Rheology of Complex Materials

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Lecture - 13
Introduction to tensors

So, in this lecture we will get introductory ideas about tensors. We have seen already that in rheology we have to understand the relations between stress tensor on one hand and strain tensor or the strain rate tensor. We also saw that for example, to find the traction at a particular point the stress tensor could be used as well as the unit normal vector had to be used. So, in general we will find that in course of rheology we will have scalars, vectors and tensors involved and the governing equations are nothing, but inter relations among these different variables. So, in this set of lectures we will first get introduced to the idea of tensors and then look at some of the operations in which we will use them so that we can write the governing equations more compactly and then also we can seek solutions more easily.

So, just to look at the outline of what overall we are going to do, we will begin our discussions by looking at what is meant by a frame and a coordinate system and then quickly review what we know about scalars and vectors and what we will see is these are tensors of order 0 and 1, respectively. Then we will spend some time talking about the tensor of order 2 of which stress and strain rate are examples and then the important aspect of how do we denote different quantities which are involved in the governing equations, and when we look at the summary of notation times then we will be ready to look at some of the example operations.

So, we will first review again the operations with vectors which we are familiar from earlier school as well as initial courses in undergraduate education and then we will also summarize the operations with tensors which are especially relevant for the course on rheology. We will finish up the overall discussion on tensors by summarizing some of the operations that are involved with derivatives.

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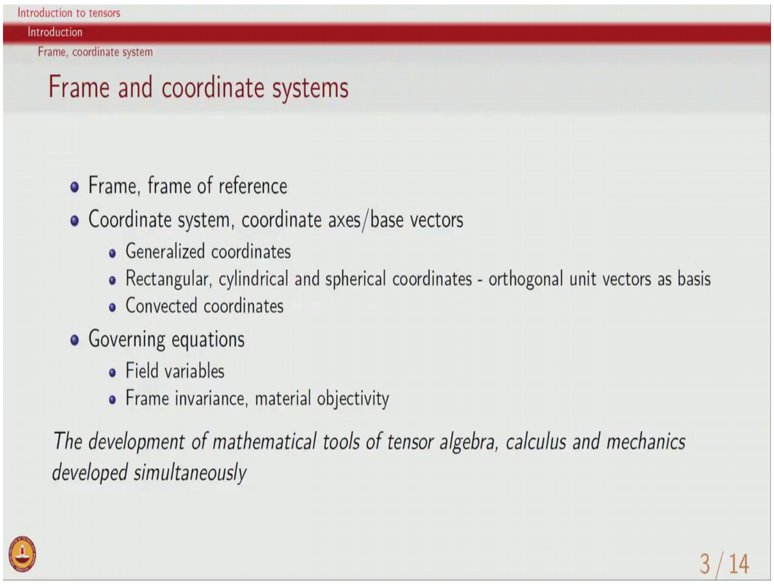

So, with this now let us begin in terms of what is the basis for describing the physical variables that are part of the governing equations in rheology. And as we discussed earlier we will be looking at field variables which are functions of space and time and so to describe the status the attributes of a material particle we will need a frame to describe that, and by frame we need it is a frame of reference and by that what we mean is the basis the reference point from where we view the overall deformation phenomenon that is occurring in the material. Quite often when we say we have chosen a frame we are basically defining sort of the origin of the viewpoint and also how does the origin change or remain static as a function of time. So, that basically specifies what is the frame in which we are going to observe the phenomenon.

Once we have decided the frame then to quantify the spatial coordinates what we do is we have a coordinate system which then describes the matrix in the frame. So, of course, most commonly we will be dealing with 3 dimensional space and therefore, we have the coordinate axes and the base vectors which are integral part of defining a coordinate systems. Since it is a 3 dimensional coordinate system we have basically 3 coordinate axes or 3 base vectors. More often than not these base vectors are unit vectors and they are orthogonal to each other. But in general when we say that there is a coordinate system that can be used to describe quantitatively a frame then we have several options in terms of choosing the coordinate system.

