Rheology of Complex Materials
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Strain measure and convected rates: conceptual framework
Lecture – 46
Strain and Convected Rate- 3

The last lecture on Strain Measures and the Convected Rates; we had started looking at the conceptual framework and what we had done was looked at again how a position of material particle changes and how displacement and displacement gradients and deformation gradients are defined.

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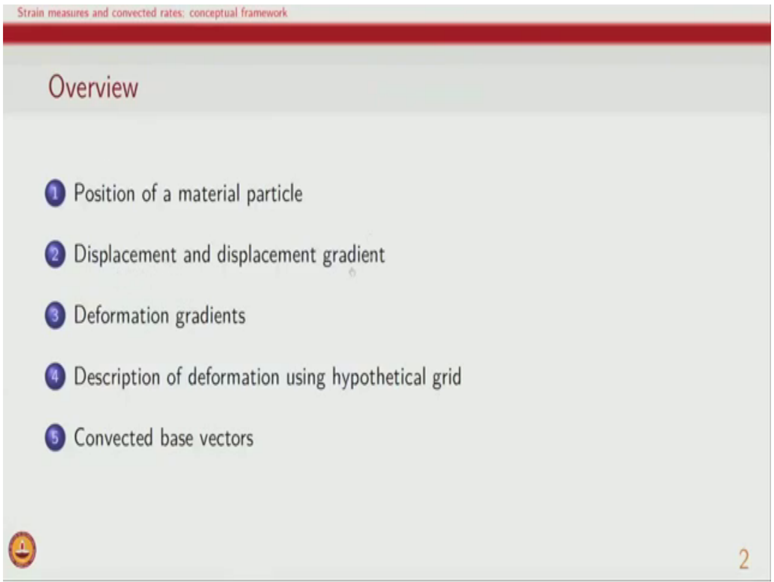

And then we looked at what happens to this deformation if we embed a hypothetical grid in the material, and we looked at examples of both shear as well as extension. And so, today we will continue the process and look at what is how these this hypothetical grid can be used to describe a convected set of base vectors, and how then rate quantities which are evaluated in this convected frame how do they look. So, let us just quickly look at where we had stopped, if you recall that we had talked about a deformation.

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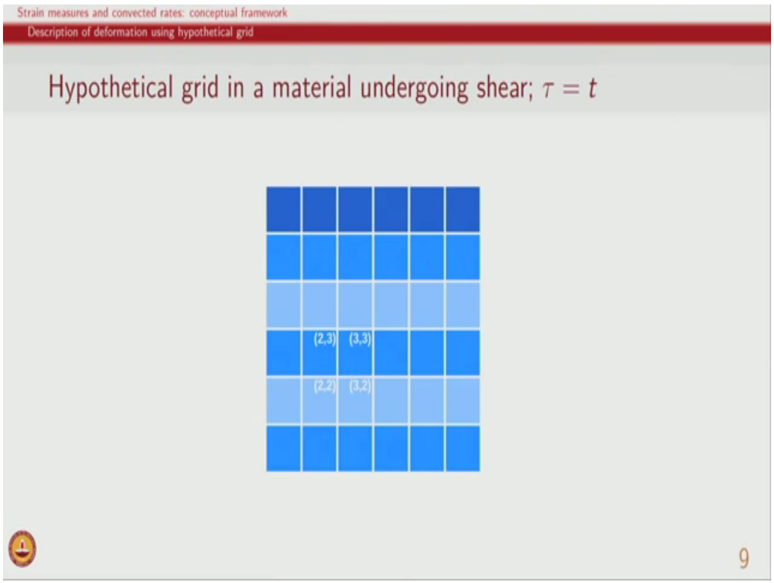

For example a simple shear deformation that a grid which is embedded in the material, at some time in the past may look like this.

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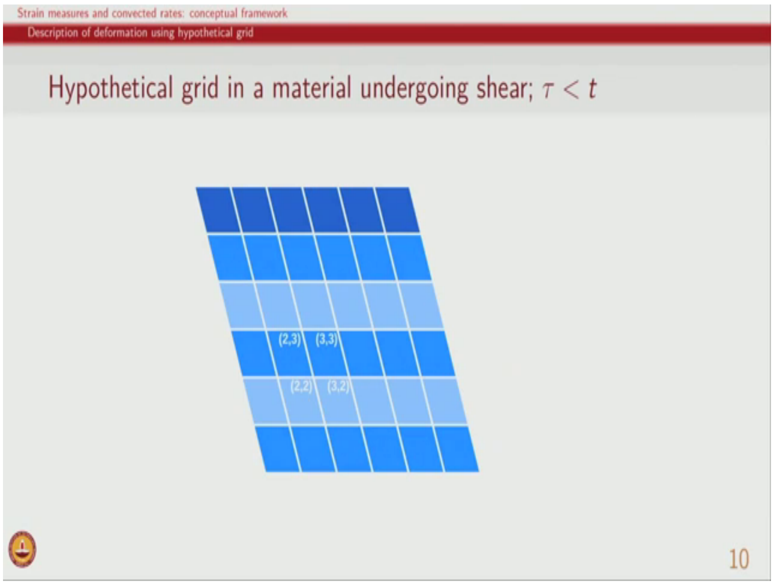

And sometime in the future may look like this.

