Applied Time - Series Analysis
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Lecture – 76
Lecture 34A - Spectral Representations of Random Processes 6

What I want to do is I may spend about 20-25 minutes on the spectral part; particularly the harmonic processes and then briefly talk about spectral factorization. If you recall yesterday, we were talking of harmonic processes which are random processes that have a periodicity in them and as I mentioned yesterday there are different models for harmonic processes, one of the standard models is a sign embedded at embedded in noise.

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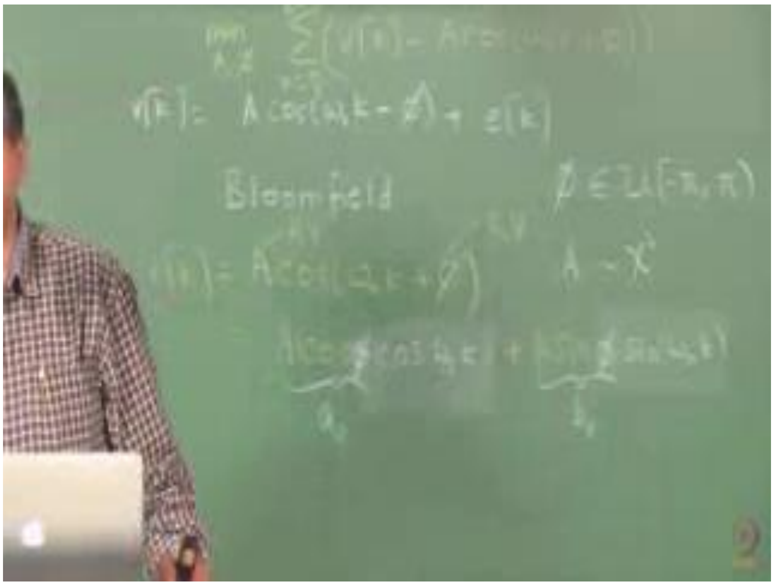

So, for example, you could think of v being made up of a cosine plus some e k where this part is a completely deterministic there is no randomness in them, and discovering the frequency omega naught from measurements of v k is a classic problem of the sinusoids embedded in noise discovering measurements from sinusoids embedded in noise. And the problem with this kind of a model is that v k is not stationary nevertheless this problem has been widely studied and there are numerous solutions that are available. A fantastic book which a very easy to read also gives you a very nice historical account as well as good insights into the different solutions that are merged and generalization of this model is the book by Bloomfield which is a one of the most widely reference book and in fact, I do give this as one of the reference towards the end of this lecture as well. Do read that book; it is a very nice book, very easy to read, no complications, beings with a very elementary develops material to begin with.

Now we will not perceive this model per se because we are dealing with stationary processes this model is not embedded to stationarity now gradually people said well, why do I have to consider a mixed model like this? I would rather consider another model which is as I wrote yesterday, simply consider v k as a cause omega naught k plus v, there is nothing specific about cosine it could even have a sin here there is nothing about no sine no sanctity about using a cosine function here. But the difference between this model and this model is now we can allow phi to be random and in addition we can also have the amplitude to be random.

Yesterday I did mention that amplitude could be fixed, but in generally also let the amplitudes be random that is the size of the sin or cosine that you are using to synthesis, the periodic random process as well as the phase that is when a particular wave sin sinusoidal or your building block begins with reference signal also can be a random variable. Now you have to understand that a and phi are random variables well while v is a random process what is making v random; a and phi are making v k random a and phi do not change with k that is something that you have to understand.

What this means is for one realization a and phi are fixed, the different possibilities of a and phi as use span across, as you walk across the space of a and phi that is a sample space of a and phi you would have generated an on some of v k of course, you know walking along time as well. So, you have a phi giving the randomness in v k whereas, here the randomness is important by this white noise. So, depends on what model you are looking at and that is something that you should keep in mind when you are analyzing a random periodic process simple straight a do not jump to using periodogram or any other tool first spend a moment and ask what is a model that you are imagining for the given signal and so in the case with the a periodic processes well what is a model that you are imagining whether you are looking at any AR model, MA model or an ARMA model and so on and then you decide to fit a corresponding model and then you draw the necessary inferences.

Here when you assume that a is a random variable and so is phi you given measurements, now the problem is that your given measurements of v k and we are suppose to estimate, let us say a omega naught and phi these are the three parameters that we are suppose to estimate for a given realization whereas, in the previous problem you are given v k again the same story what is given to you is the same its always the finite length data record. But what you determine here again is a omega naught on phi and phi, but in addition you may have to estimate sigma square e the only difference is a omega naught and phi are deterministic here a and phi are random omega naught always remains fixed of course, you can have multiple frequency the sum of cosines and so on.

