Process Control - Design, Analysis and Assessment
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Lecture 11
Analysis of transfer function models Part 2

We will continue with the eleventh lecture in the course on process control, analysis, design and assessment.

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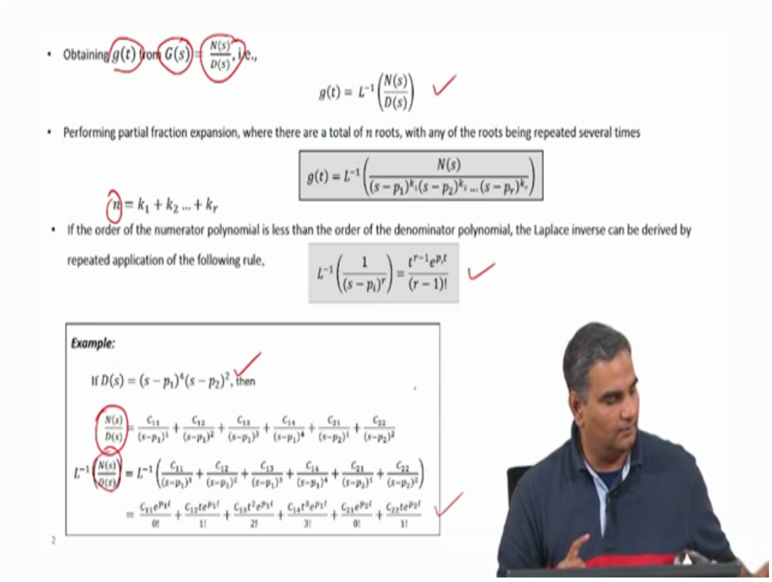

So we have been talking about analysing transfer function models and in the last few lectures I have been talking about getting time function out of Laplace function and remember we talked about an input that comes in gets converted to a Laplace domain function gets multiplied by the transfer function and you get an output Laplace domain function which could be anything and we could call it Y of s and basically what we want to do is while we have the result in the Laplace domain we want to still get the result back in time domain because that is what makes physical sense, so how do I go from Y of s to Y of t.

Here we are using G of s to g of t and remember we also said that if I do a Laplace of input that is going to be some numerator by denominator function and then I have the transfer function which is also a numerator by denominator, so any output is also going to be a numerator by a denominator polynomial which is of this form right here. So as long as the denominator polynomial is for greater order than the numerator polynomial we are looking at coming up with one general method using which we can do this inversion.

Once we know how to do this inversion we do not have to really worry about what the actual forms are we can always get a solution in time domain. So ultimately we are looking at getting a g of t which is Laplace inverse a numerator by denominator and one technique which will always work as long as you have this form is this partial fraction, so there are no cases where you can say I cannot do partial fractions you can always do partial fractions with this notion that the denominator is higher order polynomial and an important thing to note that is that once you have these polynomials, these polynomials could have real complex roots.

So you do not want a technique which only works for real roots or complex roots and so on actually what we are going to show it does not really matter whether it is a real root complex root it will always work and that is the reason why we are going to use a notation P as a pole of a transfer function. So whatever I am deriving here and whatever result I am showing here you can simply follow that result whether the root is does complex or real, it really does not matter.

The only thing that will happen when you have complex roots it is there will be a complex conjugate root pair whenever you have a complex root that is the only thing and basically when you have a complex root and a complex conjugate pair you think of them as two roots of the system and then simply proceed the way that I am going to show you in terms of how we should do it.

So let us start and then say I have l l inverse of n over N of s or D of s, now let us keep the numerator polynomial the same but the denominator polynomial I am going to write in what I am going to term as root resolved form which basically says that there is an nth order polynomial here, so we are generalizing this now and this nth order polynomial has let us say r roots if no root is repeated then it will have n roots but if there are certain roots repeated then it has r roots.

Let us say that p 1, p 2 p 3 and so on up to p r are the r roots and let us also generalize this and then say p the root p 1 is repeated k 1 times the root p 2 is repeated k 2 times and the root p r is repeated k r times and so on then the denominator polynomial you can always write it as s minus p 1 power k 1 times s minus p 2 power k 2 and so on, s minus p r power k r. So the condition is that since I cannot have more than n roots for an nth order polynomial the condition is that n equal to k 1 plus k 2 all the way up to k r.

