Process Control - Design, Analysis and Assessment
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Analysis of transfer function models Part 1

We will continue with our lecture on process control, in the last two lectures I talked about Laplace transforms how we could use Laplace transforms in solving a differential equation models also call by SSS state space models and I also introduce this notion or idea of transfer function models for the first time in this course we saw that the transfer function model is just a function in the Laplace domain that is used to understand how the input affects outputs and so on.

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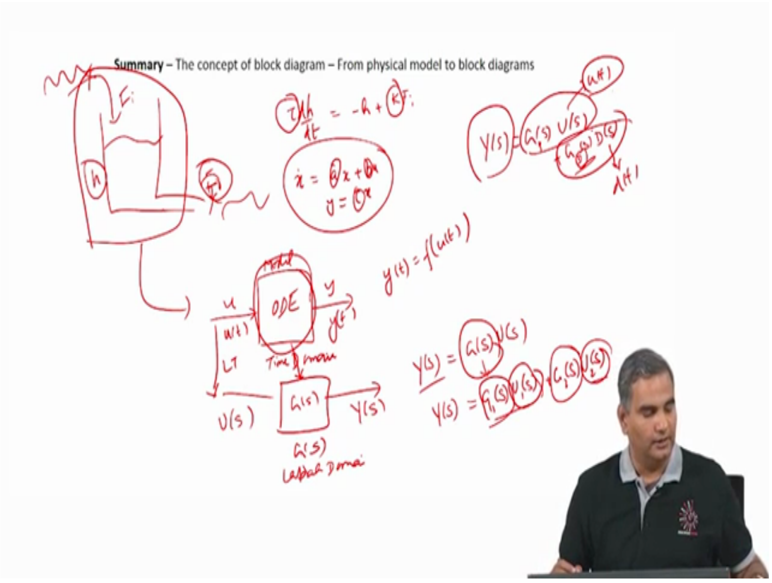

This lecture I am going to start with a summary of what we have learned till now the reason for the summary is the following while we go forward with this control course and we start doing more manipulation with Laplace transforms and differential equation and so on, I do not want all of the material to look too abstract completely mathematical, I just want to take break here and then summarize and say look all of these started from a physical process and all of this modeling and the math should serve as in understanding the physical process and in understanding how to control the physical process.

Now this tank was a very nice example we saw that you have this basic flow in flow out and height as the three variables and then we saw that basically physical model of this should allow us to understand how when things changed either way if not changes or if I changes how height changes and so on. So the idea of the model itself is to be able to explain what is happening here.

So the basically root fundamental conservation equations constitutive relationship and so on for this and then we got to differential equation of this form tau d h by d t equal to minus h plus KF i where F i if I assume is the index flow rate and that is what I here and then you also had a relationship between the outlet flow rate if not and h and so on. Now the thing to remember is that this while is now become a parameter in this equation this embeds all the physical parameters of the system like area and if you are linear acing from a nonlinear equation then the steady state value of height and so on and similarly this k which is now a parameter in this equation also represent a same thing.

So an equation of this form and then a equation that relates f not h together we said if you want to do this in a general form we can call this as the state space model which is x dot equal ax plus b u and y equal to cx again this a, b and c or basic translation of tau h k and so on, so it is basically constant which comes from the first principles modeling that we have done.

So if you really drop picture like this and then say how do i understand changes in flow effecting let us say changes at the outlet flow rate your motion is basically there is some time profile that is happening for this and I want to know what (hap) happens at if not, ok. Now this when we do this mathematical model something like this then in the time domain if I put in a block like this here then I might say there is a u that comes in and the y that goes out and when I write a block like this then basically what I am saying is this u is some time function and this y is some time function I want to know how y changes when u changes, ok.

So though I just write u and y if this is u of t and y of t, the reason why I am putting this in block is because this is a block in the real tank block so here again there is a some input f y and actually some output if not that we abstract out as a model of the process and if this is an ordinary differential equation like what we have been using till now then this model is basically in the time domain, ok.

