Applied Time-Series Analysis
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Lecture – 74
Lecture 33 B - Spectral Representations of Random Processes 4

We have learnt a lot of useful things how to arrive at the spectral density, interpreting the spectral density and so on. Now, what we will do is I am just going to go pass this breeze through this proof that is the proof of the relationship between spectral density and the transfer function operator. What you do is to arrive at this relation.

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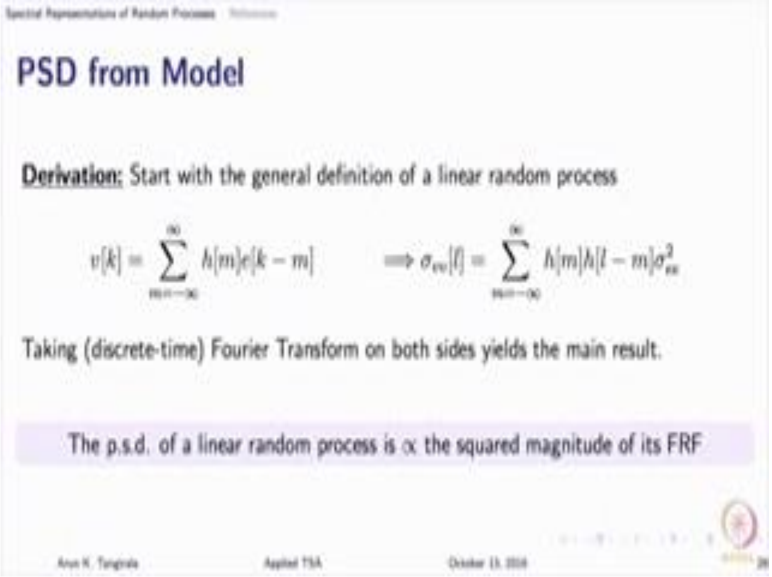

PSD from Model

Derivation: Start with the general definition of a linear random process

$$v[k] = \sum_{m=-\infty}^{\infty} h[m]e[k-m] \qquad \Longrightarrow \sigma_{vv}[l] = \sum_{m=-\infty}^{\infty} h[m]h[l-m]\sigma_{ee}^2$$

Taking (discrete-time) Fourier Transform on both sides yields the main result.

The p.s.d. of a linear random process is $\propto$ the squared magnitude of its FRF

You start with the definition of a linear random process.

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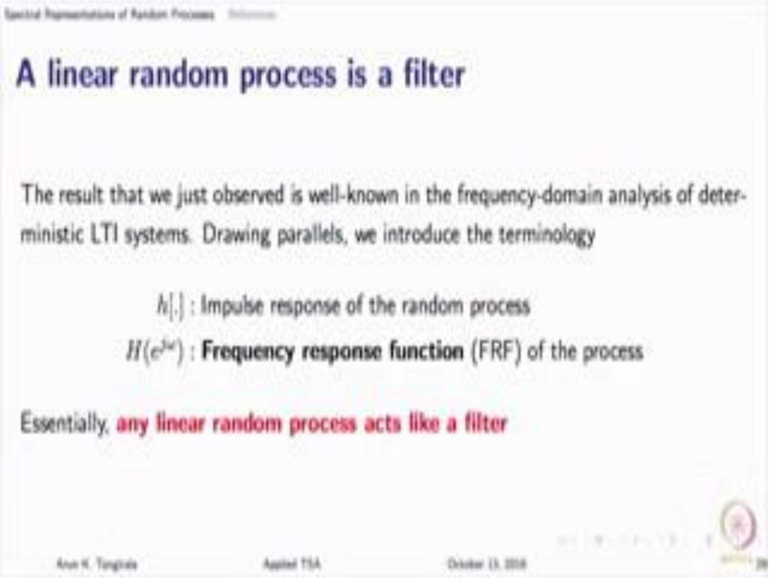

A linear random process is a filter

The result that we just observed is well-known in the frequency-domain analysis of deterministic LTI systems. Drawing parallels, we introduce the terminology

$h[.]$: Impulse response of the random process

$H(e^{j\omega})$: **Frequency response function** (FRF) of the process

Essentially, any linear random process acts like a filter

And then what you do is you take first from the given linear random process you construct the auto covariance function. So, what you have on the left hand side is the standard expression that we are written for linear random processes. And what we have on the right hand side is the resulting expression for the auto covariance function. And then you simply take discrete time Fourier transform on both sides of the right hand side expression. The left hand side will give you when you take the d t f t of the left hand side what do you get. This spectral density of course, differing by factor of 1 over 2 pi that is you can multiply both sides.

On the right hand side you can do some algebraic jugglery and show that the right hand side expression is of this form.