Now what we are going to do is we are going to repeatedly apply just one row of the Laplace inversion table which is this, so whenever I have a term of the form 1 by s minus p i to the power r the Laplace inverse of this Laplace domain function is the following time domain function which is t r minus 1 if this is r this is r minus 1 e power p i times t divided by r minus 1 factorial, so very simple formula.

So if it is 2 for example then 1 by s minus p I square will be t to the power 2 minus 1 which is just t, e power p i t divided by 2 minus 1 factorial 1 factorial will be 1, so it will simply be t e power p i t. Now let me illustrate this with let us say a denominator polynomial let us assume it is of this form here where there are two roots p 1 and p 2 the first root is repeated four times and the second root is repeated two times, so that basically says that this is a sixth order polynomial which then can be written in this form which is s minus p 1 to the power 4 times s minus p 2 square if we make the leading term as 1.

Now any N s by D s ok can be expanded if D s is of this form like this so in partial fractions whenever some root is repeated what should you do is you add a term for every repeat, so this is repeated four times so I have the first term which is c 11 by s minus p 1 the second term is c 12 s minus p 1 square c 13 s minus p 1 cube c 14 s minus p 1 to the power 4, so the notation I have used is the first number here is the number of the root this is the first root and the second number is the time it repeats right, so the second repeat will be see one to third repeat will be c 13 and so on and correspondingly the denominator will be the corresponding powers which is s minus p 1 square s minus p 1 cube and so on.

And similarly since p 2 has repeated twice there will be only two terms corresponding to p 2 I have c 21 s minus p 2 to the power 1 so 2 is this is the second root the first repeat which is s minus p 2 second root second repeat s minus p 2 square. Now you might wonder what happened to N s does it not matter of course N s matters and what N s is would then define what these constants are.

So once you put the correct constants you can know that by taking the denominator multiplication you can simplify the numerator and that will turn out to be N s, so I am going to show you how to compute the c 11, c 12, c 13, c 14, c 21, c 22 for general problems but as of now this is a procedure, right if one someone told you how to compute the c is then you quickly inverted this Laplace domain function and all you need is just this there is nothing else that you need.

So Laplace inverse N s by D s is Laplace inverse of this large number of terms and using the linearity property of Laplace transforms we can do Laplace inverse of these sum of six terms is the sum of six Laplace inverse terms, so this will be Laplace inverse c 11 by s minus p 1 to the power 1 so if this is in version of this with r equal to 1, if r is 1 this is going to be t to the power 1 minus 1 t to the power 0 is 1 so it will have simply be e power p 1 t c 11 by 0 factorial, the second term will be using this c 12 t per p 1 t by 1 factorial and the third term is c 13 t square e power p 1 t by 2 factorial and so on.

So what has happened is that this Laplace domain function has been converted to a time domain function and the only thing that we still do not know is how to compute the constants but otherwise the procedure is set and irrespective of whatever a functional form you get N s or D s you can always do this partial fraction expansion. We will for now just assume that the order of the D s polynomial is greater than the order of the N s polynomial and just to reiterate the constants basically if you compute them correctly will help you retrieve N of s however the denominator terms right dictate really how many terms in the expansion you have, so in this case it is a fourth sixth order polynomial and there will be six terms the sixth order polynomial could be in different ways here I have p 1 repeated four times p 2 repeated two times, so there are four p1 related terms and 2 p 2 related terms.

If for example this D of s where s minus p 1 cube times s minus p 2 cube then there will be 3 terms related to p 1 and three terms related to p 2 and so on, if all of them are a non-repeated roots then you will correspondingly have six terms, every one term will represent one pole or one root.

