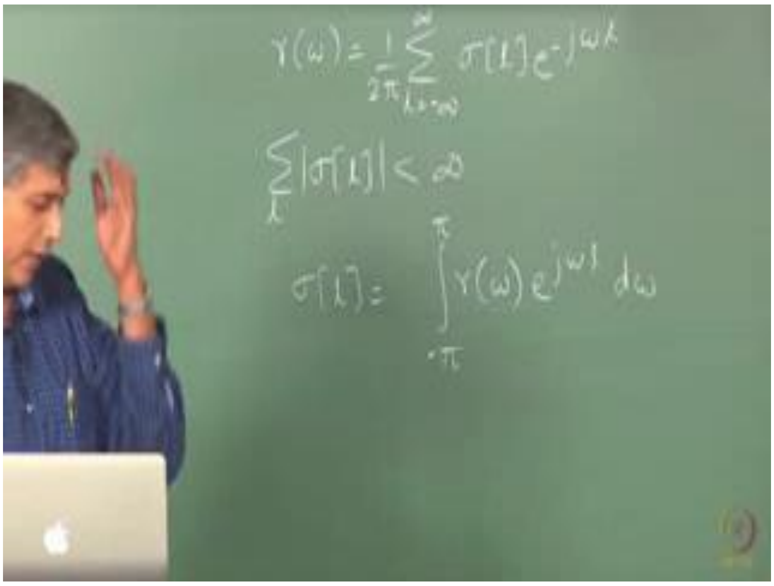
Applied Time-Series Analysis
Prof. Arun K. Tangirala
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture – 73
Lecture 33 A - Spectral Representations of Random Processes 3

We looked at spectral representations of random processes, and we learnt if you leave aside all the theoretical rigor. What we learnt is that we can define a spectral density for random process that is stationary, and whose ACVF is absolutely convergent. And there are three different ways of defining or arriving at the spectral density: one is of course the veneer generalized harmonic analysis which is fairly rigorous we did not pursue that line of thought. The other one was a semi formal approach where we started with the periodogram of a finite length realization. And then we gradually built the so called spectral density in a somewhat informal manner.

(Refer Slide Time: 01:11)



And we showed that the spectral density exists if an only if the ACVF is absolutely convergent. We did not prove it in a very rigorous manner, but it suffices to know that the spectral density exists for all stationery processes whose ACVF is absolutely convergent. And as I mentioned in the last lecture and even previously there are stationary processes whose auto covariance does not necessarily satisfy this condition.

And that is the case one such example is the periodic process or the harmonic process which we will briefly talk about today.

And the most important result that is useful in practice and everywhere even in proving so many other results and so on is the relationship between the auto covariance and the spectral density which is that the spectral density is the discrete time Fourier transform of the auto covariance function. And again as I said many textbooks present this as a definition is not really necessarily the definition, but if there exists a stationary process whose absolute ACF is absolutely convergent then a spectral density exists and it is related to the auto covariance through the Fourier transform.

And there are many useful results that follow from this and that is where we learnt why the white noise is called the white noise process. In fact, the other equation that is the inverse Fourier transform; so this is your forward Fourier transform, the inverse relation allows us to reconstruct the ACVF given the spectral density. In fact, I told you that in a many algorithms that compute auto covariance function they would probably compute a spectral density first and then do an inverse Fourier transform to get the auto covariance function from finite data.

And a word of caution here we have a 1 over 2 pi is a factor in front of the summation. Now, some definitions do not use the 1 over 2 pi so you have to be careful, then accordingly you will have to modify the inverse Fourier transform. What we are following is the traditional kind of the definition that is followed in many traditional developments, but there are some texts which would ignore 1 over 2 pi. Of course, if you rewrite this in terms of the cyclic frequency then 1 over 2 pi is missed.