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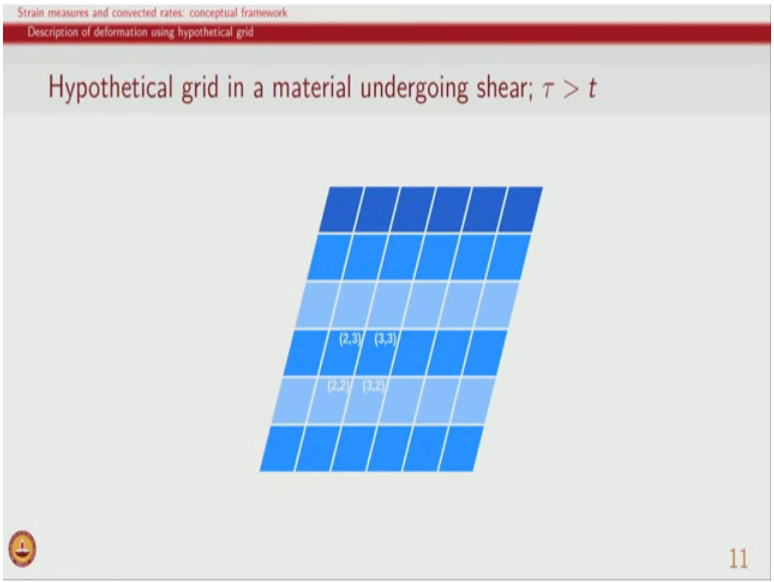

Because the top part of the fluid is moving much faster towards the right and therefore, each and every point moves towards the right as the time goes on.

And since this is sometime in the future we have the overall set of points moving towards the right. Similarly we had also looked at the same grid, but if there was an extension.

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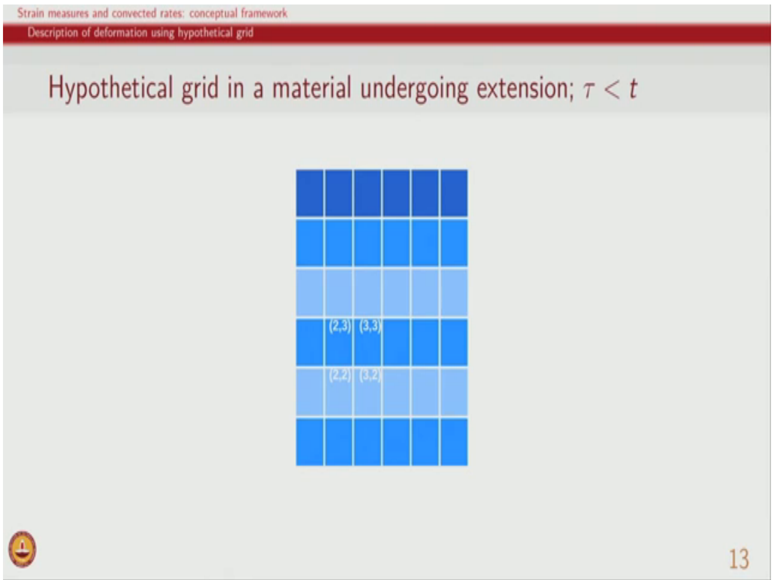

And so, sometime in the past since the overall material is coming down and then moving rightward and leftward.

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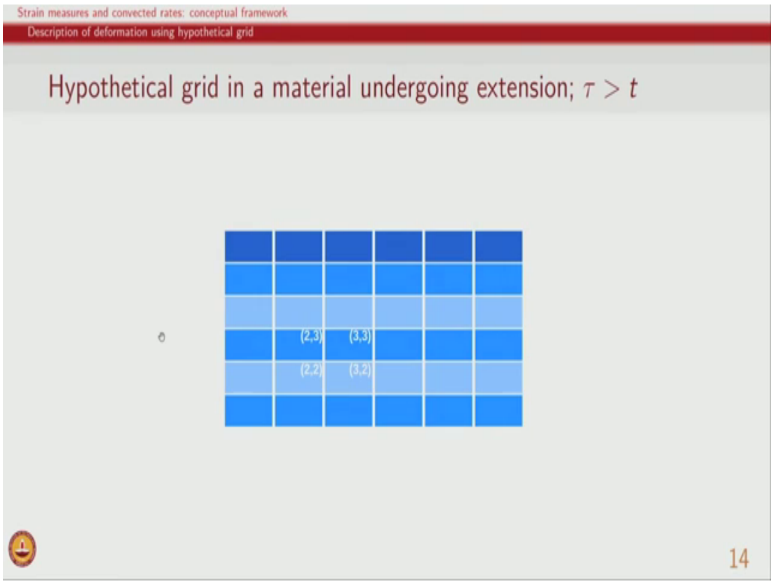

What we have is the deformation where the overall grid itself gets stretched in one direction and then it gets contracted in the other direction.

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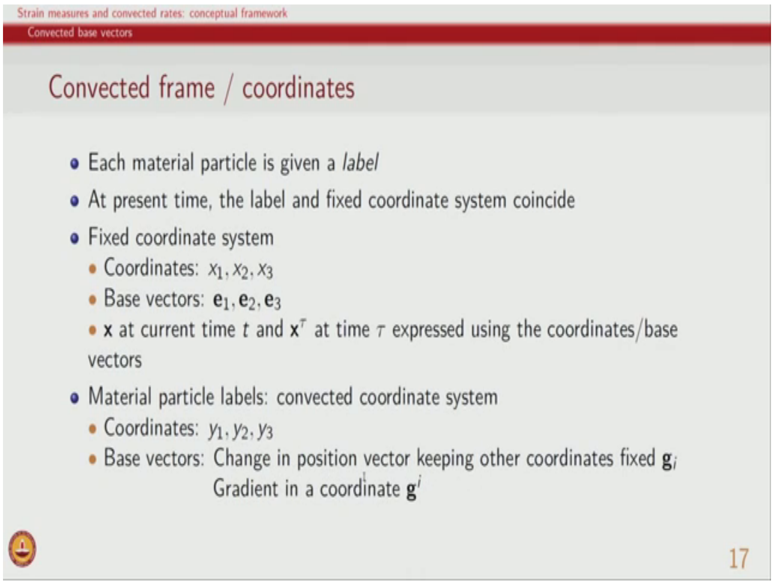