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Essentially the same thing but I am going to write it more explicitly. H of e to the minus j omega times H star of e to the minus j omega times sigma square e by 2 pi, but the 2 pi comes about by deliberately multiplying both sides with 1 over 2 pi. That is just about two steps you just have to do a change of variable. And then what you will have on right hand side is the double summation, because you are going to take the discrete time Fourier transform. That double summation can be expressed as a product of two summations that is the essence.

Now, this gives us a filtering interpretation to the process. And now we can imagine any correlated process to be acting like a filter. What it is doing is it is only selectively allowing certain frequencies in a white noise to pass through preferentially; it does not nullify necessarily a band of frequencies that is very very important. a spectral the spectral density of any stationary process can be 0 at finite points perhaps, but not over a continuum; which means this process cannot really kill a range of frequencies. If at all it can actually kill only nullify only 1 or 2 or some finite set of frequencies; which means your H of e to the minus j omega cannot be 0 on an interval; That is something that you should remember.

So the message is that any linear random process is a filter and one more example for you. Here we have an ARMA process I mentioned earlier it is nice to use this expression to compute the spectral density for particularly for ARMA processes, because the auto co variance has a complicated expression, here the transfer function operator is given. And it suffices to compute mod H of e to the minus j omega square to get a feel of gamma, because the rest is a proportionality constant right sigma square e over 2 pi is a proportionality concept that is not going to change the shape it is only change the magnitude of the spectral density.

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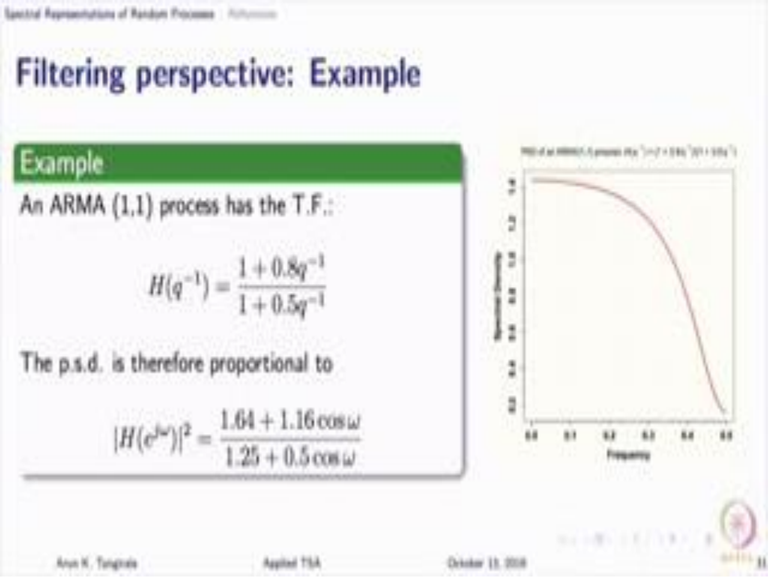

Filtering perspective: Example

Example

An ARMA (1,1) process has the T.F.:

$$H(q^{-1}) = \frac{1+0.8q^{-1}}{1+0.5q^{-1}}$$

The p.s.d. is therefore proportional to

$$|H(e^{j\omega})|^2 = \frac{1.64+1.16\cos\omega}{1.25+0.5\cos\omega}$$

And you can see here it is a rational function of cosines. Once again you should do the standard checks right and this is how the shape looks like. Of course, for this example it can look different for other kinds of ARMA processes.

Now, at this point I want to tell you and give you a peak into estimation of spectral densities. We have now learnt two different ways of arriving at the spectral density. In practice what do we do? You have to ask, these are all theory like someone said- oh this is lot of theory yes it is, but this foundation is extremely essential for practice. So, in practice now if I want to compute the spectral density I want to know what kind of a filter my market is what kind of shocks it is able to filter out whether low frequencies or the high frequencies and so on.

Believe me there is so much use to this filtering interpretation of the random process, but it finds applications in almost every field. So, I would like to know what kind of shocks it can absorb a random process can absorb and what it cannot. Therefore, given time series data I would like to construct the spectral density. I have two different ways: one is using the Wiener-Khinchin relation. What I do is, I estimate the ACVF we have learnt not formally though, but we have looked we have used this sample ACF expressions. I take the sample ACF and I compute the spectral density.

Alternatively, what can I do? Time series, but am I given the model right, am I given the model, yes or no?

Student: (Refer Time: 05:36).

So, what do we have to do?

Student: (Refer Time: 05:37).

