative term, so it is s into Laplace of y of t minus y of 0 equal to 2 by 5 into unit step is here, so Laplace transform is 1 by s so it is 2 by Phi s and not to get scared by this big term but this is simply okay let me calculate the value. So it is simply 0.1125 so 2 by 5 East 0.4, so 0.4 into 1 by s minus 0.1125 into Laplace transform of y of t so that is all there written there.

Now since this term I computed the Laplace transform but the Laplace transform of y of t is certain in the as a statement like Laplace of y of t comma t comma s which is not quite pretty to see. So I have let me substitute the this term as y underscore LT and the value of the initial condition also I will substitute here itself so initial condition is nothing but 0. So again I will define the new variable y underscore LT as a symbolic variable and my variables are y of 0 and the Laplace of y comma t comma s I want y of 0 to be replaced by 0 and Laplace of y comma t s to be replaced by y underscore LT.

So I will substitute these values (and the way) for the variables and find the equation so the equation LT is now reduced as s times y LT which represents the Laplace transform of y of T equal to this is 0.4 into 1 by s and this is minus 0.1125 times Laplace transform of y of t.

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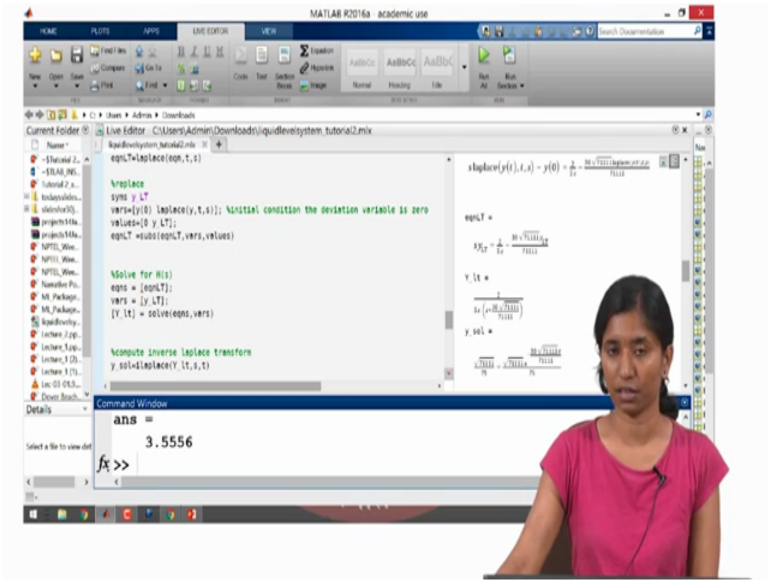