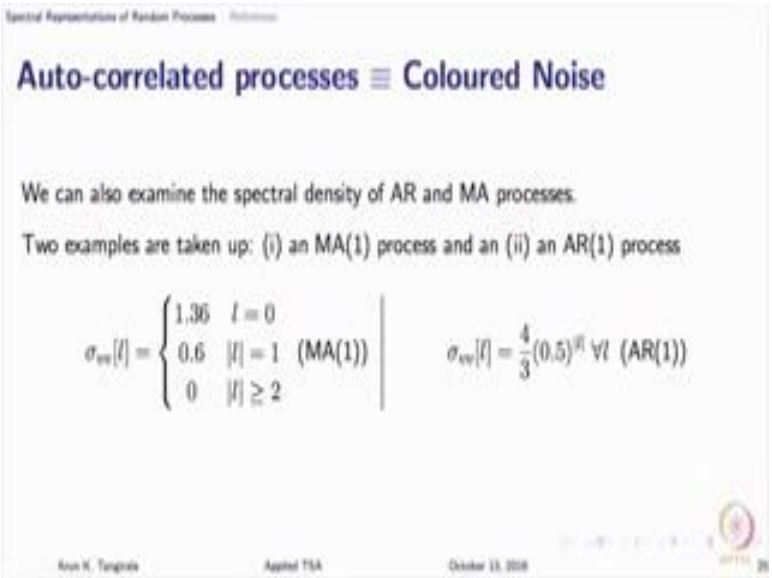
So, if you leave aside everything else what I should tell you is that the spectral density is the discrete time Fourier transform of the auto covariance function, even if you just remember that. And of course, along with that this condition which is the standard condition that is required for any sequence to have a Fourier transform. If you remember that then you have remembered half of the story and the useful part of the story.

And we use this condition now last time in the last lecture we used this veneer tension relation to derive the spectral densities of the white noise process and the correlated processes namely MA 1 and AR 1. I show you again just for your own recollection if you recall the white noise has a flat spectral density. And time and again you should ask yourself do I understand what is spectral density; and if you do not, then you have to tell yourselves that the area under the spectral density over a certain frequencies gives me the contribution of those frequencies that is the band of frequencies to the overall power of the processes.

Remember this random process exists forever by definition. So, there it should have a finite power to keep generating this random sequence. At some point it should not be such that it has ran out of power and random sequence has died down it cannot happen. So, that is why we are talking of power and power spectral densities.

So, white noise has a flat spectral density and by analogy with white like which contains uniform contributions from all frequencies engineers called this process un ideal uncorrelated process as white noise process.

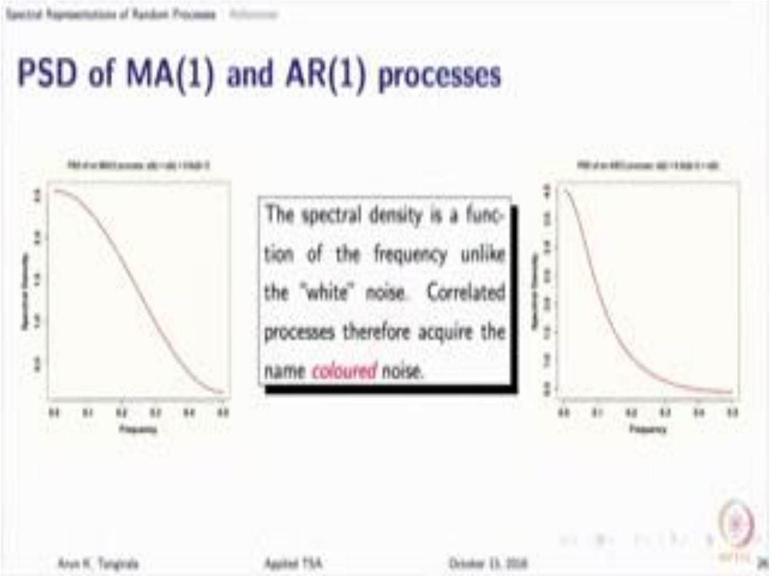
(Refer Slide Time: 05:57)



So finally, the mystery these kind of reveal; this term white noise we have been using almost from day one, but it took us this long to really understand why white noise is called white noise.