Now, what happens is in general, when you are trying to estimate let us say a and phi given measurements of v, you could actually pose this as a regression problem, you could pose this as a regression problem, what we mean by regression problem is suppose I expand this I have here a cosine omega naught k and sin phi time sin phi plus a what do we have, sorry I have cosine phi plus a sin omega naught k sin phi this is simple by trigonometric expansion is it.

Now if you look at from a regression view point let us say I give you omega naught you kind of know the periodicity and then the goal is to determine a and phi there is the amplitude and phase you can pose this as a regression problem we will shortly learn how to detect omega naught. In fact, we have already done that we look at the power spectrum here you would look at the power spectrum as well, but only difference is you will looking at random signal.

Let us say you know omega naught then you can you can consider regressing v k on to 2 regresses cosine omega naught k and sin omega naught k right and then you would try to estimate a and phi or you can just say well find a and phi such that the standard optimization problem is solved expectation of v k minus a cosine omega naught k plus whole square that variance is minimized or you could solve this sample least square problem it is a minimize v k minus a cosine omega naught k plus phi whole square, k running from 0 to n minus 1, this is what you would probably you would be solving in practice. Whether it is this model or this model you will be solving this optimization problem 8.

Now the point is when you use this model the decision variables are a and phi you have to optimize them and you look at the regression problem is it a linear regression problem in the unknowns or a non regression problem, its non-linear we do not go we want to avoid non-linearities as much as possible. So, can we now convert this to a linear regression problem what do you think, how do we do that?

We can now denote a cos phi, let uss actually rewrite this we can say a cos phi times a cosine omega naught k likewise here, we write here a sin phi sin omega naught k and now denote this with sum a naught and this by b naught right instead of estimating a and phi I am going to estimate a naught and b naught and that makes the regression problem easier of course, in this case what I would do is I would indirectly estimate a and phi. But if I just want to know what are the contributions of cosine and sins I do not need to go back to a and phi how do I recover a and phi from a naught and b naught simply use the trigonometric identities right trigonometric identity is a would be square root of a naught square plus b naught square and phi would be simply tan inverse b naught by a naught. Of course, that is a non-linear mapping there is no escape to that all we are saying is instead of solving a non regression problem let me solve a linear regression problem.

The problem at the time of estimation is simplified now from a naught and b naught if we truly want to recover a and phi then we have to go through a non-linear mapping. But if you do not want that then this is a much simpler model to work with and that is exactly what you see on the screen. Expect that now we have generalized only that there we have generalized now this idea to mixture of cosine and sins, I mean you could have multiple periodicities and now a n that is a small a n and small b n are random variables given that already a is a random variable and phi is a random variable. And generally you can show that if phi falls out of a uniform distribution and a has a Chi-Square distribution then you can show that small a n b n follow the Gaussian distribution.

Now, we did discuss additional requirements on a n and b n for v k to be stationary remember we want v k to be stationary whereas, this model does not allow v k to be stationary. So, we will discard that model in favor of this one and what are the conditions that we imposed on a n and b n the first condition was that there average is should be 0 and again here we are talking of averages of a n and b n within their respective sample space, you have to understand there is nothing there is no notion of time when it comes to the coefficients for a given realization they are fixed. Notice that there is no dependency of a n or b n on k they do not change with time for one realization they are fixed again one another realization a n and b n would take on another value.

The requirement are 2 fold, first a n and b n have to be independent random variables, there is a nice question asked by one of students yesterday when I said a could be fixed and phi alone being random then how could you guarantee independence and that was correct, but now we have corrected that by saying a should also be a random variable. And now you can effort to think of a n and b n being independent random variables independent would essentially mean here uncorrelated that is of course, independence is a very strong requirement, but a consequence of independence is uncorrelatedness.

You given that a n and b n are uncorrelated what does it mean the way I choose a as no bearing, on the way I choose b the cosines and sins can add in their own way just because I included a cosine of a particular magnitude it should not mean that a sin of similar magnitude or related magnitude should be included they can be on their own and with equal of course. We assume equal variances they do not have to be equal variance that is only for convenience sake we assume equality.

2 requirements independence and 0 mean with that assumption you can show that the auto covariance function is simply what I have given you on the screen it is sigma that is the sum of sigma square n. In fact, sigma square n is a variability of a n and b n when we said equal variances earlier for a given omega n a n and b n have equal variance.