So, for example, the most general development in continuum mechanics and also in other fields of mechanics is done in terms of generalized coordinates. In which case we basically work with the fundamental hypothesis that to describe 3 dimensional space we need 3 coordinates and 3 base vectors. We do not assign any other particular property to these particular quantities, only thing that is required is these are 3 independent coordinates and 3 independent base vectors. So, any linear combinations of these can be used to describe the physical space effectively.

For engineering problems and as well as many other scientific problems of interest we have a geometry in mind we have a typical configuration of the material which we are interested in looking at and therefore, more often than not we either look at rectangular coordinate system a cylindrical coordinate system or a spherical coordinate system. In each of these case we know very well that the 3 orthogonal unit vectors are actually the basis of these coordinate systems and of course, in each and every case we have 3 coordinate axis, x y z for example in case of rectangular or r theta phi in case of spherical So, these 3 coordinates and of course, they are the 3 unit vectors are used to describe any other features associated with material that is being described in these coordinate systems.

So, generally given a frame of reference and given a coordinate system of choice then we can write down the governing equations in more detail as we have already said that these governing equations will be involving field variables. One aspect of physical intuition that we have when we write these governing equations is the fact that the response of the material or the result or the analysis that we are doing and the results that we are obtaining should not depend on what is the frame that we have chosen. So, the material response should not be a function of how it is being described. So, that is called frame invariance and then therefore, there are multiple frames in which we can describe a problem and we should get identical response. Similarly we also say that we expect material objectivity to be observed and that is again saying the same thing that even if there are different frames in which the material response is being described, the material objectivity implies that there is the material response is same in all the frames.

For the purpose of our course many times we can also use this different coordinate system and expect that the overall response to be the same. So, for the purpose of argument we can either choose two different frames. For example, a frame in which there is a rotating table and we are observing something while rotating or it could be a stationary frame where we are standing outside and something else is happening on a rotating table. So, for example, what is happening on a rotating table can be viewed from these two different frames. We could also just look at 2 different coordinate systems.

So, therefore, we have a rectangular coordinate system or a spherical coordinate system to describe a problem because there may not be a natural choice in terms of either rectangular or spherical coordinate system to choose. But the expectation that we have is the material response will be independent of these choices. So, this important expectation is behind a lot of development in terms of what specific governing equations do we use because we would like the governing equations as well as the a constitutive relations that we use later on in the course they all have to be frame invariant or they have to follow the principle of material objectivity. One aside note that at this point is helpful to remember that the mathematical tools of tensor algebra calculus and mechanics were developed actually simultaneously.

So, these expectations that we talk about in terms of frame invariance or material objectivity or these description in terms of different coordinate frames and generalized coordinates and the derivatives with respect to many of these field variables. All of these things came about simultaneously because to describe many of the natural phenomena these mathematical tools were required, and because the ma use of mathematical tools corresponded well with what was observed physically we could formulate the governing equations very on a very general basis.

So, similarly the idea of tensors and the tensor algebra also arose because while people were analysing problems related to mechanics it became clear that there are certain physical variables that do not fit the notions of variables that was known then for example, either scalars or vectors. So, therefore, clearly a new mathematical set of tools were required to actually describe the quantities that were involved in mechanics.

So, with this introduction to frame and coordinate system now let us look at tensors more specifically. So, an important aspect of tensors is that these are necessarily describing physical variables. So, we will use them in continuum mechanics and they will be involved in governing equations. By this we also mean that the tensors are not necessarily just a collection of numbers. For example, if we write a 3 by 3 matrix and a matrix itself as if it is a 3 by 3 it has 9 elements and those 9 elements actually can have any arbitrary number and therefore, that is a matrix. But when we say we are describing a tensor we have a specific physical meaning to all the components that are therefore, a tensor and therefore, the components will have a certain relationship with each other they ought to since they are representing a physical reality we will see that there are certain rules that these tensors will have to obey and therefore, we are only describing physical variables using these tensors.

Since we are describing physical variables quite often we will also know that physical variables have certain magnitude or there may be some invariants associated with a physical variable and this is because as we saw earlier that there is frame invariance or material objectivity. So, clearly the physical variables which are being described in different frames or different coordinates or to also have certain invariants.