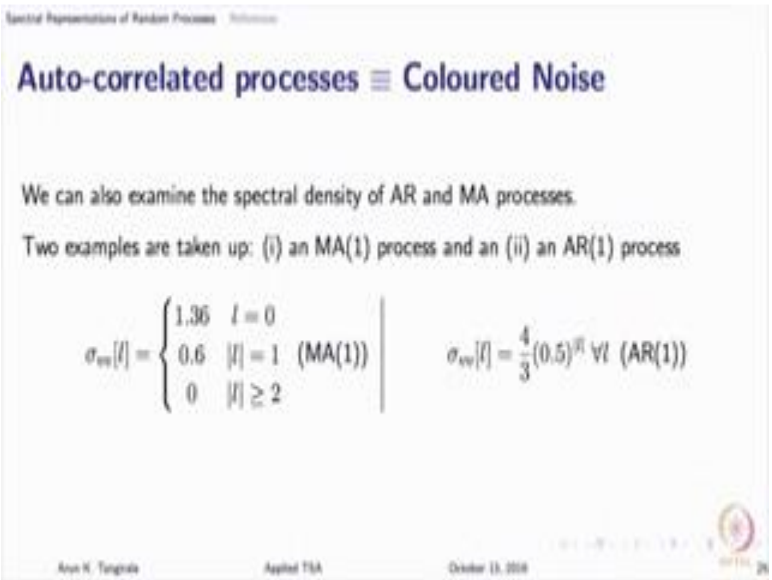
And then we turn to correlated processes and we said that these processes have a better name called coloured noise or an alternative name.

(Refer Slide Time: 06:19)



And the term coloured noise comes from the fact that you have non uniform contribution from different frequency bands. Again in analogy with colours like red, pink and so on. In fact, a lot of times in the literature you will come across the terminology red noise, pink noise and so on. And what they are referring to essentially is the frequency band over which predominantly you have the spectral density being significant compared to other frequencies.

(Refer Slide Time: 06:56)



And of course, the spectral densities that I show you here are only specific to this MA 1 and AR 1 process. The shape can change depending on the sign of the coefficient the magnitude of a coefficient and so on, and of course the order or the process, but the fact remains that the any correlated process will have a non flat or non uniform spectral density.

So, you can think of this as an alternative way of describing the random process. Although it is not so natural because what is natural is time, as I kept saying and I keep saying what is natural is not necessarily the most convenient and a classic example to motivate yourself for frequency domain analysis is the recovery of a sinusoid embedded noise for example, or any signal embedded in noise for that matter where you cannot really separate the signal from noise in time domain, but the moment you move to frequency domain this seperability is enhanced by leaps and bounds and that should give you enough motivation to look at frequency domain. Apart from the fact that so many communication devices everything is characterized in terms of frequency.

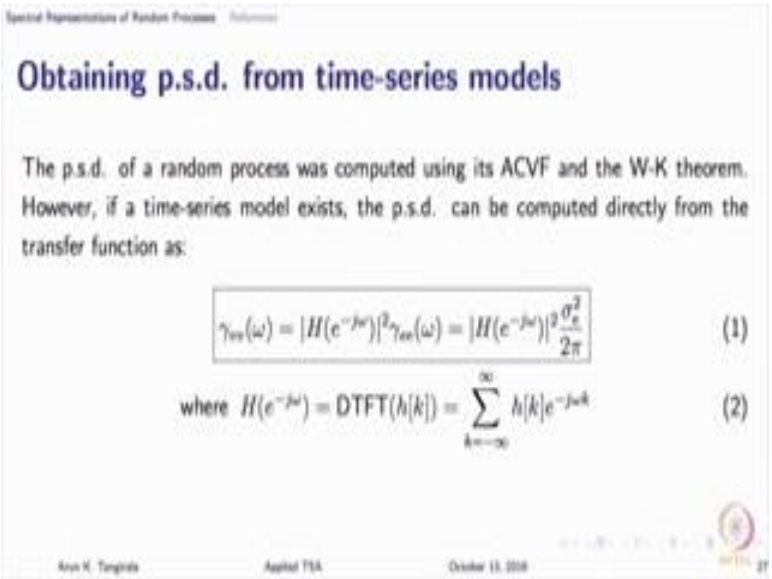
In fact, the other day when I was standing at the beach you may think I am crazy, but when the sea when the waves are hitting you standing at the beach you can actually count the time that takes for the big wave to come and hit you then the harmonic to come and hit you and so on. So, there is a certain physics associated with it. In fact, you can use your physics to predict how long it will take for a next wave to come and hit you the big wave and then comes the silent killer and so on. So, I was just reminded of the frequency domain analysis periodicity, essentially if you do not like the term frequency think of periodicities. So, everywhere you see frequencies really coming and saying hello to you.