Now, how do you arrive at this expression well fairly straight forward first you derive the mean right we know already the mean is 0 because a and b have 0 averages the small a and small b and then you apply the definition of auto covariance function and use the trigonometric property of trigonometric functions. During that you can show that the auto covariance function as a Fourier series expansion as an remarked yesterday in the previous lecture the ideal situation should have been that you should have complex exponentials in the Fourier series expansion, but because auto covariance function is symmetry you only have even function participating in the Fourier series expansion.

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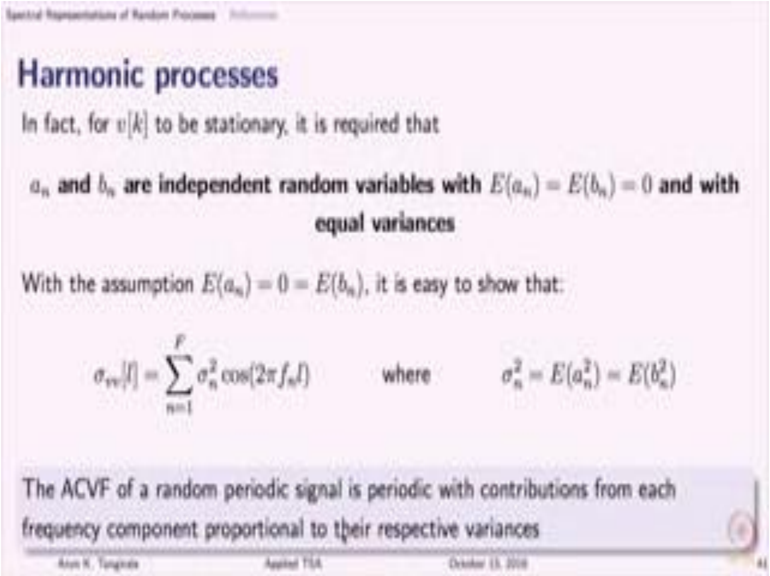

What it says essentially is now you are model is indeed generating harmonic process right because the definition of a harmonic process is that the auto covariance should be periodic with the same period as the signal itself and here indeed of course, in this case you the period what would be the period when you have a mixture of cosines like this?

Student: (Refer Time: 14:53).

There is common multiple and that is important. So, your omega or your f should be such that there should be least common multiple and you should have an integer period all of that is implicitly assumed. So, that is it. So, we have we have guaranteed that this model generates a stationary periodic process in other words harmonic process. You can come up with your own models and so on, but as far as harmonic process is concerned this is by far the most widely used model, and in practice we would be given realization a single realization of v k and the goal will would be to discover omega naught and of course, the coefficients a n and b n their estimates point estimates only from the given realization.

Now, of course, that part will discuss in the estimation part, but what we now can remember is that for a random periodic process or a harmonic process the ACVF as a Fourier series expansion, whereas for an a periodic process which is what we have discussed at length the auto covariance function has a Fourier transform and that there it would gives as a spectral density here we do not get a spectral density at all and just for comparison you should obtain that.

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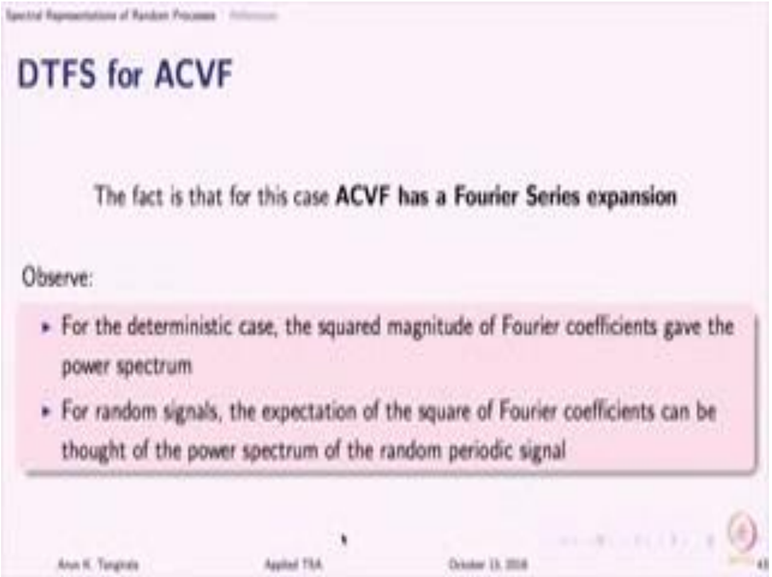