So, we had in the previous class we had looked at that we can define a convected frame and therefore, a convected coordinate system, where each and every material particle is given a label and at the present time the label and the fixed coordinate system coincide. And of course, the fixed coordinate system is let us say a rectangular coordinate system which would be just xyz or in a generic term it will be indicated as x 1, x 2, x 3. And therefore, the base vectors are e 1, e 2, e 3 of course, this is an orthogonal coordinate system, which we usually use when we use fixed coordinate and therefore, these e 1 in e 2 will be perpendicular to each other and e 2 and e 3 will be perpendicular to each other and so on. And the position of the material particle is described as x at present time and x tau at any other time tau which could be in the past or in future. And the label itself of the material particle which coincides with this x will be indicated as y 1, y 2, y 3.

So, in a 3 dimensional space we need 3 coordinates to describe the position and of course, we also need 3 base vectors and the 3 base vectors can be defined based on 2 alternate definitions. We can look at the change in position vector or how a position of material point changes when we keep the other coordinates fixed. So, for example, if we are looking at y 1 the base vector in the y 1 direction, then we are looking at how to keep how which is the direction which in keeps y 1 fixed.

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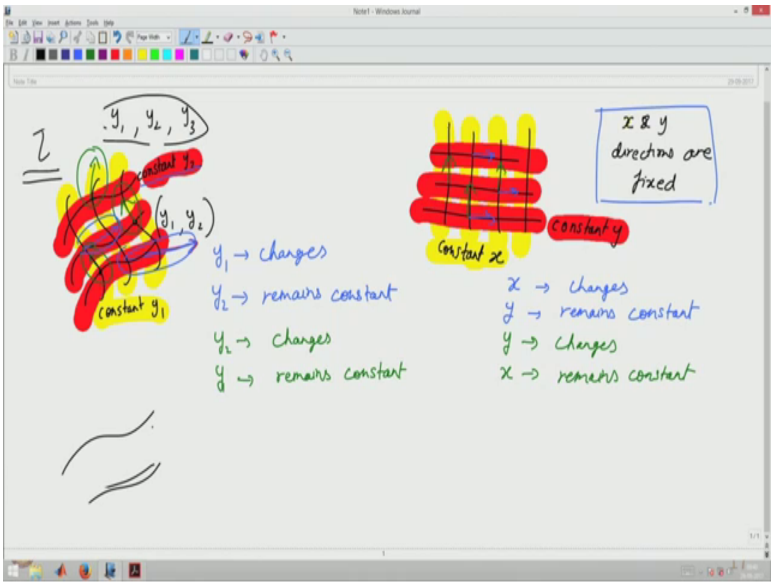

So, just to understand this further, we have a set of generalized coordinates and the grid which is representing because there is deformation happening, the overall set of points will be moving and let us say in 2 dimensional representation this in the y 1 y 2 point that we are talking about. So, the sets of questions that we are asking are 2 sets of questions. One what happens to change in position of a material particle when other coordinate remains fixed. So, in this case in this is 1 y 1 same y 1 constant y 1 and these are constant y 2 surfaces; so just to highlight the fact. So, we have constant y 1 surfaces going this way and so, all along these curves y 1 remains constant and then the other is where the constant y 2 is there

So, this is of course, similar to what we generally do when we talk about a rectangular coordinate system. In a rectangular coordinate system also we have again a grid and again this is where constant y is there and this is the surface the axis along which there is constant x. So, if we were to indicate these as constant y surfaces then these are the constant y surfaces and similarly if we were indicated to indicate constant x, then these are the constant x surfaces.

So, only thing is these quantities remain fixed in case of a rectangular coordinate system. So, the x and y direction are fixed x and y directions are fixed. But now we are looking at the y 1, y 2, y 3 which keep on moving along with the material and since deformation is an arbitrary therefore, the positions of y 1, y 2, y 3 can also change and in 2 dimensions therefore, we have indicated it using any set of curves along which either y 2 or y 1 remains constant. So, the 2 definitions that we are asking one of them we are saying that the other coordinates are fixed. So, therefore, we are looking at a direction where y 2 is not changing.

So, for example, in the case that we have drawn, we can clearly see that this is the direction this is the direction along which there is no change in y 2 because these are constant y 2 surfaces. Similarly if I choose another point this is the point along which. So, these are along the surface where y 1 changes and y 2 remains constant. Similarly I can also look at an alternate set of vectors for example, here or here or in this case here these are all the directions in which y 2 changes, but y 1 remains constant. Very similarly we could also write here for example, this is the direction along which x changes y remains constant and this is the direction along which y changes, but x remains constant.

So, we can see in the this case of course, the y direction is fixed and it is the same in all cases and similarly the x direction is fixed and its same for all the points and more importantly it also remains fixed with respect to time. In this case this is 1 snapshot at particular time instant tau and we can see that the vectors are the base vectors are not in the same direction.

So, we have this in one direction, this is another direction and this is another direction. Similarly the other base vector also which is pointing out this is in one direction this is in another direction. So, therefore, not only are these base vectors not in the same direction, they also will keep on changing when the time itself changes. So, therefore, the change in position vector keeping other coordinate fixed will indicate one set of base vectors. Now we can look at the other possibility also, we can draw the same grid again which is indicating basically the material which is deforming.