So the reason why I am doing this as I just want to emphasize again that this time domain model has constant which reflect reality they are not some abstract numbers that comes out of somewhere. Now in Laplace transform what we do is a following, so we say look if I have this Laplace transform then I can convert this u to u of s ok then this goes into this transfer function model which is somehow this ODE converted into the transfer function model and then I can get my y of s here.

Now notice the way this blocks works, so when I want to get y of t from u of t in this block then basically it is actually some integral equation that I have to solve to get y of t as a function of u of t, so it is not a simple u of t is something time y of t something like u of t it is actually some function which you have to compute given u of t will compute y of t this function is something that we showed in the last lecture whereas when you look at this block, this block the interpretation is if I have a transfer function model here G of s this block is in the Laplace domain and if I want to get my output variable y of s it is very straightforward in a simply multiplying G of s times u of s and this G of s again has parameters which come from this ODE and those parameters can come from the physical system.

So G of s is also representing the physical system albeit in Laplace domain whereas this ODE model is representing the physical test system in the time domain, so let us not lose sight of this fact that while we keep talking about transfer function models and will just talk about ns by ds numerator by denominator and so on those are not just mathematical concepts they are actually representation of physical system, so that something that I would like the ex-students watching this to keep in mind.

Now this is single input single output now if you have more inputs because of linearity assumptions things are quite simple for example if there are two inputs which affect the output then there will be two transfer function models we will see this in more detail later but I just want to explain this here so that you understand how these blocks works, so y of s if there are two inputs could be something like G 1 s U 1 s plus G 2 s U 2 s, so this basically says the effect of the inputs U 1 and U 2 are added because underlying model is linear and the effect of U 1 s is moderated by some transfer function G 1 s and the effect of U 2 s is moderated by some transfer function G 2 s.

Now the G 1 s and G 2 s the parameters in those would reflect the actual physical parameters of the system that is being model. Now it might be that there are not two inputs but there is one input and one disturbance in which case we will write y of s is G of s U s, so the standard notation is I will do subscript here G p for process and then I could write G d of s the subscript is to talk about the disturbance that affects the system.

So the total output that I see from the physical system is a combination of the effect of the manipulated variable things that I am changing affect y and also an affect due to the disturbance variables which are changing by themselves and the effect of the disturbance variables are (my) moderated by G d of s and we can always write this is additions because we are assuming that the underlying system is linear and because it is linear and we can use the linear superposition principle we are able to write this equations.

So ultimately when we go and then write an equation like this, this D of s has a physical D of t which is actually a disturbance variable changing in time and this U of s has a physical U of t which is actually some manipulated variable changing with time, so in this tank example if I put an a control del here and then manipulate the inlet flow rate then f i will become this u t the physical variable that is changing and if I do not control anything here based on the pressure difference and the resistance yet if not will change but I love no control over it, so it will become a disturbance variable.

Instead if I were to put in a control valve here and then I manipulate f naught then f naught will become u of t and f y which is uncontrolled and any amount of load can come in I really do not have control over that will become D of t, so this U of s and D of s are actually Laplace transform of real physical variable something that I want you to keep in mind.

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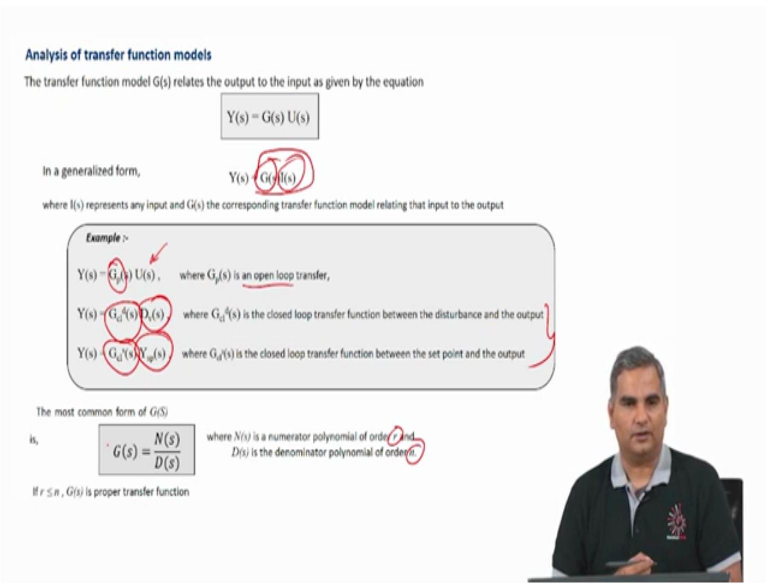