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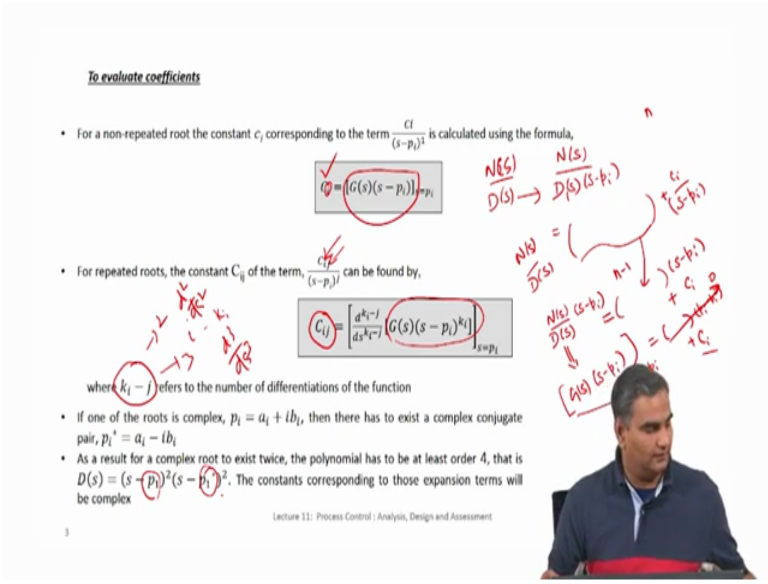

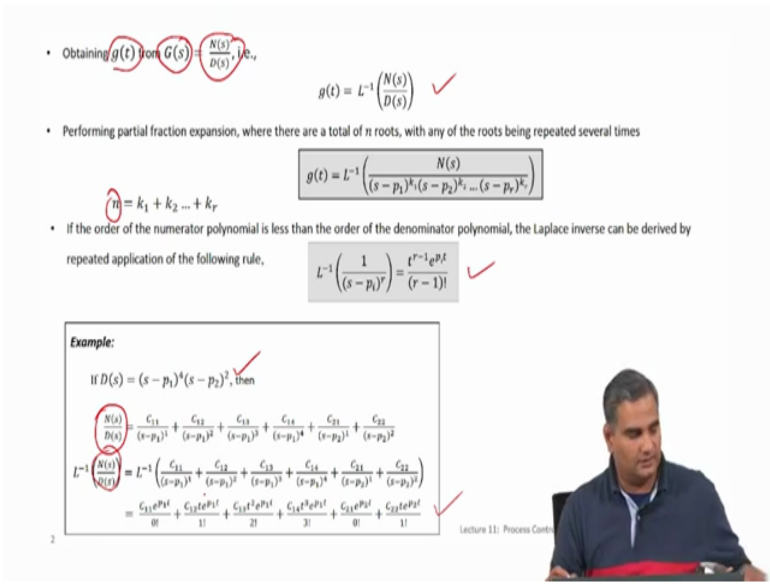

So if I have let us say something where the root is not repeated ok then how do I compute the coefficient corresponding to that? So the idea is the following so supposing I have let us say N s by D s and let us say this D s has one pole which is not repeated, so basically this is going to be N s divided by some D 1 s times s minus p i ok, this is not repeated then remember when you do the partial fractions there will be several terms which will all be functions of the poles other than p i and then I will have this term which is s minus p i ok, so this is my N s by D s.

So we do not have to even expand this I am just showing you that there will be n if the order is n of this polynomial there will be n minus 1 terms here right whichever way they are organized depending on how many roots are repeated or not repeated in D 1 s and so on. Now a neat trick that you can do to compute the c i is the following, so what you can do is you can multiply this N s by D s right by s minus p i, so if you multiply both sides by s minus p i so N s by D s times s minus p i if you do this then this whole term where there is no denominator with s minus p i because that I have separated it out which is this here will be multiplied by s minus p i plus when I multiplied this by s minus p i s minus p i and s minus p i get cancelled so I have c i.

So this will be the equivalent representation when I multiply this, this will be equal to this. Now notice something interesting so this is G of s times s minus P i now if I set s equal to p i so that basically means this is G of s s minus P i ok if I set this with s equal to P i then what will happen on the right hand side is I will have this term and I will p i minus p i plus C i because already the s minus p i while I did this we had removed.