You should observe that in the deterministic case, the squared magnitude of Fourier coefficients gave us the power spectrum, if you recall of the Fourier coefficients of the signal expansion whereas, for random signals the expectation of the square of Fourier coefficients gives us the power spectrum of the random periodic signal. What we mean by this is if you go back to this expression here you see that you have sigma square n as the coefficient of the Fourier series expansion correct and what are the sigma square hence they are the contributions of the individual sinusoids or cosines you can say to the overall power right. In fact, how do you say it is given it is a contribution to the overall power from this expression? How do you say that?

Student: (Refer Time: 17:13).

Exactly, at lag 0, what is expression that you get? Sigma square v is simply the sum square sigma square n, correct. So, this is what you would have.

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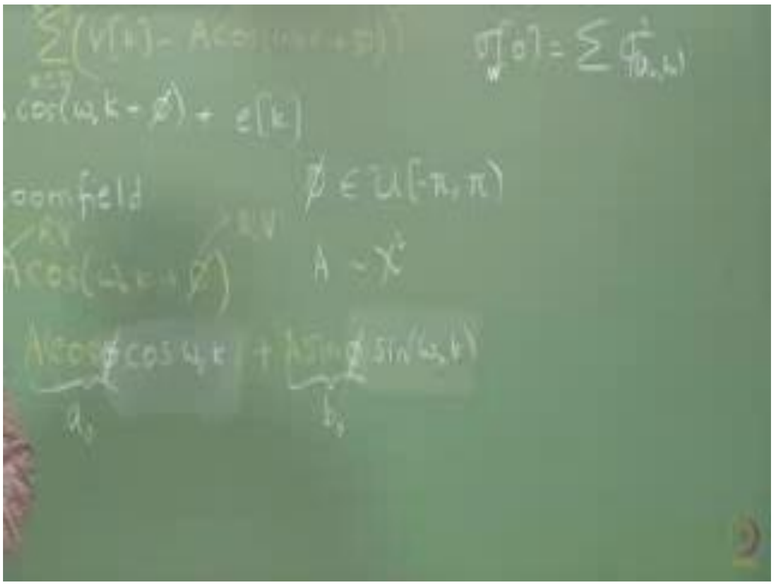

Sigma at 0, this is for the signal to be more specific a n or b n even just to highlight that this sigma square is for the coefficients and we know already cosines and sins give you they are orthogonal functions. So, each contribution is unique. So, we have now a power spectral decomposition look at the difference here in the deterministic world, the moment we move to a periodic signals, we always talked about energy signals where as in the random signal world whether it is a periodic or periodic we are always talking about power.

The fact is and the reason is because they exist forever, the signal never decays with time even if it is a periodic. So, in both cases, it is power you do not have to be confused at all and. In fact, when you use d f d also it is always empirical power spectral density that you consider. So, every where it is about power, the energy spectral density is kind of take a back seat when it comes to a practice. So, this equation tells us that the power of the signal is decomposed into the respective contributions from the frequencies they need not be necessarily here kind of harmonics you just have some bunch of frequencies adding up, alright.

Now that is what gives us this interpretation that we just made that for random signals the expectation of the square of Fourier coefficients when I say square of Fourier coefficients the Fourier coefficients are actually a n and b n remember. The only difference is in the deterministic world those a n's and b n's would have been fixed or deterministic where as for the random signals a n’s and b n’s are random variable remember we made this observation earlier also. So, that is why we are talking of expectation of square of Fourier coefficients so that is the difference always the expectations coming for random signal anyway.

Now we kind of a unified everything whatever we have looked at until that is the world of aperiodic signals and periodic signals in the case of aperiodic signals we talked of spectral densities correct and when it comes to harmonic processes we talk of power spectrum alone of course, I would have shown you a number of signals and so on, but one thing that I will show you in R is the use of ARMA ACF which will allow you to compute the theoretical spectral density. But I reserve the discussion on estimating power spectral on working with real life examples to the section where I will talk about estimation of power spectral or power spectral densities and that time we will take up some real life examples and at least real life data sets and show you how to interpret the resulting power spectral density and so on.

To summarize what we have is for periodic signals we have power spectral density and for harmonic processes we have power spectrum. Again this is an analogy with the random variables for continuous valued random variables we have a probability density function where as for discrete value random variables we have probability mass function correct. But regardless of the situation we have a probability distribution function likewise here whether we have a spectral density or not we will have a spectral distribution we can think of a spectral distribution function.