So in general we could write the output as G of s U of s if you had a disturbance you could write it as G d of s D of s or in a generalize form we can write the output as some transfer function multiplied by some input that makes a difference as far as output is concerned, so for example if this is an disturbance variable we will call it D of s which basically says what is effect of the disturbance on the output if I of s is U of s then we are asking about what is effect of the input on the output, if I of s is something else which we will see later, then we have something called the close loop transfer function which we will talk about later but I want you to introduce this notion that the output is basically a product of a transfer function and some input and if there are multiple inputs that affect the output then because of linear superposition we can simply add these effects.

So if you write an equation like this where Y of s is G p of s U of s where we are trying to see the effect of the manipulated variable directly on the output then this transfer function is typically called as the open loop transfer function, ok so this is open loop transfer function. Now if we write the effect of the disturbance variables on Y of s here I have some terminology which will come back to later.

This is some moderating function which is a close looked transfer function an on how the disturbance affects output we are not get talked about close looked transfer function so we will come back to that later but for now you might simply assume this is some transfer function which moderates the effect of the disturbance on the output, there is another input which we are not talked about which is a set point so why set point of s is that is the input then the effect of that on Y we call that as a close looked transfer function, right and these two we will see ones we define what a controller is and how you handle controllers and so on.

Nevertheless the reason why I want introduce these ideas here is that any time you do an analysis of transfer function ultimately all you are doing is something like this G of s times I of s, now this G of swill be derived based on what system and what structure that you are using however if you are simply looking at how you would analyze control systems with transfer function models once you learn how to analyze control system with transfer function models which are generic G of s I of s then you know how to address any type of system, so that is the reason why I introduce this concepts here.

And we also talked about general form for G of s we said G of s is some numerator divided by denominator and the numerator polynomial is of order r let us say and the denominator polynomial is of order n then as long as r is less than equal to n we call G of s is a proper transfer function that means the numerator order is less or equal to this order we call it a proper transfer function and if it is strictly less than the order of the denominator it is a numerator order then we call it as a strictly proper transfer function, so there are two definition which are used in control literature.

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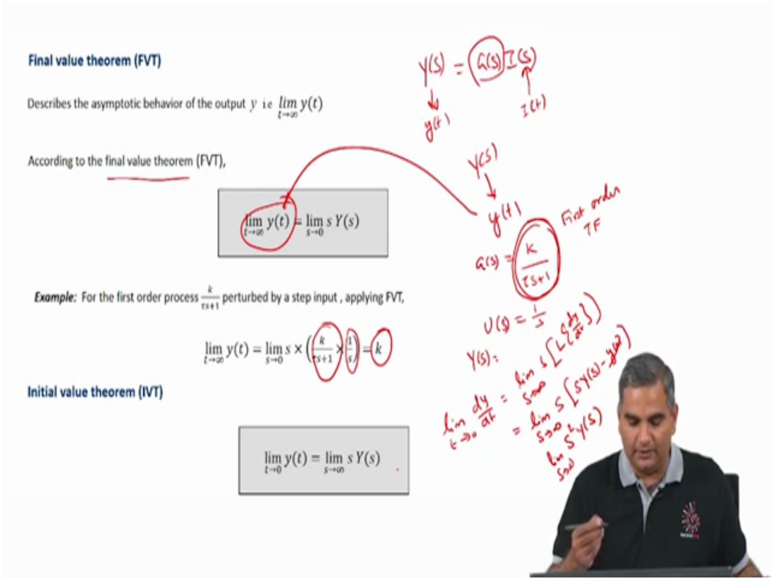