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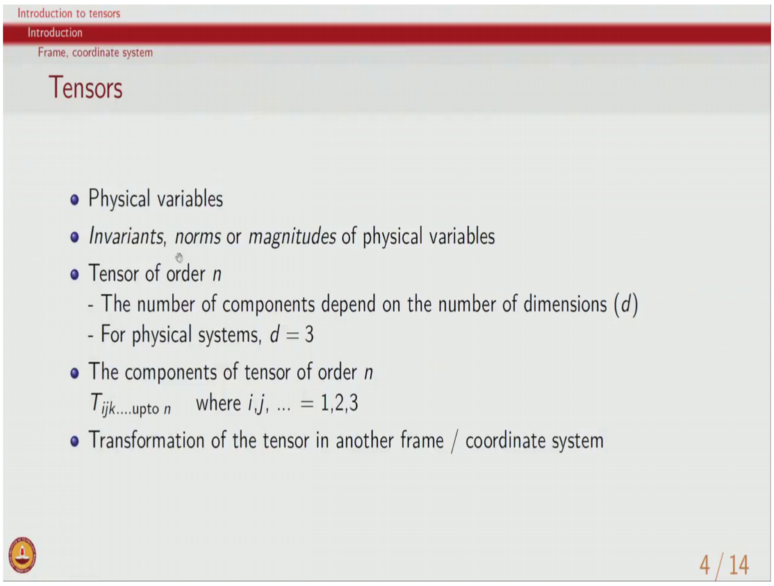

So, generally a tensor of order n basically will have the number of components of this tensor will depend on the number of dimensions and for most physical systems we have of course, 3 dimensional, so d is 3. So, given this fact what we have is the components of tensor of order n will have basically T ijk and the number of indices will be all the way up to n. So, for example, if we have tensor of order 3 then we have ijk, if we have tensor of order four then we will have ijkl.

So, basically this implies that the number of indices in the components of a tensor will depend on its order. And of course, each of the index that we are using and we will see that we will call these ijk all dummy indices, so these ijk values will vary from one two to 3 because we are talking about a 3 dimensional space. So, basically this is what a tensor is. Tensor is a collection of these components and the number of components depend on what is the order of the tensor and a key expectation that we will work with is that when we change a frame or a coordinate system this tensor gets transformed and by that we mean is the components individual components may change. However, we expect that there will be certain invariants or norms or magnitudes that will remain the same even if the tensor is getting transformed. So, we will define in each and every case what is meant by the invariance of a tensor.

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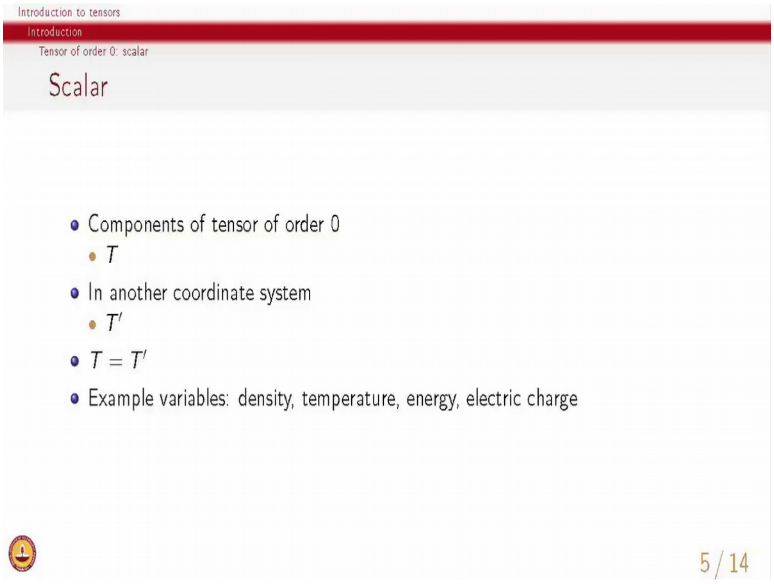

So, with this brief outline now let us look at the simplest tensor we know which is tensor of order 0 and since in this case there are no indices the order is 0 itself and, the number of components basically is just 1. So, in another coordinate system or another frame when we transform the quantity can be represented as T prime let us say and of course, we know given the scalar that it is a scalar quantity that T and T prime are the same. In other words even if we change a frame or coordinate system the quantity does not change and of course, we know that the example variables of this are density temperature and specific heat and these variables do not change whenever we have a change in coordinate system.

