Applied Time-Series Analysis
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Lecture – 72
Lecture 32B - Spectral Representations of Random Processes 2

There are essentially 3 or 4 steps in the semi formal approach, all of which are fairly intuited. So, let us take the first step.

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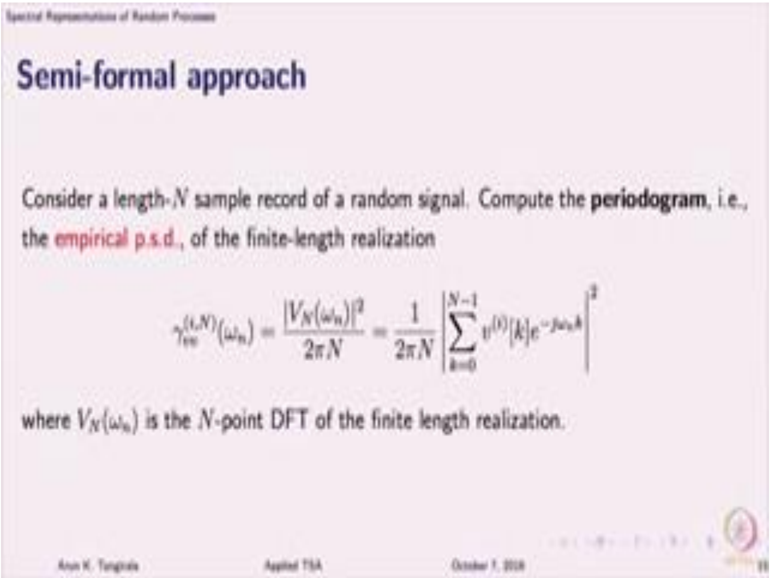

We said that we cannot think of a power spectral density in the usual sense for the random signal, because it exists forever, it has infinite energy and so on it is a power signal. So, what do we do? What we can do is we can start with the finite length realization that means; I may have 1000 observation or 500, 524 whatever I have n observations of a single realization of the random signal right.

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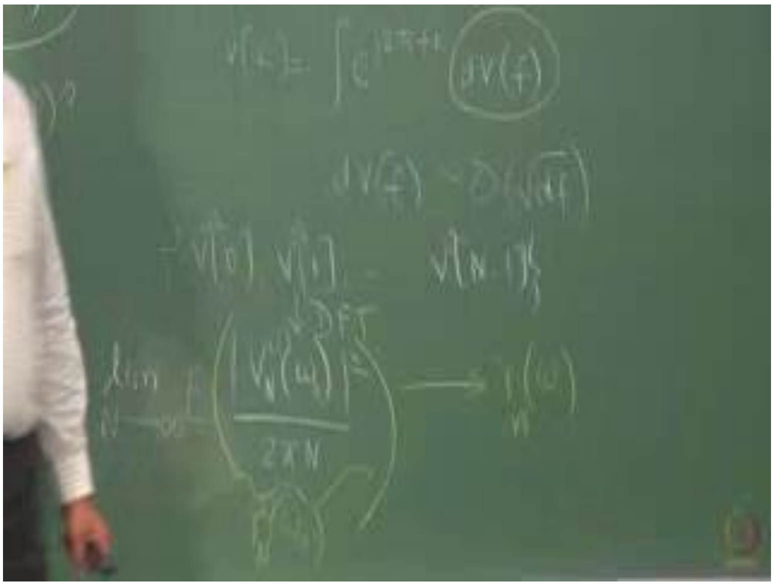

So, that is what I have with me, V 0 to V n minus 1, this is what I have in practice. Can I take the Fourier transform of this finite length realization that I have? I can; they nothing stopping me right it does not violate any of the requirements. So, we can take the Fourier transform and denote that by V n of omega we have switch into angular frequency now from cyclic, but we make go back to the cyclic frequency so.

So, V n of omega is simply what is it? It is the DFT we have already talked about DFT right, V n of omega subscript n. So, I can take the DFT and arrive at V n of omega n right and I can construct the empirical power spectral density like the periodogram, if you recall what is the definition of periodogram: mode V n of omega n square, that is subscript n indicates that I have used n, I have constructed an n point DFT from n observation.

So, I take mode V n of omega n square and divided by 2 pi n. I can call this as the empirical power spectral density for the finite length realization, until now things are ok, but now if I want to connect this periodogram which is giving me an empirical power spectral density, power per unit frequency to the process itself. What is the difference between the process and what I have here? First of all this is the finite length realization this is not even a signal realization in it is entirety. Secondly, the process is collection of realizations right. So, to be able to jump to the property of the process from the periodogram, I need to take two steps two further steps: one which is to extend this realization to it is infinite realization, this is a just finite length one and secondly then I have to look at all possible realizations.

So, if this is coming from the ith realization, then this is the periodogram of the ith realization. I am right and in fact, for the finite length part. So, now, first what we do is we ask what is the limit you can actually do to two things, you can ask for the example limit and then take the expectation or you can say no let me look at all the finite length realization and then let n go to infinity. So, you have to have some imagination not too much; your true process is a collection of infinitely long realizations and what we have is just a finite length realization.

So, somehow from this finite length I have to jump to the full process and in doing so I must guaranty that this periodogram make sense right? Right now for the finite length realization does a periodogram make any sense to us? It does right because it is just a empirical power spectral density. If I take the average of all such realizations, finite length realization let us say, would it still make sense? That would be the average power spectral density for all possible finite length realization and now when I let n go to infinity. So, here I have not taken expectation. So, let us put in the expectation here.