So now it let us say you have this Y of s is some G of s I of s which we have now what we are going to start doing is clearly to get y of t you need to know G of s which is something that will come out of the modeling and you also need to know what I of t is, so whatever is the variation and that happens with this input then you can convert this to I of s and then you can get Y of s is G of s times I of s and then once you get a Y of s then inverse Laplace transform will give you y of t, so that is a general flow that we are going to see as far as control is concerned.

There are certain interesting things that we would like to compute about y of t, so the first idea is what is called a final value theorem where you are saying, ok if Y of s is some G of s times I of s I want to know what happens as t tense to infinity that basically means if I let the system run for a long time will my output value settle, if it will settle at what value it will settle and so on.

So mathematically this question you post as limit t tending to infinity y of t, now this is already proved so what this theorem says, say if you want to know limit t tending to infinity y of t you can get this value by equivalently looking at limit s tending to 0 s Y of s, right what is importance of this theorem typically if you want to get this you will have to first convert Y of s to y of t and then put these into this and then say limit t tends to infinity but this is simpler approach if you want to get the final value you can directly use the variable in

Laplace domain itself and then get the limit, so that is the reason why this theorem is interesting.

So for example if we have let the G of s and we will talk about this a quite a bit later is of the form k tau s plus 1, so this is what is called as the first order transfer function, ok and then let us say I give an input which is a step so G of s is this U of s for a step input we have already seen that the Laplace transform is 1 over s now I want to know once a step my input what will be the final value of my output.

So the steps to follow is first we will get Y of s which is G of s times U of s and then we will use the final value theorem which says limit t tends to infinity y of t is limit s tending to 0 s times Y of s, Y of s is G of s times U of s, now this s and s will get cancel and when you substitute s equal to 0 here you will get this value, k. So if my transfer function is of this form then the final value that Y will take whenever I part of the system is a step input is k which is what is given here equivalently you can ask the flip question if I want to know what will be the initial value of y of t and this is not really useful for Y of t itself but we are going to use this later where we can find the initial value of d y by d t.

So basically that will give me what is the initial slope that y is going to take or initial rate at which y is going to change, so if I want limit t tends to infinity y of t then I can calculate that as limit s tends to infinity s Y of s, right. Now you can see how there are kind of symmetric t tends to infinity s tends to 0 t tends to 0 s tends to infinity and s Y of s is the same, ok. Now as I said before this might not be useful for Y of t itself but we could ask questions like limit t tending to 0 what is d y by d t.

So this will be limit s tending to infinity s we will say Laplace of d y by d t, ok we already know that Laplace of d y by d t is s Y of s minus y of 0 and if you are assume that y of 0 then this would be simply limit s tends to infinity s square Y of s, ok. So you could apply in this theorem for not only y of t you can apply for d y by d t, d square by d t or any other function that you choose.

So when we start analyzing these transfer functions and then we start using these transfer functions these are important concepts that will come in handy as we go through in control, these two particular theorems are going to be quite important because I am going to use this later when we describe offset and so on and this we will look at in much more detail when we talk about zeroes and poles later in terms of performance limiting factors in control, ok.