So, going on to the next tensor that we very are very well aware of is a vector quantity.

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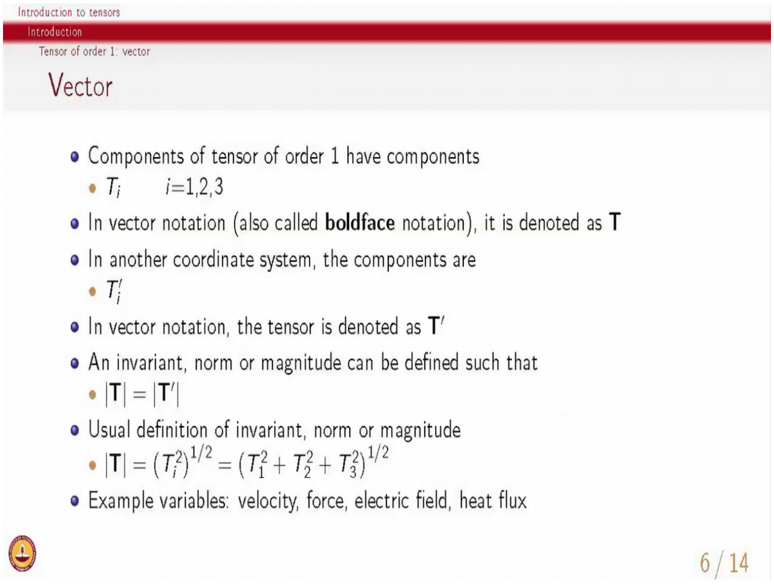

So, now vector is nothing, but a tensor of order 1. So, which means we have only one index which will describe the number of components. So, since i is 1 2 3 we basically have 3 components of a tensor of order 1 and we call in this in common parlance as vector. We denote the tensors and vector quantity in what is called a boldface notation or its also called a vector notation and therefore, vector T will be indicated as a boldface T. Of course, since we have talked about that the fact that we expect the vectors to transform from one coordinate to another coordinate system in another coordinate system the components of T will become T I prime. Again meaning that there are 3 components, but the 3 components may not be the same as what these components were in this coordinate system. In vector notation the tensor again in another coordinate system can be indicated as T prime.

We know that we can define a magnitude of vector and we also know that this magnitude does not change even if we describe the vector using 2 or 3 or any number of coordinate systems. So, way of saying that is the fact that magnitude of a or the norm of T and similarly norm or magnitude of T prime is same. So, regardless of the coordinate transformation that takes place this norm that we define or the magnitude that we define is an invariant. So, the invariants of vector quantity is nothing but this magnitude. And of course, there are multiple ways in which we can define the norm and the most common definition of the invariant is summation of square of the components of the vector.

So, in summary what we have is T 1 squared plus T 2 squared plus T 3 squared and square root. So, this is the magnitude of the vector and since we know that it is invariant we would also expect that we have basically in one coordinate system - T 1 squared plus T 2 squared plus T 3 squared to the power half and when we express it in another coordinate system we have T 1 prime squared plus T 2 prime squared plus T 3 prime squared and again to the power half.