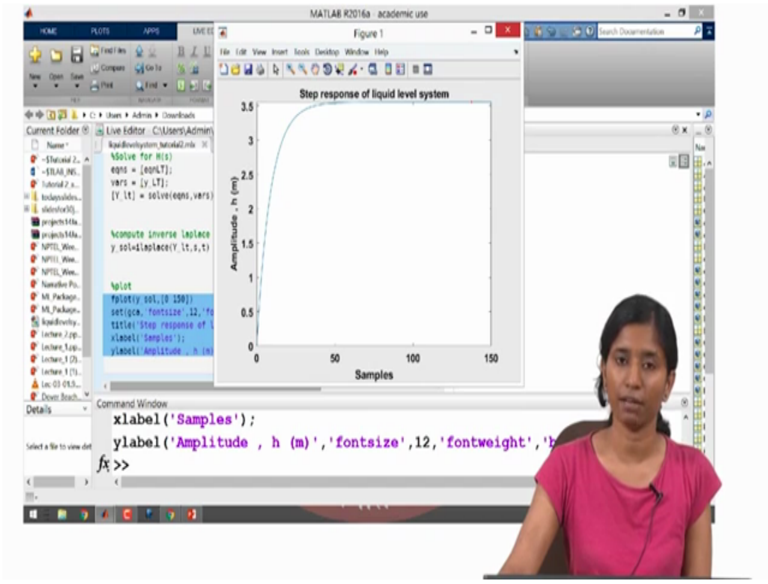

Now we have to rearrange and find the solution in terms of the dependent variable Laplace transform of the variable so solve for the Laplace transform of dependent variable so we can find that when we rearrange this we will get the Laplace transform of dependent variable to be 0.4 by s into s plus 0.1125 and if we want to find the time domain solution we have to take the inverse Laplace transform. So if we take the inverse Laplace transform using the command I Laplace of function that is y underscore LT comma s comma t here the transformation variable is t so I have named it as y underscore sol by solution.

So it is found to be this one let us just verify whether this is the same so so the first term is 3.5556 - again 3.5556 times e raised to minus 0.1125 times t. So this is a solution that we obtained time domain solution that we obtained using the Laplace transform approach. Now let us plot the solution for different time instants and see if it matches the response that we obtain by solving the differential equation previously.

So this exactly matches what we had obtained before as a steady state value is 3.55 so solution that we obtained using Laplace transform approach as well as the solution that we obtained by solving the differential equation are both same, but using Laplace transform approach it makes our job very easier and (it can) we can use this to look at the system response and derive a system response quite easily.

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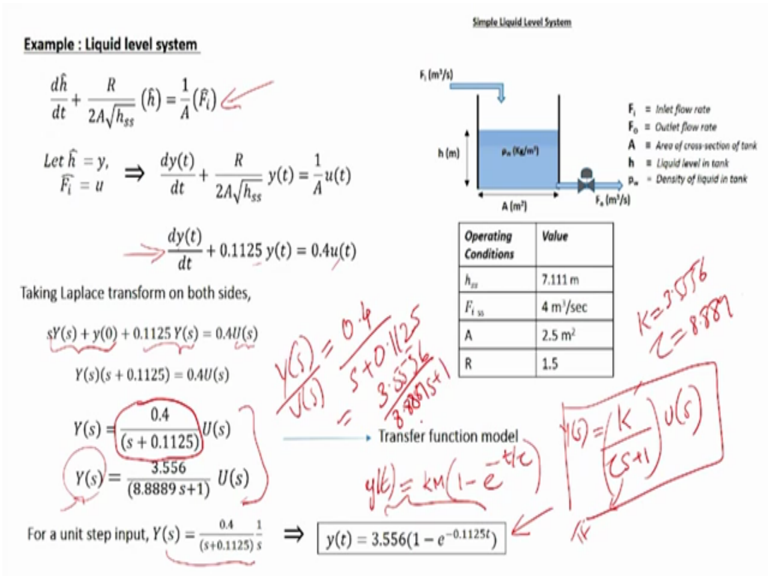

Now let us go back to the term that is that we looked at before that is a transfer function model. Suppose there is a change in the initial condition or any a change in the forcing function then we have to do the entire Laplace transform procedure that is taking the Laplace transform, rearranging and then taking the inverse Laplace transform all this we have to repeat.

But if we have an expression like this that is Y f s equal to 0.4 by s plus 0.1125 times U of S, then irrespective of what forcing function is we can always find the response to other system and not only that one, this is independent the initial condition in a way that we had derived this using deviation variable. So the dependency of initial condition is already not there and if we have expression like this we do not need to sit and derive all the Laplace transform take the inverse Laplace and so on.

So whatever be the input input we can simply plug it in to this transfer function model and find the response, this makes our job very easy so this particular form for a first order system can be generalized as Y of s equal to K by tau s plus 1 into U of s but K is the process key, tau is the time constant, for this system the value of K is 3.5556 and the time constant is 8.88, so this is a general form of representing transfer function this is the general form of representing a first order system as a transfer function model, this term is the transfer function so Y of s by U of S is written as 0.4 by S plus 0.1125 or in time constant form it can be written as.