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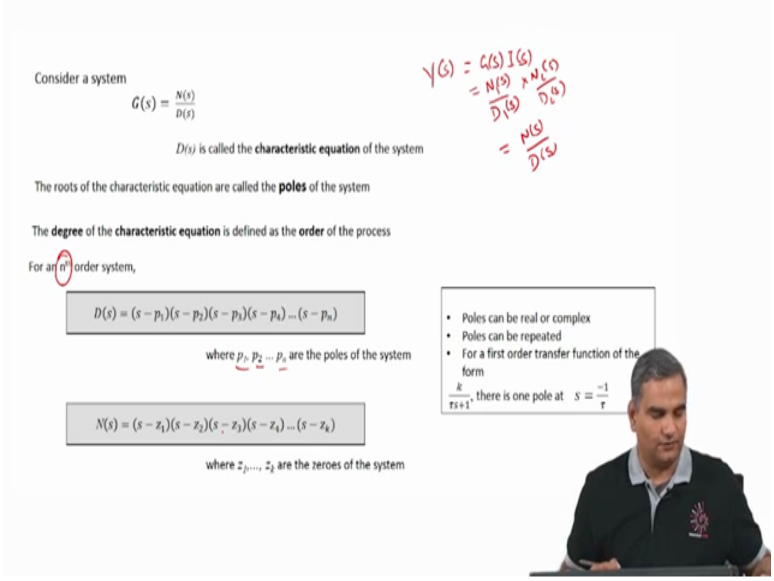

So that basically tells you that there are theorems which you can use to compute some basic numbers without actually going through the whole inversion process however in general what we would like to do is we would like to get Y of s G of s I of s and then from this we want to get a value for y of t. Now typically each one of these let us say G of s in a standard form will be some numerator polynomial by denominator polynomial and I of s will also be some numerator polynomial by denominator polynomial.

So ultimately whether it is G of s I of s or Y of s we have to really look at a numerator polynomial by denominator polynomial because I could say this is again N of s by D of s where N of s is N 1 s times into s and D of s is D 1 s times D 2 s, so ultimately it does not matter what you are trying to invert whether you are trying to invert G of s you are trying to invert I of s you are trying to invert Y of s you are going to try and find the inverse Laplace transform of ratio of polynomials numerator polynomial by denominator polynomial.

Now whenever we talk about polynomials we know that we can write them in root resolve form if we keep the first coefficient as unity then we can actually write the expansion in this form, so we talked about an order of a system and then we said the degree of the denominator polynomial is order of the system. So if it is Nth order system then the denominator polynomial is going to be Nth order and we know from a algebra that Nth order polynomial have N groups, so we have D of s written as s minus p 1, s minus p 2, s minus p n this is something that we had seen before and when I write an equation like this for denominator this p 1, p 2 all the way up to p n or call the poles of the transfer function or the poles of the system itself, ok.

So the roots of the denominator polynomial of the poles, similarly the numerator polynomial can also be written in a root resolve form which is s minus z 1, s minus z 2 all the way up to s minus z k if k is the order of the polynomial numerator polynomial, in this case the roots of the z 1, z 2, z 3 all the way up to z k or called as the zeros of the system or the zeros of the transfer function.

So if you set the numerator polynomial to zero and then compute the roots then those roots are call the zeros of the transfer function and if you set the denominator polynomial to zero and calculate the roots those roots are called as the poles of the transfer function.

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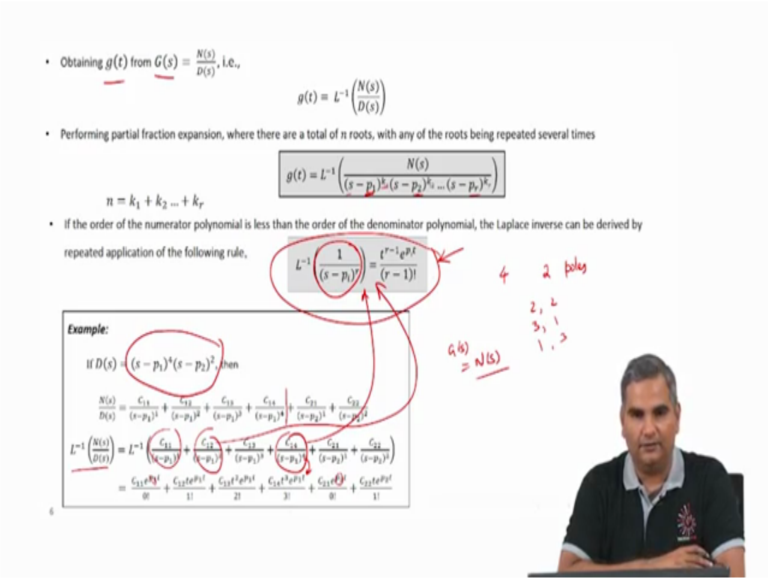