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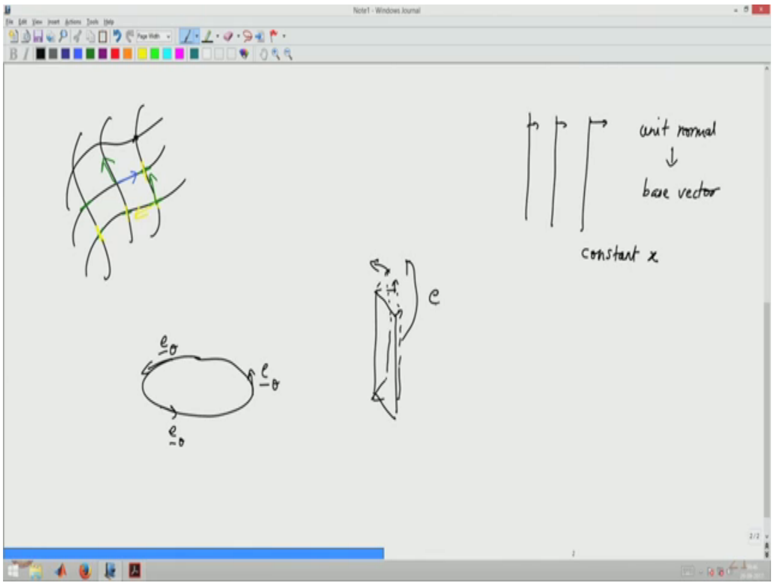

So, if we have the material which is deforming and all these material points are moving. So, in this case we can also look at an alternate set of base vectors and those are defined as what is the gradient in coordinate and that will define another set of base vectors. So, for example, we saw already that y 2 and y 1 are indicated along the due to this grid points and so, these are all constant y 2 surfaces and constant y 1 surfaces. So, which is the direction in which constant y 2 surfaces go? So, if you look at here this is the direction in which constant y 2 surfaces are there generally, and this is the direction in which constant y 1 surfaces are there.

So, therefore, rather than looking at the direction of the way we had done in the previous slide where we had looked at the direction based on the coordinates changing, we can look at what is the orientation of the plane itself? So, if you look at this particular point the orientation of the plane are all changing.

So, therefore, this gives us an idea about what is the overall base vector similarly at the same point the bae the orientation of the other plane also can be looked at and again these need not be orthogonal to each other. So, this way we are going to define alternate set of. So, this will be one base vector this will be another base vector. So, we will look at this process in little bit more detail first for some simplistic situations.

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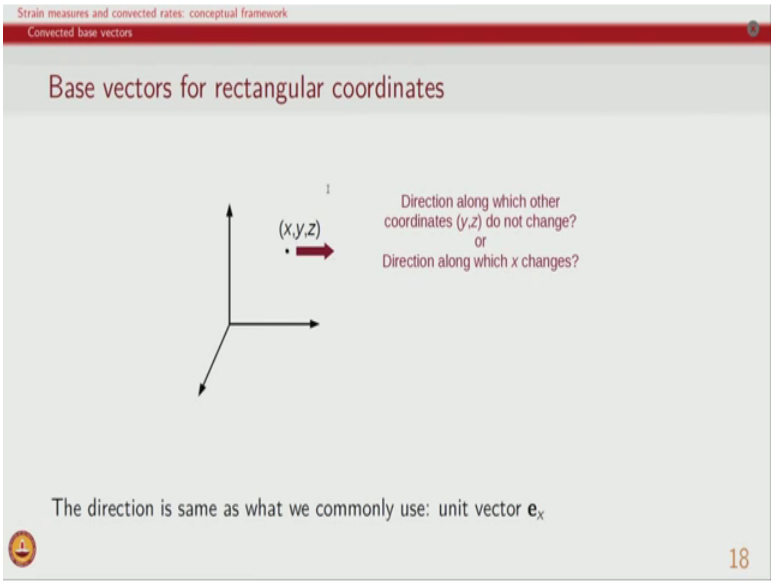

So, now let us look at for example, what are the base vectors for a rectangular coordinate system. So, in rectangular coordinate system we denote a point using xyz and the question that we can ask is what is the direction along which other coordinates do not change or direction along which exchanges and clearly that is this direction because we generally indicate x y and z. So, therefore, xy the direction along which x changes is this direction given by the red arrow. And so, of course, this direction is same as what we commonly use and we call it the unit vector ex. Now we could ask the next question the direction of surfaces on which coordinate x is constant.

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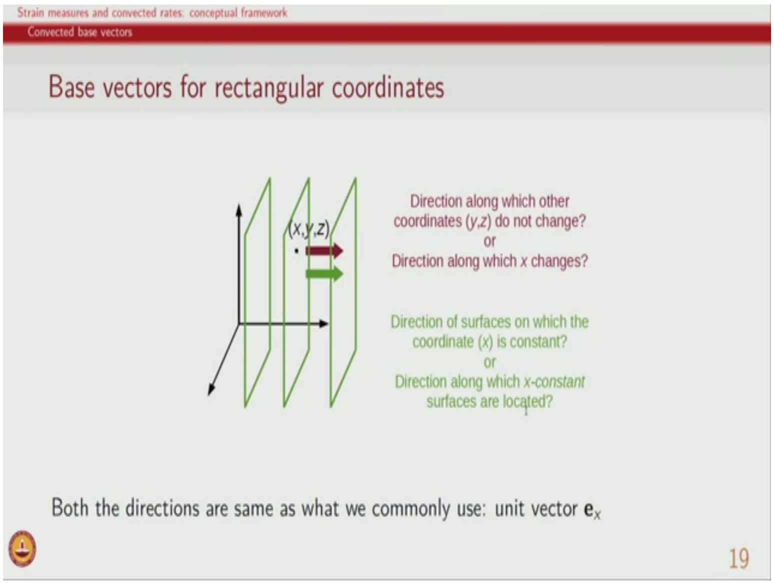

So, these planes that are drawn in green are in fact, indicating. So, all over this plane value of x is 1 value. Similarly on this value of x is another value; however, all along this plane value of x remains constant and so, these 3 are 3 different planes with 3 different values of x being constant. So, you can see that to see the x constant surfaces one has to again go in this direction or each of these plane can be specifying using a unit normal, and the unit normal is given by the green arrow.