Let us proceed now and unless you have any questions on this. So, what this relation tells us now is that given the auto covariance description of a process one can compute the spectral density, and spectral density gives us an alternative picture of the process. What we learn very quickly is; how we can derive the spectral density, given some other information about a random process. What do we mean by another description or some other information of the random process? One way of describing a random process is through the auto co variance function, although it does not allow you to recover the signal, but it gives you the statistical description.

What is an alternative way of describing a random process, any idea? So, you give the transfer function that is I give you the time series model, instead of giving you the auto covariance function I give you the time series model; can I derive the spectral density? Well, what do you think yes or no? At least for those of you to whom it is not so obvious given the transfer function we can derive the auto covariance function and from the auto covariance function I can derive the spectral density. But of course we are not going to take the truth, what we are going to do is we will look at a way that allows us to compute the spectral density directly from the transfer function. And that result is given by this.

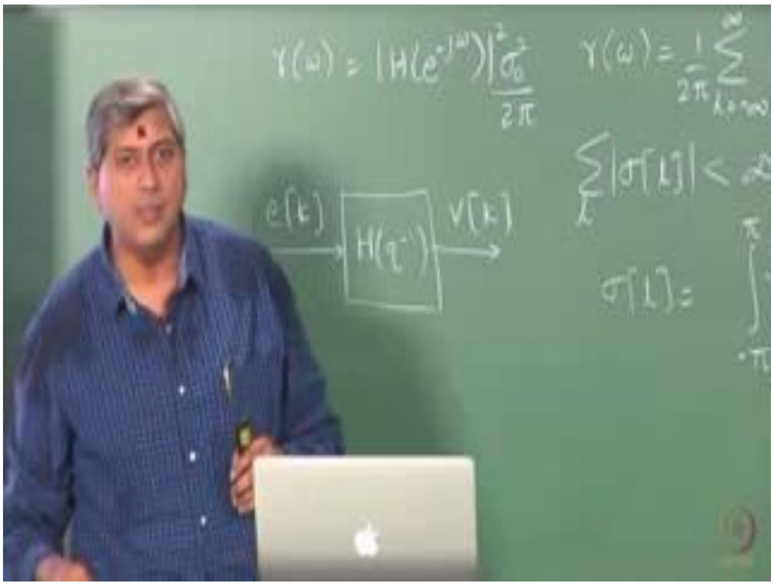
We have already seen this kind of a result before in the deterministic world if you recall. In the in the deterministic world we had a similar relation, but for energy spectral densities. Now what we have is a relation for power spectral density; that is only difference. Of course, that makes the big difference, but as far as the form of the equation is concerned it is not anything new and that is why I kept saying at certain points in the deterministic world that certain expressions will come and will appear in a different form to you, and this is one such expression.

(Refer Slide Time: 11:07)



So, the power spectral density can be computed from the transfer function operator using this expression.