So as I mentioned before generically whether it is g of t y of t I of t we have to do inversion and we have to do (info) inversion of N s by G D s, right, so let me assume that I am trying to get a g of t from G of s, right this could be y of t from t of s I of t from I of s does not matter, so this is going to be as a numerator divided by denominator. So the Laplace inverse of this I can write as a N of s divided by this, now I made a little more sophisticated because I am going to talk about a general solution to this, so this looks little busy with mathematics but it is not that difficult it is a quite simple straightforward algebra, so you have to just follow the notation little carefully.

One of the problems with you know when we talked about roots of the polynomial and so on in the previous lecture to simplify matters I had assumed all the roots are distinct or different but supposing you have let us say 5^{th} order polynomial it could be that all the roots are the same they are repeated 5 times, so for example if you take a polynomial of the form X minus

2 to the power 5, if you set it to equal to 0 x equal to twos only root but that is repeated 5 times, ok.

So what we then say is to generalize this every root in the denominator polynomial which is a pole let us say p 1 to p r each of this poles can be repeated many times with they are not repeated at all then I will have n poles but let us say if one pole repeats and nothing else repeats I will have n minus 1 poles and so on. So we generalize this by saying let us assume a pole p 1 is repeated k 1 times pole p 2 is repeated k 2 times and so on if that is the case then the denominator polynomial can be written as s minus p 1 power k 1 times s minus p 2 power k 2 all the way up to s minus p r power k r.

The important thing to notice here is there cannot be more than n roots, so k 1 plus k 2 all the way up to k r has to be n, so for example if I have a 4th order polynomial and I have only 2 poles, ok either each pole has to be repeated twice or one pole is repeated thrice and the other one is single or this is repeated not repeated the other pole is repeated thrice and so on, so this are the possibilities if you have two poles but notice always that two poles there are 2 numbers 2 plus 2 is 4, 3 plus 1 is 4, 1 plus 3 is 4 so there are always these number of repeats have add up to n, ok.

So now if we want to invert this what we are going to do is a following we are going to repeatedly apply this rule you can go and look up the table for this rule and understand this well, so what we are going to say is in general if we take Laplace inverse of 1 by s minus p i to the power r the inverse is t to the power r minus 1 e power p i t by r minus 1 factorial, ok so this is already computed we are going to keep using this again and again there is nothing new here once you understand what this is then it is going to straightforward application of this to find the inverse Laplace transform.

So how do we use this I am going to give you very simple example let us assume that I have let us say some G of s which is a numerator polynomial N of s divided by let us say 6th order denominator polynomial and the 6th order polynomial has 2 roots p 1 and p 2, p 1 is repeated 4 times p 2 is repeated 2 times, so 4 plus 2 6 so that is the reason why I call it 6th order. So when I write this like this now when I do a partial fraction expansion the simple rule is you will get as many terms for a root or a pole as it is repeats.

So for example p 1 is repeated 4 times so I am going to get 4 terms for p 1, so the first term will have in it is denominator s minus p 1, the second term will have a s minus p 1 square, the third term will have a s minus p 1 cube and the fourth term will have a s minus p 1 to the power 4 and p 2 is repeated twice, so the first term will have a s minus p 2 and the second term will have a s minus p 2 whole square, so it is very simple pattern that keeps following, so if it is repeated five times I will have one more and so on and whenever I have something like this then as I told you before in one of my previous lectures that the information from the numerator is all hidden in this constant c 11, c 12, c 13, c 14, c 21, c 22.