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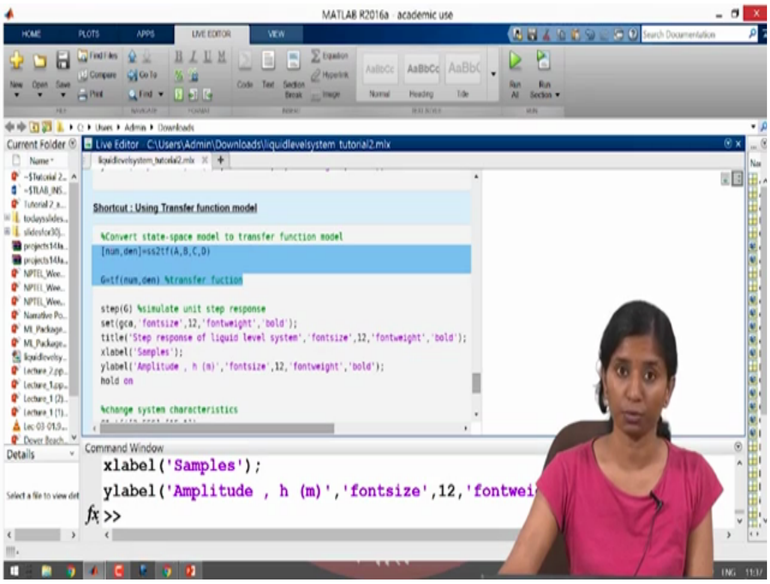

Now let's see how the how we can simulate the response of the system using the transfer function model in MATLAB. So the transfer function model can be either arrived by doing the Laplace transform and finding the expression which relates the Laplace transform of the input Laplace transform of the output to Laplace transform of the input or if we have the state space matrices we can simply use the command ss2tf of A, B, C, D ss2tf will convert the state space model to transfer function model and it will give us the numerator and the denominator of the transfer function model. So I and you see in the command tf of numerator common denominator I can generate the transfer function.

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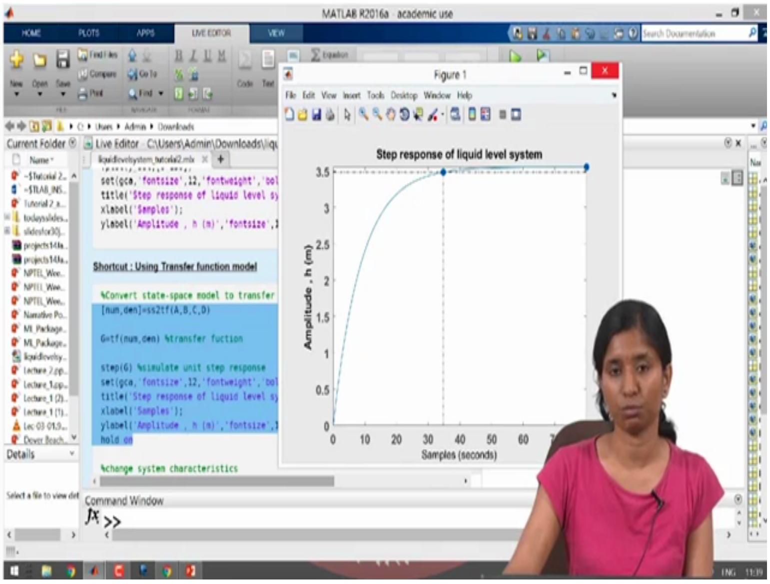

So let me see let me show you how to do that let me run this section and once we have the transfer function we can use the command step of G, G is the system name to simulate the step response. So this is the step response that I obtained using my transfer function I can move around and see the value of the (())(40:32) for different time so that as t is time as time is very large the system has the steady state I can show you the characteristics system characteristics that is the steady state value shown here, final value as 3.56, this is my settling time 34.8 which is the same that I obtained previously.

So once we have the transfer function it is enough to explain the entire dynamic behaviour of a process and for all the further analysis that is yet to come in our course we will be only using this transfer function to model the system.