We have to fit a model. Once you fit a model what are we doing we are parameterizing the impulse response function, remember that. You recall, we said the general linear random process assumes that there is an impulse response sequence that is absolutely convergent, but does not assume any shape. The moment we parameterize the impulse response function we get a time series model. That amounts to parameterizing the ACF we have said that, now that amounts to parameterizing the spectral density, correct. The spectral density of a stationary process if it exists can have an arbitrary shape so long as it is satisfies the four important properties that we have said. There is yet another important property we will talk about it very soon

But, the spectral density like the ACF can have an arbitrary shape so long as it satisfies the basic properties. Why should it have a specific equation, but if you think it as a specific equation then that means, you think that you have there is a time series model representation. So, what I am trying to tell you here is, fitting a time series model amounts to parameterizing the ACF and the spectral density simultaneously, same kind of parameterization they are linked which means you are assuming that the spectral density as some shape. Is it necessarily true for all random stationary processes? No.

So, what happens when the reality is that the spectral density exists and does not fit into your parametric assumption? What you end up doing is you end up only approximating the real spectral density function. And there is a theorem that says if you choose an auto regressive model or a time series model of sufficient high order then you can fit, you can actually approximate the spectral density to a fair degree of accuracy arbitrary degree of accuracy.

It says only arbitrary degree but it does not say exactly, which means the difference between the approximated spectral density; how do you construct the approximated spectral density? From the time series model, you fit a time series model use this expression construct the spectral density that is your parametric way of arriving at the spectral density. The error between what you have fit and the reality can be shrunk to an arbitrarily small value in magnitude but not 0, if the underlying process does not have a spectral density that can be parameterized. That is the way you are parameterizing.

So, fitting a time series model means a lot of things; you are parameterizing the ACF or you are paramterizing the spectral density and so on, but then ultimately say who cares I do not know the reality anyways, you cannot prove that I am wrong. Ultimately, what is the test of your model?

Student: Prediction.

Prediction that is why there are many people who say- I do not care about frequency domain so much. My model does a good job of prediction you are happy I am happy who cares about happens to the frequency domain. Well, that is true to a certain extent, but then that is the shout of a more of a practitioner rather than a theoretician, because the theoretician tells you under what conditions, what you are doing is correct and under what conditions your theory works, whatever you are doing works and under what happens when things fail where to look for. So, that is why we are learning this theory.

Otherwise, I could have straightaway shown you the procedure of fitting a time series model and you are happy I am happy, but you know education is not about such kind of happiness always. We are supposed to also learn the theory so that we know the framework in which whatever we are doing works. Otherwise how do you know? Today what you are doing may work for the process that you are looking at tomorrow it may not work and you do not know what to do you so- oh my god you know today my stars are not correct, that is not the way you look at it. That is why we are going through this theory. And hopefully now you will appreciate this theoretical discussion a lot better.

So, to summarize there are two different ways of estimating spectral density, and why do we want to estimate spectral density, to get a feel of the filtering nature of the process. Of course, many a times we want to look at the spectrum to also detect periodicities; that is a unwritten kind of advantage of working with spectra. But here we are talking of spectral densities we have not included periodic processes. We are talking of aperiodic random processes.

So, the one approach is to compute the ACF and the other approach is to fit the time series model and then arrive at the spectral density. Each of it has it is own merits and demerits and we will talk about it during estimation. The one final thing that I want to talk about with respect to this before we move on to periodic processes is that given spectral density can you arrive at the ACF? Yes or no? Right this expression tells you that you can do that.

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So, what happens in general? When I give the spectral densities and I ask my students to compute the ACF they religiously use this expression. They pay obeisance to this I know this expression whether I can work out further that integral or not I will give you the expression if you like it give me some marks and move on or I will work further and give me more credit.

Typically, if you given the spectral density at least in a parametric form. If you are not given if you are given a spectral density in some kind of arbitrary form then that is different, but if you are given a parametric form suppose I give you the answer that you have that is I just give you the mod H of e to the minus j omega square times some 1 over 2 pi let us say. And I ask you to arrive at the ACF what would you do, or even you take this expression I give you this expression and I ask you to arrive at the ACF.

Yes, theoretically you can start with this integral and stay with this integral for your life, because these integrals can be painful for cert most of the processes they are nice for theoretical analysis. Practically it may be a good idea when you are given spectral density in a parameterized form you look at the spectral density and by visual inspection guess the order of H and even the coefficients.

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So, you if are given this and you are asked to compute the ACF what would you do? You can guess that the spectral density has to come from an MA 1. And you know if you have worked out a few problems on your own then you will know the relationship between the MA coefficients and the spectral density and you should be able to guess, because always you would have cosines and so on.

So, some practical common sense alertness will help you overcome the necessity of evaluating an integral. Once you have the H then arriving at the ACF is a lot easier. So, keep that in mind. Of course, given a spectral density you can also compute the variance as we have talked about earlier. So any questions before we move on to harmonic processes, very good.