So the notation we are followed here is p1 is repeated four times so four terms are there, there have to be four constants so c 11 is the first constant where the power is 1, c 12 is a second constant where the power is 2 and so on and similarly this is for the second pole c 21 is for second pole with power 1 and c 22 is for the second pole with power 2 notice that I have not said how we are going to compute this c 11, c 12, c 13, c 14, c 21, c 22 yet however I can guarantee that if I have this numerator by denominator where the denominator is of this form I can always write this in this partial fraction form and this constants will exists such that when I reduce all of this I will get back my numerator N of s by D of s.

So let us follow the logic that we have been using until now so I said we can write any analysis of transfer function as Y of s is G of s times I of s, G of s is a numerator by denominator polynomial I of s is a numerator by denominator polynomial and Y of s is also numerator by denominator polynomial, so as long as we learned the technique so how to invert numerator by denominator polynomial find the Laplace inverse of this numerator by denominator polynomial in to time domain then we can solve all kinds of problems we do not have to really worry about how we will do the Laplace inversion again, the only question will be how do I get the transfer function, how do I get the I of s and so on.

So what I am trying to show here is a generic solution for N s by D s and I said that minute we have a polynomial we have n roots but it is not that all roots need to be distinct, so we have multiple roots repeating and we have come to state where the inversion of the N of s or D of s is simply finding the Laplace inverse of this again this to reiterate I have not shown you how to compute the c 11, c 12, c 13 and so on which we will see in the next lecture nonetheless I can guarantee that if the denominator is of this form and the numerator order is less than the denominator order I can always get a partial fraction of this form with a correct c 11, c 12 and so on, ok.

So now when I do the Laplace inverse of this I have a Laplace inverse of the whole sum and again appealing to the notation of linearity Laplace inverse of the sum will be Laplace inverse of an individual term and add them all are so this will be Laplace inverse of the first term plus Laplace inverse of the second term and so on and if you notice carefully and interesting thing happens (mi) right now if you notice each of this terms, ok to find a Laplace inverse of all of these terms you have to only use this rule here.

So if I take the first term so this is Laplace inverse of 1 by s minus p i power r, r is 1, ok so I put r is 1 here t power 1 minus 1 is 0 e power p 1 t divided by 0 factorial which is 1, so the Laplace inverse of this is simply c 11 e power p 1 t by 0 factorial. Let me take just another term where is a little away from here and you can work out other terms so let us say I take this term here and I go here I see that r is 4 so this should be t power 4 minus 1 t cube e power p 1 t divided by 4 minus 1 3 factorial.

So if you see the Laplace inverse of this it is c 14 which is a constant which is had to be determine I have to tell you how to determine that nonetheless if you had determine that already this is going to be t cube e power p 1 t divided by 3 factorial and similarly you will get everyone of this terms because everyone of this terms looks like this here and it is not anything specific to this example that this terms look like this anytime you have the transfer function and a numerator by denominator form I can always guarantee that I can put this in this partial fraction form and the minute I put it in the (prob) partial fraction form it is very trivial to write the inverse Laplace transform and the inverse Laplace transform keeps using this again and again it repeatedly apply this to get the solution.

So ultimately when I want to get the inverse Laplace of N s or D s which could be Y s G s or I s then I get this here now notice that p 1 is the roots so we already know what the value of the root is based on the denominator polynomial p 2 is also root we already know what the value is based on the denominator polynomial and t is again a time function here t, t square, t cube and so on.

The only thing that I am not at shown you is how to get this constants out of for this partial fraction to work which we will see in the next lecture but nonetheless the key point that I want to make here is irrespective of whatever is your polynomial this technique will always work, ok. So the question of inverting this transfer functions becomes understanding only one formula right here right now, so that we can do this quite easily.

So I will stop here as far as this lecture is concerned and then we will go on to working how to get this constants and how to generalize this further in future lectures, thank you.