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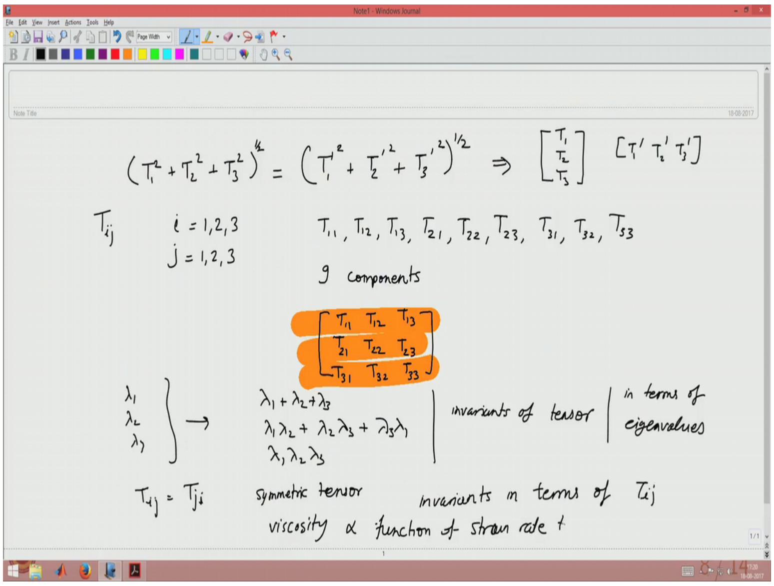

So, therefore, we have the expectation that this magnitude remains constant and which is what is expressed in this relationship here. So, therefore, you can find this magnitude using the definition and example variables in this case are velocity and force and electric field and these variables have been used in variety of problems related to mechanics.

So, now, we go on we have looked at tensors of order 0 which we have always known as scalar quantity and the tensor of order one which we have known as vector quantity now we can go ahead and start looking at tensors of order 2. The tensor of order two in most common language is basically referred to as just tensor. So, we generally refer to tensors of order 0 1 and 2 as scalar vector and tensor respectively. So, given that the tensor of order 2 has two indices so T ij and both i and j will be 1 2 and 3. So, therefore, we have basically 9 components.

So, given we have T ij and given that i goes from 1 2 3 and j goes from 1 2 3, what we have is T 11, T 12, T 13 and so on, T 21, T 22, T 23 and the last 3 set of components. So, what you can see is the fact that basically the tensor of order two has 9 components and of course, we can ascribe one of them as a row vector and one of them as a column vector. So, generally the matrix notation which we use to describe other quantities can also be used to describe the tensor of order 2 and therefore, what we have is a description of tensor in matrix notation and so that just has 9 components written as rows and columns. Similar to this of course, we can write a vector quantity also as a column or row vector. So, we could write T 1, T 2, T 3 as a column vector. So, which is a 3 rows, but only one column. Alternately we can also write this as a T 1 prime, T 2 prime or T 3 prime in this case there is only 1 row, but 3 columns.

So, therefore, we can use matrix notations to describe and write down some of these quantities, but we should remember always that tensors are describing physical variables and we should keep that in mind while looking at the properties of tensors.

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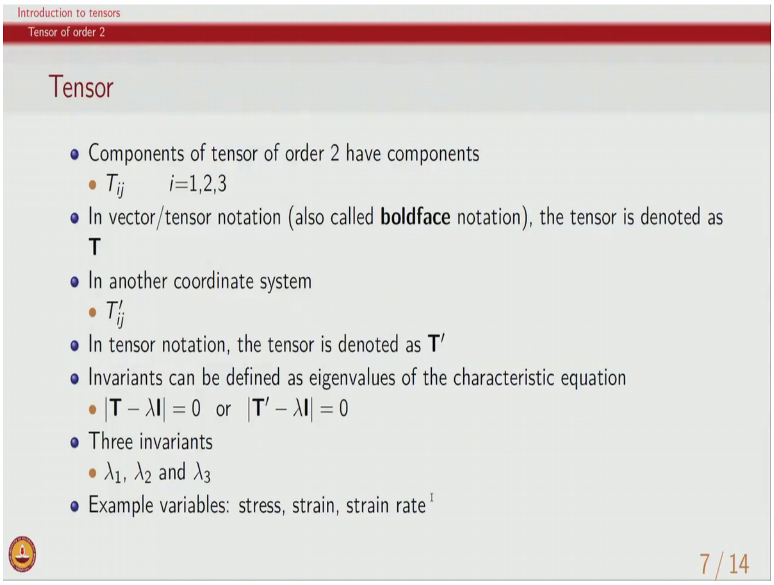

So, in given that there are 9 components of these tensors in a vector or tensor notation we describe or the boldface notation the tensor is you again given as just boldface. So, as you see both vectors and tensors or both tensors of order 1 and 2 are indicated using just bold typeface. So, from the context and from the variables that we are dealing with we will know whether it is a vector or a tensor. From a notation point of view in boldface notation both of them are typeface bold.