So, what we are saying is in case of rectangular coordinate system given that these are all the planes which specify constant x and. So, unit vectors in the unit normal in all these cases will specify the base vector. So, therefore, in all the cases also as based on what green arrow is indicating, the direction of the base vector is again going to be the same.

So, therefore, both the questions that direction along which x changes or a direction along which x constant surfaces are located are same and therefore, we use ex. And so, generally when we learn of cylindrical coordinates or rectangular coordinates therefore, we do not really distinguish between the 2 different base vectors which are possible. Since both of them are same we use them without really taking recourse to mentioning with set of base vectors we are using. We can continue and understand the same variation also for a cylindrical coordinate system.

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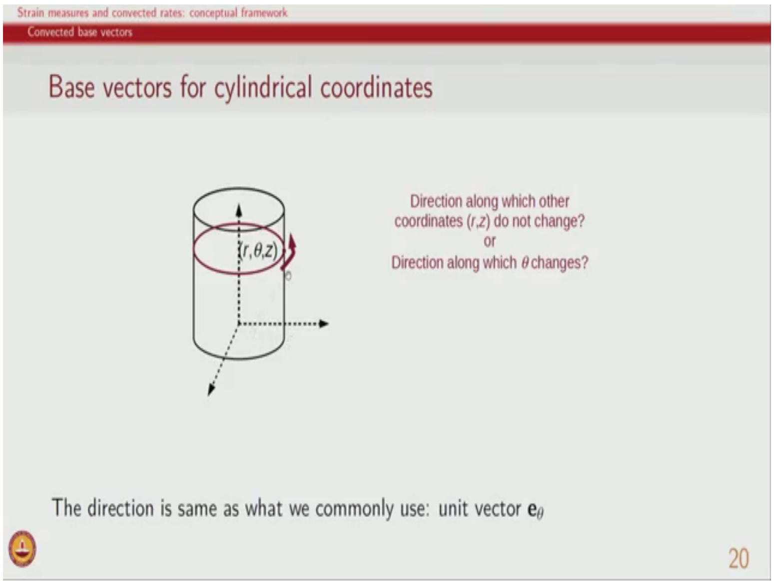

So, in a cylindrical coordinate we have a point which is r theta z, r is meant measured from the origin of the circular cross section and then theta and z z is of course, distance in this direction and theta is the orientation in the azimuthal direction and so, the question that we could ask for example, is the direction along which other coordinates let us say r and z do not change or alternately direction along which theta changes and of course, that is clearly this.

So, this is the direction in which theta changes and of course, we know that this unit vector e theta gives indicates what is the direction of the where theta changes and so, at any point e theta indicates. So, for example, if we have at any point e theta is what is used to indicate the direction in the theta coordinate and so, therefore, this is the direction along which theta changes.

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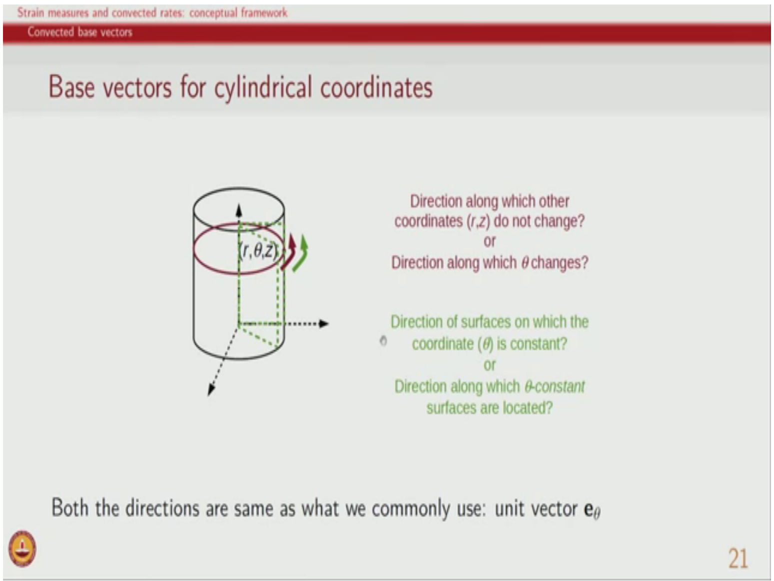

So, now we could ask the next question which is related to surfaces and here we could again ask the direction along which theta constant surfaces are located. So, this is one theta constant surface and then this is another theta constant surface and you can see that if we go in this direction we will encounter different theta constant surfaces or the normal vector to these surfaces is again going to be in the direction of e theta.

So, in the second case we are looking at constant surfaces. Therefore, we are looking at one theta surface then the other theta surface and then the other theta surface. So, basically again in each and every case the unit normal is again e theta. Therefore, both the questions which can define base vectors one related to the change of the coordinate itself or the other one related to the surfaces constant surfaces both of them give us the same unit vector.