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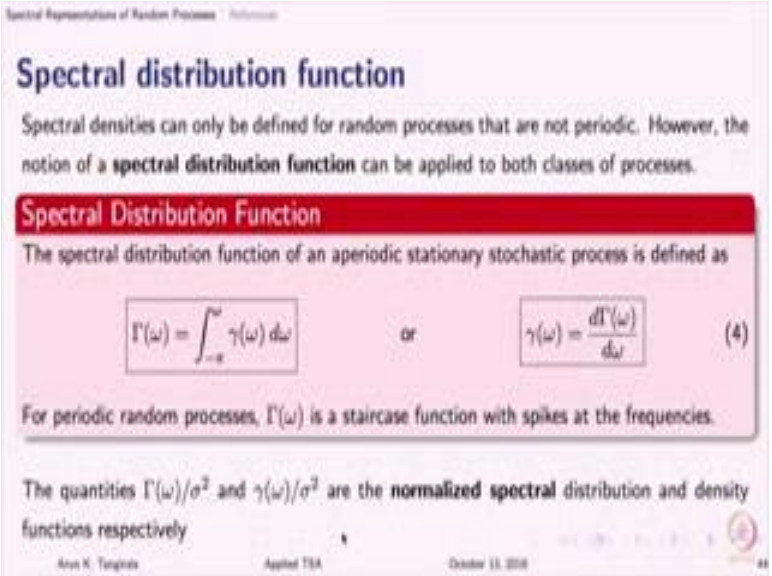

Of course for an a periodic stationary process you can write the spectral distribution as an integral as an area under the spectral density and that is what I am pointing out and exactly like your probability density if you are confuse this recall the probability density notion right. Whereas, the spectral distribution for a periodic process will not have this kind of a or a harmonic process will not have this kind of a relation, there is no spectral density at all. How will the spectral distribution look like for a periodic process?

Student: (Refer Time: 21:49).

Step like whereas, here it would look continuous, exactly the same case as you see in the probability in the random variables when you have continuous value random variables this a probability distribution would look continuous and for discrete valued random variables the probability distribution as a shear stress like shape. So, with this we give a, we just studied the unified Wiener-Khinchin theorem is nothing new in this.

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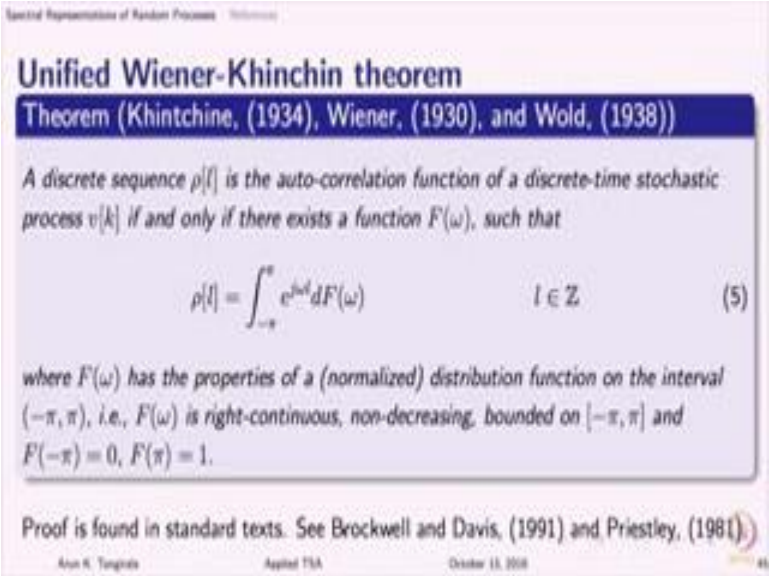

All we are doing is we are unifying 2 results, one from for the aperiodic case and other from the periodic case and this is basically on the lines of Fourier stieltjes integral. If you recall we said there is a Fourier transform for a periodic deterministic signals finite energy signals and then there is a Fourier series expansion for deterministic periodic signals and then if you recall we said the Fourier stieltjes integral unifies both those results. So, that it becomes easy to look at both class or signals in a single framework likewise this unified Wiener Khinchin theorem unifies the relations between autocorrelation or in fact, here this auto correlation and the spectral distribution now. So, what is result says and it is a very profound result in the theory of random process. It says a discrete time stochastic process or sorry discrete sequence rho of l.