When we express this same tensor in another coordinate system then of course, the components will not be identical to what they were here. So, we can indicate again that the components are same the 9 components can be described using T prime ij and in tensor notation the tensor can be described as T prime.

So, now just the way we had with vectors when we transform a tensor quantity T to another coordinate system T prime we expect that there will be certain invariants which we can define and these will not change when you describe the tensor quantity in different coordinate system. So, therefore, the invariants can be defined for this particular tensor on the basis of the characteristic equation as eigen values. So, we know the governing equation to find the eigen values and so T minus lambda i where lambda is the eigen value and T prime minus lambda i both of these will give us the same because these are invariants and we know that this eq1uation when we write it down its a cubic equation, therefore, there are 3 roots and therefore, we have lambda 1, lambda 2 and lambda 3 as 3 eigen values or the 3 invariants of a tensor.

So, using these eigen values we can of course, construct many other invariants. For example, we could combine these lambdas which are the eigen values of the system as different combinations we could say lambda 1 plus lambda 2 plus lambda 3 and this will again be an invariant we could also combine them by product which is another way of combining them. So, this is also another invariant or we can also have their product. So, these are again invariants of the tensor.

So, therefore, we have several examples especially in the course related on rheology that are tensor and we have already seen that stresses because we have the direction of the force as well as the surface on which it is acting because it is a contact force stress. Tensor is one example. Strain again is another example because material particles have a relative displacement which is a vector and then this relative displacement can happen in 3 different directions. So, therefore, there is a displacement gradient which is again a tensor quantity.

Similarly we have rate of deformation described as a velocity gradient or strain rate and in this case again we have 3 directions of motion in terms of velocity and again the velocity can change in 3 different direction. So, therefore, we have again 9 components. So, each of the tensors stress strain or strain rate have 9 components.

So, the invariants of a tensor as we saw can be written in terms of the eigen values. So, these are invariants of tensor written in terms of the eigen values and for most of the tensors that we will encounter in the course on rheology we will have the condition that T ij is equal to T ji which means that it is a symmetric tensor. And for symmetric tensor it can be shown that all the 3 eigen values are real and positive and therefore, these 3 can be used to describe the invariance of the tensor.

We could also alternatively use the components written here. For example, we have written the components which are for example, the components which are these 9 components that we have written. So, we could also write in variants in terms of these components, invariants in terms of T ij. Why are these invariants relevant? Let us say in case of rheology we should remember that we will be talking about stresses and strain rates or strains in several situations and for example, one of the simplistic models of non Newtonian fluids is power law fluid and in this case the viscosity depends on the strain rate. And for a one dimensional simple situation we could say that viscosity would depend on strain rate, but since we have now overall engineering situation where we will have all 9 components of strain rate now how does viscosity which is a scalar quantity how does it depend on strain rate. So, while describing viscosity which is a scalar quantity as a function of strain rate tensor we have a problem in the sense that strain rate tensor would depend its components would depend on which coordinate system is being used and clearly we do not want viscosity to depend on whatever coordinate system that is being used. So, therefore, what we will do is we will use the invariants of strain rate tensor and use that as the magnitude of strain rate tensor and then say that viscosity would depend on it. So, therefore, the definitions of invariants will have to be considered from time to time during the course on rheology.

So, we will have a more chance to look at the invariants of a tensor and which invariant is useful in which context will be specified as and when we encounter the situation. In solid mechanics for example, the invariant second invariant of stress tensor is quite often used in describing the yield phenomena. Similarly the first invariant of strain rate tensor is actually 0 for incompressible fluid. So, several such invariants of tensors will be encountered and will be very useful while we are describing the several governing equations in rheology.

So, in this lecture we have seen a brief introduction on tensors. Now, in the next lecture what we will see is a summary of operations that will be useful when we write governing equations using these quantities.