(Refer Slide Time: 11:17)



So, how does this expression come about? Of course, one can look at this result in analogy with the deterministic process where there is a process whose transfer function is h or operator is h and whose input is white noise and output is the process of interest. And then one can use analogies to arrive at this result, but more than analogy one can actually prove rigorously. But I will just, before we do that let me point out what is h of e to the minus j omega is; it is nothing else but the frequency response function, exactly what we had seen in the deterministic world. In the deterministic world what is the interpretation of the f r f; it is a filter it tells me for example, if I inject a sinusoidal input of a certain frequency whether the process is going to attenuate it or amplify it. And that concept is extremely useful in communication devices, filtering devices, and so on.

What about here? Can I give the same interpretation to h? Yes I can, but is it useful because even in the random process world we have a concept of impulse response function that is the sequence h of n or h of k and then the frequency response function like in the deterministic world is the discrete time Fourier transform of the impulse response sequence. And we already know for the process to be stationary what is the most important condition I mean for it to be stationary and linear what is the condition that we had given on the coefficients that the impulse response should be absolutely convergent

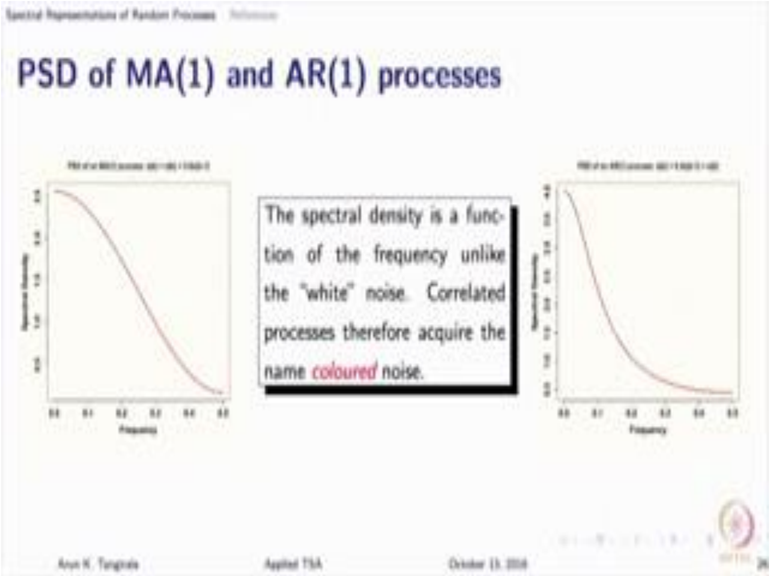
(Refer Slide Time: 13:21)

.

So, this guarantees that its Fourier transform exists right, discrete time Fourier transform exists which means a frequency response function exists. But the interpretation here that that interpretation we had for the deterministic world is not so useful because I am not going to be imagining a random process being excited by sinusoidal input, because what we generally think of is that there is a white noise exciting the process. You can nothing is going to preventive from imagining the process to be excited by sinusoidal input, but the useful interpretation is always the process being accompanied by a white noise input.

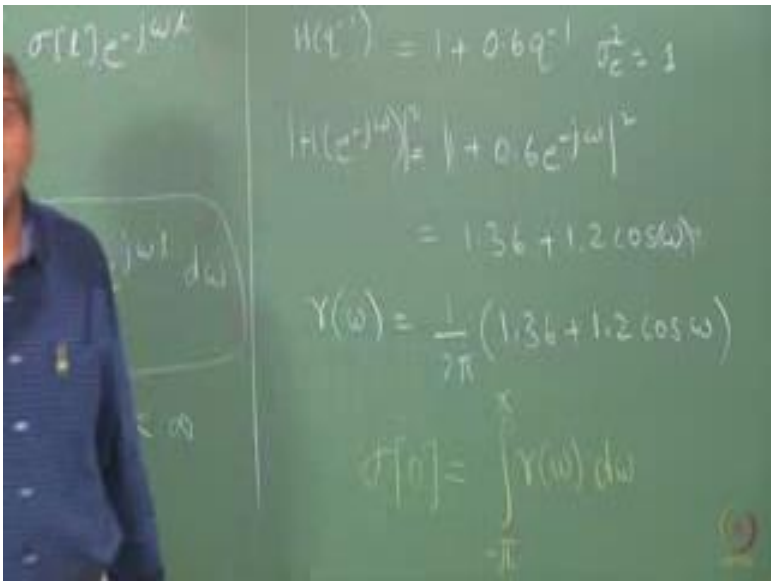
So, what is the use of the frequency response function here? It is used in arriving at of course the spectral densities, because then once I look at the spectral density then I can comment on the nature of the filtering effect that the process has. So, for example we had this MA 1 and AR 1 you take any of that.