I pick, I give you some sequence and I if I want to claim that it is a auto correlation function of some stationary process it is true if and only if you can associate a distribution function f of omega such that rho of l is this integral. So, what it says is you pick any sequence discrete time sequence and you claim you can claim it to be auto correlation function or some random process. So, if you have to ask that question why is this important because in practice what is have is a realization and I am going to compute auto correlation function and I have t o be guaranteed that from this auto correlation function that I estimate I will be able to construct a model that will generate a stationary process.

What guarantees may that this theorem says if you can construct a sequence if you estimate the auto correlation function in such a way that you can associate a distribution function which is related to this auto correlation function in this fashion which is called again the Fourier stieltjes integral type of think. Then essentially the sequence that you have rho of l qualifies to be the auto correlation function of a stochastic process this includes everything symmetricity non negative definiteness everything is embedded in this. How is it embedded? Because this distribution function in fact, because we are talking of auto correlation is a normalized distribution function should have some properties you what is meant by distribution function you should ask and what qualifies to be a distribution function some of the qualifying most qualifying remarks are given here first is it should be non decreasing.

We said the same thing of a probability distribution, what are the qualities of a probability distribution function? It should be non-decreasing, right over the extremes here because it is spectral distributions the extremes are minus phi and phi where as in the probability distribution case extreme values for the random variables. So, in other words I cannot the distribution function.

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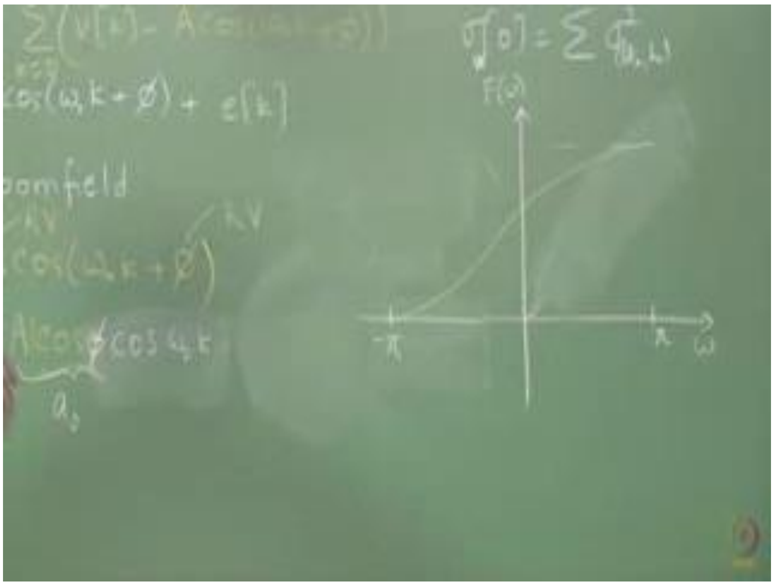

For example as to look something like this it cannot you spectral distribution cannot look like this the other curve. So, I am drawing for the 0 to the phi sorry, I should have drawn from minus phi to phi, re-draw.

Let us say here is phi and here we have minus phi. So, what this says is the distribution function if you are looking at a periodic process as to be like this. So, it as to run from minus phi to phi in a non decreasing fashion you could also have a shear stress like signal in which case you are looking at a periodic random process harmonic process. More over its normalized what does it mean, what does normalize spectral distribution mean? At pi you should reach a value of one, it is normalized what is it normalized with the variance of the process that is why we have the auto correlation and then it should be right continuous and it should be bounded. So, all of this actually makes, constitutes are normalized spectral distribution function.

If I give you the sequence and a claim it is the auto correlation function of some process or if I claim that there exist a process whose auto correlation function is what I give you then necessarily the auto correlation function should be expressible as this integral. A simpler way of looking at it is you should be for a periodic process if I tell you if you give you a sequence and I claim that it is the auto correlation function its Fourier transform should exist for an a periodic process. For a periodic process a Fourier series expansion should be possible that is what it means. So, it just unifies this.

So, this is kind of a summary of so many things that we have learnt in a single theorem, but that is why it is a very profound theorem because; obviously, people where asking this question, I have this definition of auto covariance, auto correlation then I have this definition of spectral density or distribution and then I have worry about I have to guarantee that there exist a stationary process for a given sequence, all of this questions how can I guarantee and so on, this theorem answered all those question and it says yes this is how you guarantee.

Of course, the proof is a bit involved, we do not go through the proof, but it is very important to remember this theorem. And I have already talked about the normalization; the normalization essentially is with respect to sigma square.