Now this term will go to 0 then you will get the c i is simply this, so if I have a non-repeated root I do not need to do anything at all to get the corresponding c i for the non-repeated root I take the N s by d s or a G s function and I multiply that by s minus p i and then set s equal to p i and evaluate this and this will give me c i ok, so this is for a non-repeated root.

If I have repeated roots then basically it is not only one indicator this is c ij where we said i refers to the root number and j refers to the repeat for which we are writing the term, so if there is a root i which is repeated thrice so c i one will be s minus P i power 1, c i 2 will be s minus P i power 2, c i 3 will be s minus P i power 3 and so on. So this C ij can be computed using differentiation and the differentiation formula is this so what you do is assume that this root ith root is repeated k i right, so that is a notation we had used in the previous slide where we said p 1 this repeated k 1 times p 2 is repeated k 2 times p 3 is repeated k 3 times and so on, so ith root will be repeated k i times.

So this is you do this differentiation d k the number of times this root is repeated minus j whatever jth question you want divided by D s k i minus j and inside you have the G s times s minus P i power k i, so here we multiplied only by s minus P i because it is a non-repeated root, for repeated roots we have s minus p i power k i times G s ok and then you evaluate this whole thing at s equal to P i that will give you the constant c ij so this way for every i you can compute all the j is.

Now this k i minus j let us say if it turns out to be 2 then basically this will be d square by d square you have to differentiate this G s times this 2 times, if this k i minus j turns out to be three for a particular j then that is d cube by d s cube basically you have to differentiate this three times ok. Now this is a simple formula that you can use which will give you all the c is that we saw in the previous slide and then that will basically allow you to invert any N of s or D of s very easily.

Now this might look a little confusing because of this k minus j what is this d square s square and so on, we look at an example and work this out in the next slide so that you get a very good idea of how this is done and once you understand this understand how this is done now you are empowered to do all kinds of inversion you do not have to look at any other table albeit I should warn you that when the roots are complex you can still use it but the calculations will become very tedious.

So there are tricks that people use there are tricks that are used in books to do quick Laplace inversion, so you can learn those tricks and then do Laplace in motion much faster than this laborious process nonetheless this laborious process will never fail you so if you are ready to do the work this will give you the inversion every time, more importantly another reason actually to really focus on this in this lecture is the following so if you look at this I am showing you procedure to compute the c is but irrespective of that it is either going to be real or a complex constant right, so ultimately I can simply leave this as c 11 c 12 and so on for understanding in terms of stability and other issues with respect to control.

So what this partial fraction idea does for us is it gives us a conceptual framework to understand how this inversion is done then that will tell you how you can really think about stability of this control systems performance and so on, so you will see the power of this notion of partial fractions later as I teach stability very you know simple fashion because once you understand this then the notions of stability becomes almost readily apparent.

So we really do not have to worry about this constants to understand stability however if you want to actually invert this and then do some computations and so on of course you need to calculate this constants and see they how they come about. So the point is that whenever you have complex coefficients you can still do it but let us say if one complex root is repeated twice then correspondingly the complex conjugate roots also has to repeat twice, so for example if p 1 is a complex root and I say it is repeating twice then because of the way the math works the complex conjugate root will also have to repeat twice and basically you will think of this as one root repeated twice this one has another root repeated twice because they are not the same roots p 1 and p star are different.

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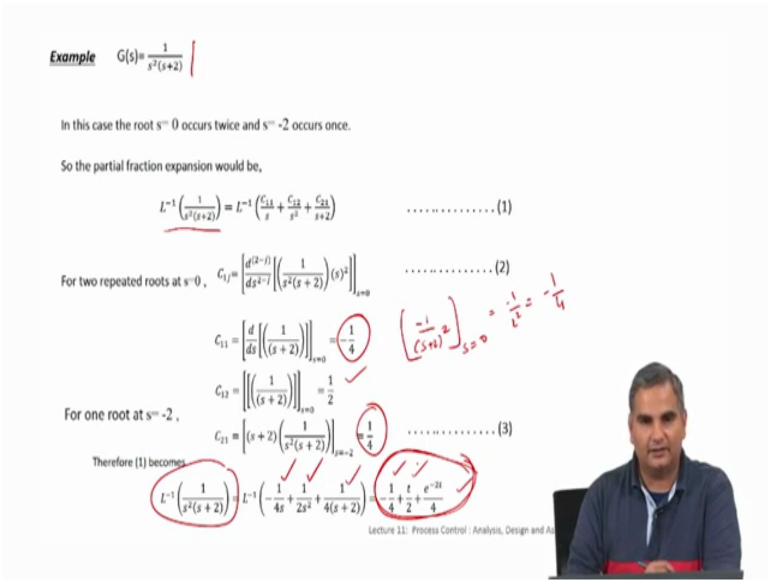