(Refer Slide Time: 14:39)



So, if you were to take the MA 1 example.

(Refer Slide Time: 14:43)



Then we had 1 plus, what is the coefficient that we had any idea 0.6. So, we had 1 plus 0.6 q inverse in the previous example, did I give you that? So, I did not give you the 0.6, but if you look at the auto covariance you should be able to infer that the coefficient is 0.6 and sigma square is 1.

So, for his process we derived the spectral density starting from the auto covariance function. Now here, we can use this relation to derive the spectral density for the same process using this relation; is an alternative way of deriving the spectral density. The only difference is now the description is in terms of the transform function operator. So, very straight forward how do we arrive at h of e to the minus j omega simply replace q inverse by e to the minus j omega. For all practical purposes its suffices to remember that although that is not the best way of introducing that I mean ideally one would like the transfer function and then derive the frequency response function.

So, here you have h of e to the minus j omega and once you take the magnitude square what do you obtain, what is the answer? So good 1.36 plus 1.2 cosine omega; how do in your answer is qualitatively correct always positive you are sure, what else? Why is it always positive? Right magnitude is between minus 1, good. What else can you use to? What else is an important property of a spectral density function? Yes, it should be non negative value because it is a density function.

What else do we know of the spectral density? Regardless of the process it should have that property periodic, good. So, what is the periodicity yes it should be periodic what should be the period of spectral density for discrete time process. Remember that sampling introduces periodicity in frequency domain, so we are looking at discrete time processes what should be the periodicity of the spectral density? Does not matter energy or power spectral density does not matter both are periodic. See it should have the same period as the Fourier transform, in this case we arrived at the spectral density by working with the auto covariance function.

In the deterministic world how do you derive the spectral density? Of course, you can derive from the auto covariance, but is there an alternative way to derive a energy spectral density in the deterministic world. Take the Fourier transform of the given sequence and then take the magnitude square. So, what do you know about the Fourier transform of discrete time sequences about the periodicity, whether you look at that sorry its periodic with period 2 pi. We do not even have to go that far, look at this expression here, what is the periodicity of gamma? Its 2 pi, you should check if it is periodic.

What else? A very important property of spectral density for real value processes. What is it even means for the function symmetric, symmetric with respect to omega which means that the spectral densities cannot contain odd functions. So, that is another check these are simple checks for you to make sure that at least you are on track. If you have a gamma of omega that is not symmetric then it is not a spectral density function, it is some other function of your liking, but it cannot be a spectral density function.

And that also has you know connections with these symmetry of the auto covariance function everything I mean real valuedness, first of all the real valuedness has got to do with the symmetry of this sigma, but a far more reaching implication is the non negative definiteness of sigma. We will talk about that a bit later.

Anyway, so these are some simple checks any spectral density function should, if you say that I pick any function and I want it to be called as a spectral density; I arbitrarily pick some function and I say construct a random process think of a random process whose spectral density is this, then you have to carry out some checks: one that it should be real value non negative value, secondly it should be periodic, more importantly it should be symmetric. All are important, but symmetricity is kind of there it should be there, and then it is periodic. So, once you do those three simple checks and at least you know you are not wrong at least.