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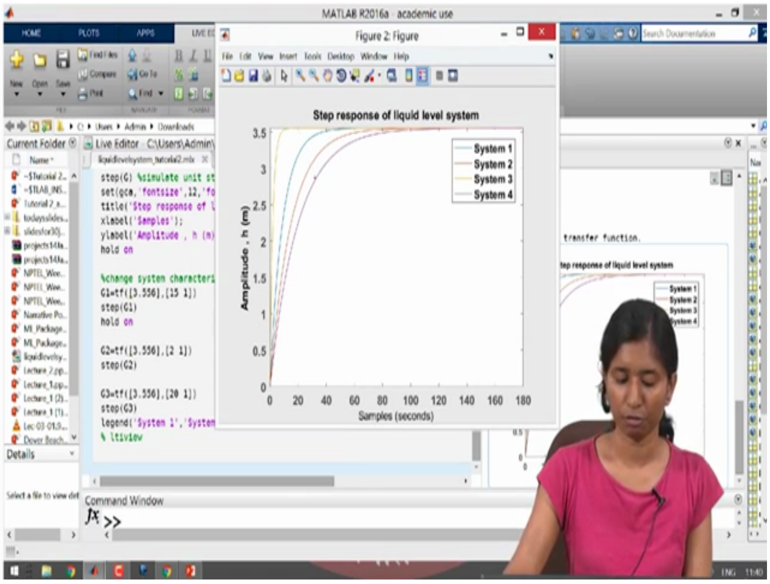

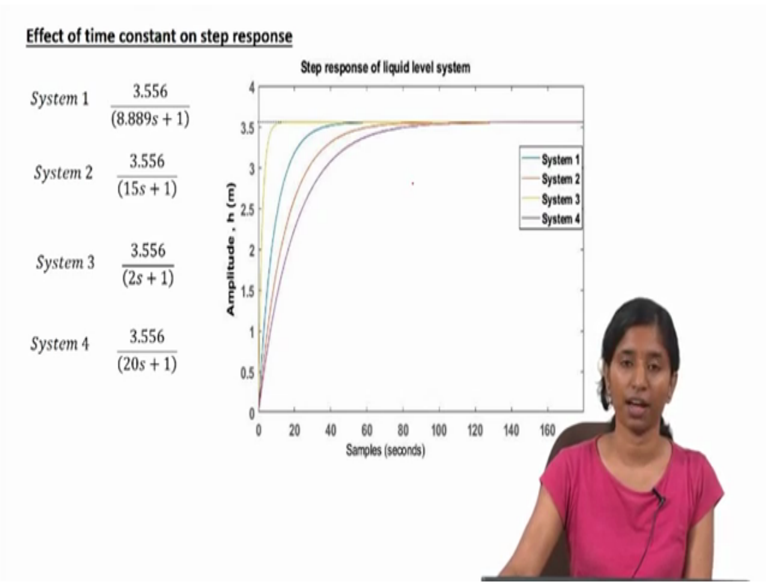

Now let me slightly change the value of the time constant and see how the system response would be. So this is my continuous time transfer function first for my original system 0.4 by s plus 0.1125 this s plus 0.1125 is the denominator and the root of the denominator is called the poles. So in this case the pole is minus 0.1125 and I have simulated 3 other systems G 1, G 2 and G 3 using different values of the time constant and let us look at the response of the system. As you can see in my system 1 the value of the the system 1 is the blue line system 1 is there, but system 3 responds very faster and quickly reaches the final value and system 4 is the slowest.

So the time constant of the system 1 was 0.889 seconds shown in blue and the time constant of system 2 is 15 seconds system two is represented by this orange line here and it is slower than system 1, system 3 has a time constant 2 and system 3 responded very fast and very quickly reached the final steady state value, system 4 has a time constant 20 and it is the slowest among this.

So in this tutorial we looked at how we can find the time domain response of the first order system and how the steady state and the time constant characterizes this first order system and how we can use Laplace transform to solve the differential equation and arrive at the transfer function model we also (found) looked at the effect of time constant on the step response of a system, I hope this tutorial was informative to you, thank you and have a nice day.