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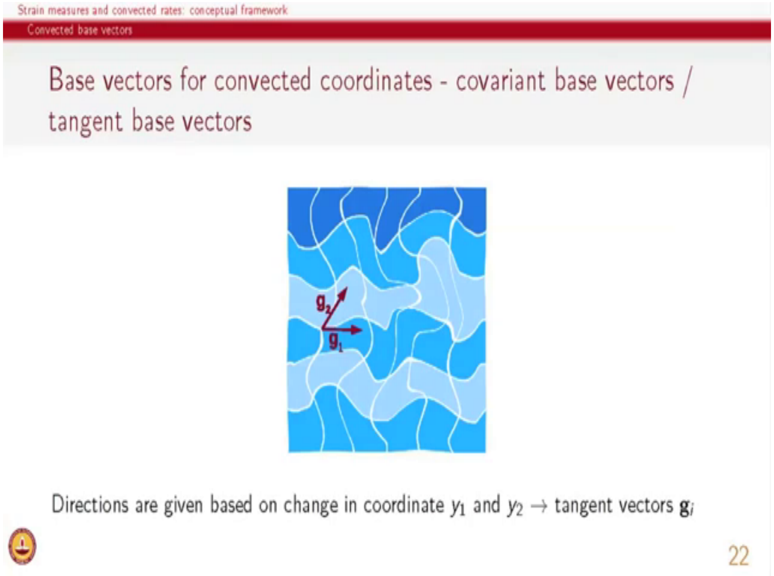

So, therefore, in case of rectangular and cylindrical coordinates, we have the base vectors which remain same and they are fixed. Other thing that we can notice is that the coordinate the base vectors in this case are orthogonal. So, they both rectangular and cylindrical are examples of a an orthogonal coordinate system. Now we can look at the same example for an arbitrary deformation in which case we have embedded the coordinates in the material and. So, as we had seen earlier that if material chord material points are moving arbitrarily then what happens to the material coordinate. And so, in this case for example, these lines are along which these curves are along which let us the y 1 coordinate is changing and these are the curves along which y 2 is changing.

Therefore, we can see at any given point this g 1 indicates the direction along which y 1 changes and so, this will keep on changing at each and every position and of course, this is at once time snaps not tau, and this will keep on also modifying when you go from 1 snapshot to the other snapshot. Similarly these are the direction along which y 2 changes and at this particular point this g 2 base vector is pointing out the direction in which y 2 will change. So, therefore, directions are given based on change in coordinates y 1 and y 2 and these are also tangent locally.

Therefore, they are also called tangent vectors. So, these base vectors where the coordinates are changing with respect to position are either called covariant base vectors or tangent base vectors. And you can see here that at any point if eh we go further the g 1 and g 2 will be different. So, in this case it will be in this direction and in this direction. So, the different if we go from different at this point for example, this is one direction and in this case if we go this will be another direction.

So, this g 1 and g 2 will keep on varying from position to position and given that this is only 1 instant of time, they will also vary from this instant of time to another instant of time. We could also ask the next question which is related to what happens to the constant surfaces. So, these are again the this is y 1 constant surface this is y 2 constant surface and so, generally what you can see is. So, this is y 2 constant surface and this is y 1 constant surface and so, you can see that the y 1 constant surfaces are this, this and this and of course, they keep on varying.

So, at any particular point the direction of this surface is given by the normal and therefore, these are called the reciprocal vectors. So, in this case also we can define 2 reciprocal vectors g 1 and g 2; g 2 indicates the direction along which y 2 constant surfaces which are this, this and this which those are the direction and locally of course, here for example, this will be the direction in which g 2 will be while this will be the direction in which g 1 is. So, we can keep on drawing depending on different deformation that is there in the material.

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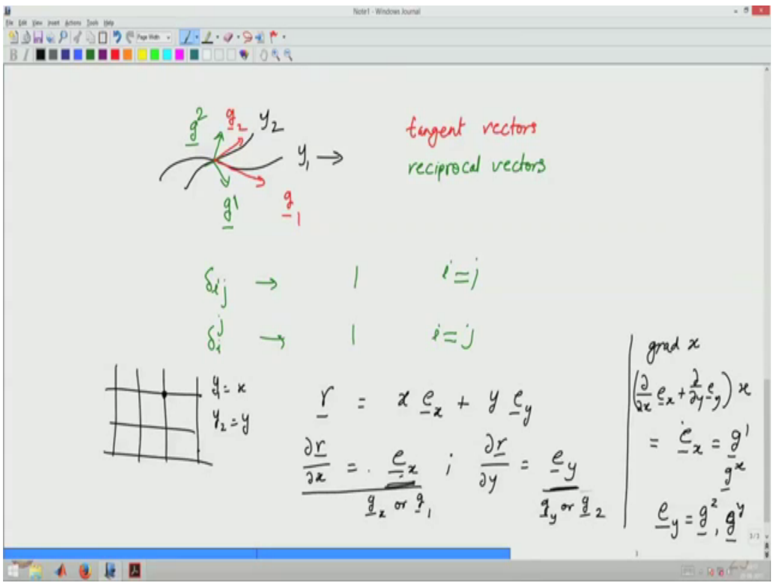

So, for example, if we have y 1 constant surface y 2 constant let us say y 1 changing in this direction and then we have another surface which indicates y 2. So, just to understand again we can draw the base vectors. So, one this is the direction in which y 1 changes and this is the direction in which y 2 changes therefore, this is g 1 and g 2. Now we also are defining based on y 1 constant surface and y 2 constant surface.

So, you can see in this case that this is the cur this is the curve along which y 2 changes and at this particular point this is the perpendicular which will define g 2 and similarly this is the perpendicular which will define g 1. So, one of these vectors are tangent vectors and the other set of vectors are reciprocal vectors. So, therefore, if we look at the both set of vectors on the same graph one thing we can notice is given that 1 set of vectors are tangent.