So, the other thing of course that you notice in this expression is it is the story is not complete we have to arrive at gamma of omega, and that is given by the magnitude square times sigma square e by 2 pi. So, the spectral density is 1 over 2 pi times 1.36 plus 1.2 times cosine omega. But that is exactly the expression that one would obtain starting with the auto covariance function. So, if you were to begin with this auto covariance function that you have on the left hand side and plug it in to this expression, you will get exactly the same expression. It should come as no surprise, obviously because everything is inter related. Any questions until now?

So, this is obviously given the transfer function operator, it is much easier provided you are not wrong with your basic complex numbers theory, it is much easier to derive particularly when you have an ARMA process for example. What is the difficulty with using the veneer tension relation directly when you have an ARMA process the auto covariance function is a bit complicated in fact a lot complicated. Whereas, if I am given the transfer function operator I can simply use this relation and straight away arrive at the spectral density.

What would be the form of spectral density for an ARMA process? Here for an MA process the spectral density is a polynomial in what, in trigonometric cosines correct. For an ARMA process what would it be? What would be the form of the spectral density? If there would be a numerator and denominator. So, it would be a rational polynomial again in cosines only in trigonometric functions, but only cosines and why do we keep saying cosines even because of the even nature of the spectral density.

And that is the difference between; in the sense when you remember your spectral density in general can be an arbitrary even that is symmetric periodic non negative function, it does not have to have always a rational form; that is I can have a random process that is stationary that satisfies this condition; that means, which can have a spectral density but need not necessarily be in a rational form. Gamma of omega can be anything, it does not have to be rational form all that we are saying is it should be non negative valued, it should be symmetric, and it should be periodic. And what else is required? Do you can you think of a forth condition for a spectral density? I will give you a hint I will give you 30 seconds and then see if you can actually think. Otherwise I will give you a hint (Refer Time: 24:06) that is understood.

Student: (Refer Time: 24:08).

Power is defined ok start from there; the hint actually is here and combined with the stationarity condition.

Student: (Refer Time: 24:22).

Area under? go ahead, area under.

Student: (Refer Time: 24:28).

One, why should be one? What is the area under spectral density going to give me?

Student: (Refer Time: 24:37).

Fine, but in terms of a time domain statistical property. Look at this relation what is the area under spectral density going to give? How do you obtain the area from this expression?

Student: (Refer Time: 24:56).

No. So, let me be more explicit then you straight away get the answer. What value of l gives you the area?

Student: (Refer Time: 25:07).

Right, and that amounts to area works out to be?

Student: (Refer Time: 25:14).

What is that? It is a variance right. So, for a stationary process the variance has to be finite correct, we have said that already. At least for second order stationarity we have said that one of the conditions is; of course mean should be invariant with time and bounded, but variance should also be bounded and auto covariance should only be a function of the lag. So, which means the fourth condition for the spectral density is that it should be integrable.

So, there are four conditions: one it should be non negative value, two it should be symmetric, three it should be periodic, and four it should be integrable. And through this discussion we know now that the area under the spectral density of course gives you the power, but it also then gives me the variance; which means, now I have a new interpretation for the variance. Variance of the process of the random process is nothing but its power. Until now we did not have this interpretation, but now we have this very nice and the sweet interpretation that the area under the spectral density is variance variant. We haave always interpreted variability to be a measure of the spread of outcomes and so on. So, what this tells us is more the power, more the variability and more is the range or the spread of the outcomes correct.

Likewise, we had an interpretation in the deterministic world. The area under the energy spectral density also gives me there the variance. So, whether you are looking at an energy pro signal or a power signal the area under the respective densities will always give me the variability or the variance. But of course, in the deterministic world the variance has got to do with what, what does the variance of an energy process tell you? Does it also tell you how far the amplitudes can go, yes or no? In some sense yes, the only difference is the variance for a random process is an ensemble property; at any time you are looking at the ensemble, whereas for the deterministic world variance is not an ensemble property, it is a property that is obtained by walking along time. So, that difference you should always observe.

And it takes some time and maybe some kind of imposition on yourself you keep writing when you have nothing else to do that this is the difference between a deterministic process and a random process. Fine.