Now let us take a very simple example and then see how this inversion process works, so let us take this example G of s is 1 over s square times s plus 2 if you just notice in this case the root s equal to 0 repeats twice or occurs twice and s equal to minus 2 or occurs only once ok. So now you start with this and I want to invert this then the idea is quite simple we know it is a third order polynomial and there are three roots 0 0 minus 2 and since 0 repeats twice there will be two terms corresponding to the root 0 and then there will be one term corresponding to the root minus 2.

So Laplace inverse 1 over s square times s plus 2 will be Laplace inverse c 11 s, so notice the notation c 12 s square so this is the first root and the first term this is first root second term and this as s square and this is the second root first term s plus 2 and there are no more terms because it is a non-repeating value. Now for the two repeated roots c 1j we said d k i k 1 in this case since s equal to 0 repeat twice that k 1 is 2 so d 2 minus j ds 2 minus J this is G s times s minus P i to the power k I, so this is s minus 0 and it is repeating twice so s square so I am going to say 1 by s square times s plus 2 times s square and differential of this and set s equal to 0 it should give you the 2 roots so this is what we have.

Now when we want c 11 that means j is 1 ok, so 2 minus 1 is 1 2 minus 1 is 1 so this is d by ds and s square and s square gets cancelled 1 over s plus 2 ok. So when I have this so d by ds of 1 by s plus 2 will be minus 1 by s plus 2 whole square so this I have to evaluate at s equal to 0 so when I set s equal to 0 this will be minus 1 by 2 square equals minus 1 by 4 so which is what I have here when I want c 12 I substitute j equal to 2 so this is d 2 minus 2 0 d s 2 minus 2 0 so there is no differentiation it is simply 1 over s plus 2 evaluated at is equal to 0, so I put s equal to 0 this is half for c 21 since it is not repeated at all, all I need to do is I have to multiply G s by s plus 2 and substitute s equal to minus 2 the roots value.

So when I do that s plus 2 and s Plus 2 get cancelled I have 1 over s square and when I put s equal to minus 2 I get 1 by 4 look at how simply we are able to do this computation so once I have this then I can put this c 11 s minus 1 by 4 so this is minus 1 by 4 s c 12 is half and this is s square so half a square and c 21 is 1 over 4 1 over 4 s plus 2 and when we do this Laplace inverse using the other formula that we have this is going to be minus 1 by 4 plus t by 2 plus e power minus 2 t by 4, so it is a very simple Laplace inversion using just that one row of the table that I showed in this slide and in the Laplace table.

So you can see how this is getting inverted into this using this partial fraction idea, now this idea is very general as I said and I and you can use this on any ratio of polynomials when the denominator polynomial order is greater than the numerator polynomial order so which is an assumption we are going to make as far as this course is concerned to do this partial fraction expansion and as I said before irrespective of what N of s D of s is you can always use this approach but you can see if I have let us say complex roots and so on then many of these will become complex numbers however when you put all of those complex numbers in and do the expansion the imaginary part will automatically vanish because the Laplace inverse of this has to be a real function because we are working with real system.

So that will happen the mathematics behind this will make sure or ensure that no imaginary part is left behind, so when you combine all of this imaginary part will go to 0 so if you are doing this expansion actually there is an imaginary term that comes in you can simply ignore this and then do the other terms because when you collect all the imaginary terms together there anyway going to go to 0, so you can get the real part of the answer which will be the answer for your inversion, ok.

So I hope this gives you a good idea of how this Laplace transform inversion is done and we will pick up from here in the next lecture, thank you.