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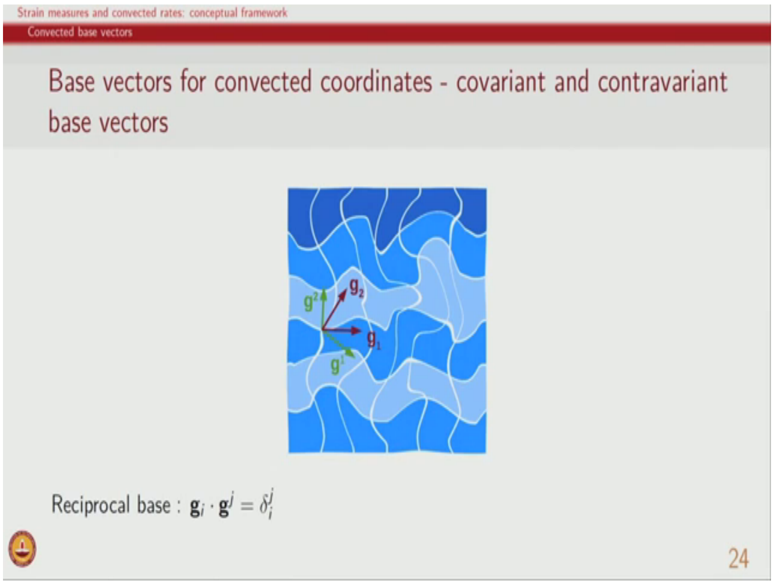

So, therefore, this g 1 is tangent and of course, the plane along which y 2 changes is in the perpendicular direction therefore, g 1 and g 2 are perpendicular to each other similarly g 2 and g 1 are perpendicular to each other. So, the reciprocal base is called so, also because when you do a dot product of gi dot gj it is same as the chronicle delta. Given that we are using g subscript i and g superscript j with chronicle delta is written as delta i j with j as superscript, but it is similar to the, so delta ij that we had written earlier where one if i is equal to j.

And so, similarly delta ij is also same that its one when i is equal to j i is equal to j and similarly here also i is equal to j and of course, its 0 otherwise. So, having looked at qualitatively the base vectors now we can move on and try to define these more formally.

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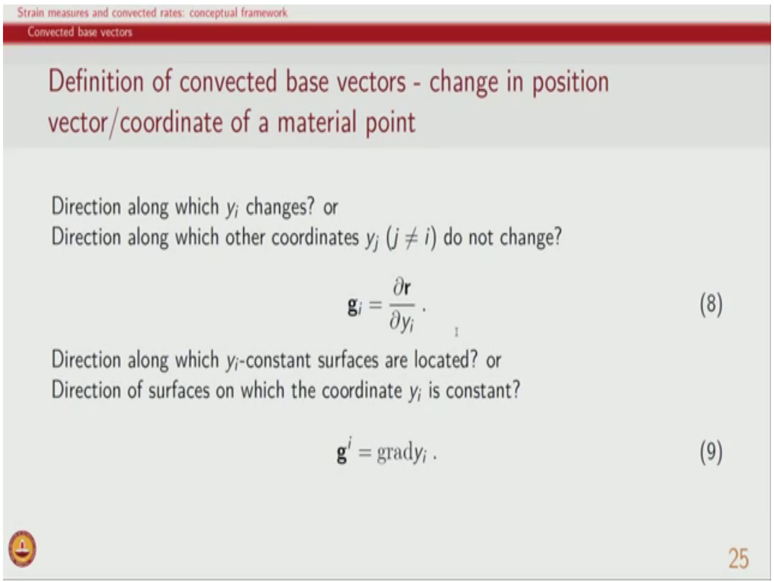

So, the both the definitions are as follows, given that the first question we are asking is direction along which coordinate why I changes by keeping the other coordinates fixed that is just looking at the partial derivative of the position of the point with respect to that particular coordinate. So, del r by del y i indicates the base vector in the i th direction. The other question that we had asked is direction along which y i constant surfaces are located and therefore, there is a change in y i.

How do these surfaces which are constant in y i how are they located and so, therefore, that relates to the gradient of y i and that is the reciprocal base vectors. So, now, using these 2 definitions we will again try to see how do we justify and how do we define these basis vectors for some simplistic situations. So, if we look at again the rectangular coordinate system, which we are quite familiar with we can again ask the same question. So, in a rectangular coordinate system for example, we have these eh. So, we have the grid which usually indicates the rectangular coordinate system and at any point here the position vector of course, is just given by x times ex plus y times ey. So, the first definition that we have talked about is in terms of how does a position of a point change if I change the coordinate.

So, in this case the coordinates are x and y. So, y 1 is x, y 2 is y and this is the overall position. So, we are looking at del r by del x and that is nothing, but e x. Similarly del r by del y and this is nothing, but e y. So, therefore, this is nothing, but g x or g 1 and similarly this is nothing, but g y or g 2. So, of course, in this case what we see is the base vectors that we already know in terms of the unit vector in the direction of x or unit vector in the direction of y, are the base vectors which are based on the definition. Now we could look at the same way gradient of x and. So, this will end up being del by del x del x ex plus del by del y ey of x and so, this again because del by del y of x will be 0 and we will del by del x of x will be. So, this again we will end up or we can write this as g x.

So, similarly we can also show that ey is going to be g 2 or g y. So, for a simple case of a rectangular coordinate system using both the definitions of base vectors we get identical results and the identical result is already known in terms of the orthogonal unit vectors that we get. So, therefore, for a case of rectangular coordinate both these definitions lead to one same set of base vectors. Now we can look at the same example for a cylindrical coordinate system. So, in case of a cylindrical coordinate we have any arbitrary point being defined as r theta and z.

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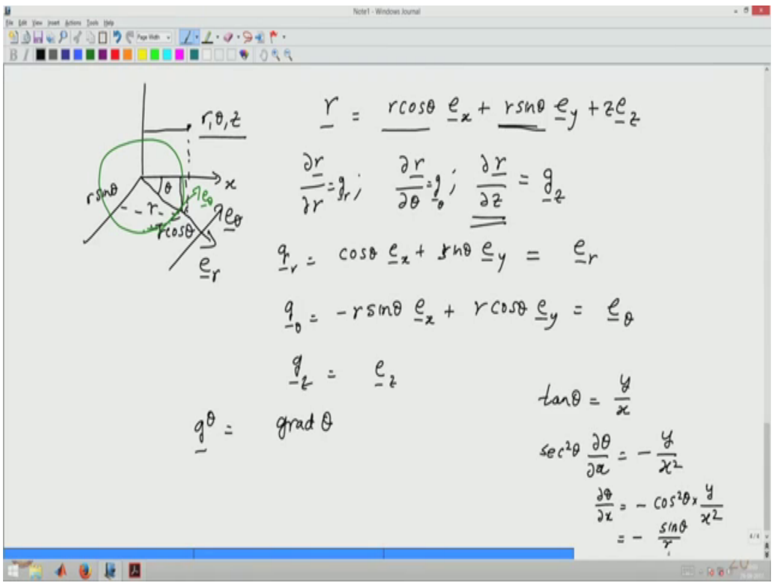

So, let us say if we draw a point and depending on the angle that it makes with respect to the axis this particular point will be described using r theta and z.

So, what we can do is we can look at its projection and see what angle does it make with respect to the axis and based on that then we can define what is its overall position vector. So, the position vector in this case will be indicated as r cos theta in the direction of x axis plus r sin theta in the direction of y axis. So, this is the r sy r cos theta because this is r and this value projected here is the r sin theta and plus of course, the ez. So, in general this is the position vector and now we are interested in finding out what happens to this position vector as a function of different coordinates. And what we have written here are nothing, but the base vectors which are defined based on different coordinates.

So, this is g theta and this is g r. And so, if we evaluate it we can clearly see that when we take the derivative with respect to r, g r is going to be just cos theta ex plus sine theta ey and of course, this is nothing, but what we normally use as the radial vector. So, in this case this is the direction of e r. Now if the derivative with respect to theta can be carried out and in that case what we are going to get which is g theta is cos theta is going to be minus.

So, therefore, r minus sin theta ex, plus r cos theta ey and this is in fact, nothing, but e theta, which is this direction because this is the projected circle that we can draw. So, this is the projected circle we can draw and e theta is in this direction and so, therefore, what we have is we again are getting base vectors which are familiar to us and we know them from before. And g z is going to be just ez because del r by del z is going to be just in terms of 1 and therefore, the unit vector will be ez.

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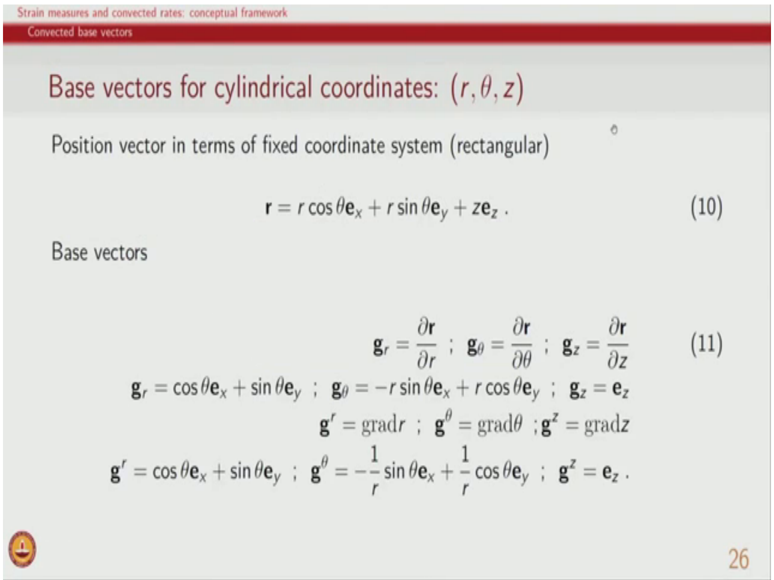

So, this is what is written here in terms of the base vector. So, g r is cos theta and sine theta, but this is nothing, but er itself and similarly g theta is in the direction of e theta;

however, g theta does not have units of or it has units of length and similarly g z has again no units and it is unit vector.

So, one important observation that we can make is based on this definition, the base vectors need not have the unit less they need not be unit list they can have different dimensions. And similarly if we evaluate the reciprocal base vectors we can show that the g r, g theta and g z are given as following and if you look at for example, the definition of g theta which is nothing, but del theta gradient of theta and we know that in this polar coordinate system or 10 theta is nothing, but y by x. So, therefore, derivative of this will lead to derivative of 10theta itself is x square theta del theta by del x will be equal to minus y by x squared.

So, therefore, del theta by del x is going to be minus cos square theta into y by x squared. So, since x is r cos theta itself, we can substitute it here. So, cos square theta x squared by cos square theta will give us the r squared itself and y will is nothing, but r sin theta.

So, therefore, this overall quantity can be shown to be sin theta by r and that is what we have here in the first term. So, similarly you can it is easy to show that the other term will be cos theta by r and so, in this case again we can see that g theta has the units of 1 over length. So, all these base vectors therefore, have units which are dependent on the coordinates that are being defined.

So, with this we have now got an idea about base vectors for familiar coordinate system, now we will continue our journey to define these base vectors for an arbitrary system.