So, that is the first modification we say that now I compute the average spectral power spectral density, it is still empirical, because you know when I take DFT it assumes signals speed periodic and so on there is nothing like a power spectral density, you have to keep asking yourself why it is a empirical power spectral density? Because I had taken the DFT of this finite length realization and DFT assumes the signal to be periodic; so that is no notion of power spectral density at all, we have defined an empirical one and this expectation by taking the expectation what I am a doing? I am actually looking at average for the entire for all possible realize finite length realizations, but still this is not we have not gone close to the process completely, we have right there is one more modification that I have to make which is to let the length of the realization go to infinity, then when I do that I would expand the entire process. So, when I do this that is now the say limits evaluate this in the limit as n goes to infinity.

So, you see there is that is why this is semi formal approach, we are only using some reasoning to take each step. Now the question is does this limit work out to something meaningful, does it makes sense to take the limit do you expect that this will give you something meaningful, what happens when you let n go to infinity? The frequency spacing now becomes a continuum because until this point delta F of 1 over n or delta omega was 2 pi over n.

Now, when you let n go to infinity, the frequency spacing becomes a continuum and as a result, you through away the subscript on omega and you replace it with simply omega itself.

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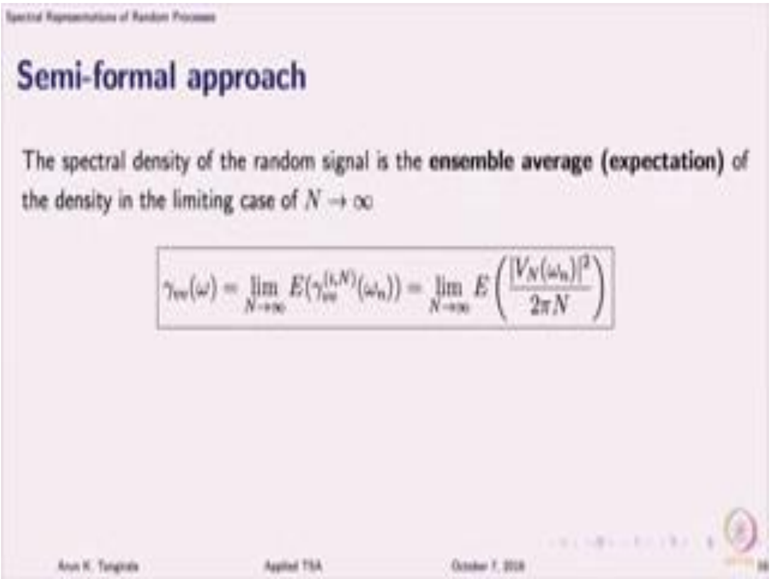

So, when you looking at this, this is you can say some gamma of omega n right for the ith realization inside and you can say that this is based on the finite length realization, but when you now do when you take the averaging and then evaluate in the limit, this now becomes are continues function of omega. What do you expect this to tell you now this gamma of omega? Look at the inner most one right this is telling us power per unit frequency, now this is for the expectation looks at all possible realizations and then the limit allows you to stretch to the infinitely long realization.

So, what do you expect gamma of omega to tell you? Will ask whether this limit exists or not and so on will come to that very quickly, but what do you expect gamma from omega tell you; what our the process what property is it revealing? It does it still have the flavor of a spectral density and what is the interpretation of spectral density? Here energy spectral density told as the contribution of a band of frequencies to the overall energy, we do not say of a single frequency it is a density correct. Likewise you should expect gamma of omega to tell you what band of frequencies contribute to the overall power of the random signal that is all. Can I ask this question this is where some depth of thinking is required and not too much, it is not to difficulty imagine, we I just now said gamma of omega can be thought of as it is a density function.

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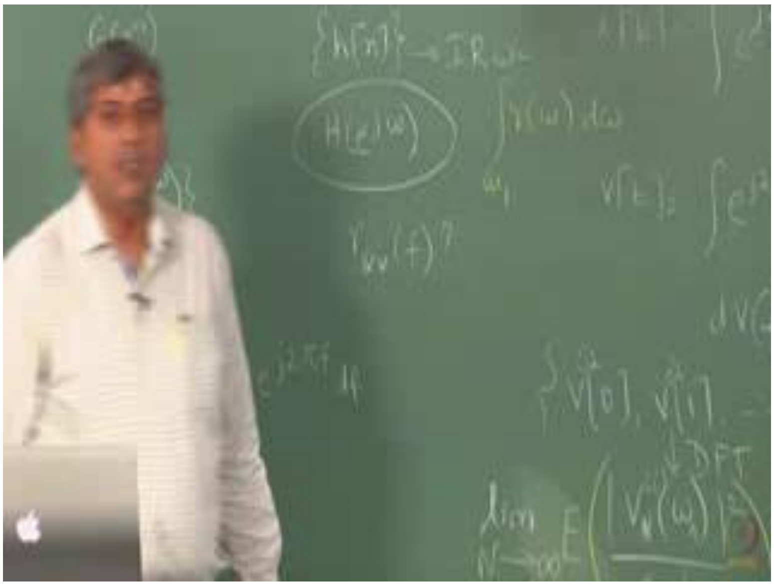

So, if I were to integrate this gamma of omega over up band omega 1 to omega 2 what would this integral tell me? It would give me the contribution of the frequencies in that band to the power of the signal of the random signal right, but does it make sense to ask to make the statement contribution of frequencies? Can we say that sinusoid and cosine are mixing up to produce V? can we is it still legal to imagine that way or not, can you can be imagine random signal being made up of sines and cosines yes or no, what do you think? The hint is something that we have discussed earlier, is it at least (Refer Time: 10:15) legal to think of a random signal being made up of sines and cosines; yes or no, then we should have been able to do take of Fourier transform right? But there was something else that we said when we were discussing Wiener's GHA the mixing co efficient; the building blocks I can always think of sines and cosines it is only the mixing coefficients now that a random variables.

So, it is still to think of the random signal being made up of sines and cosines, but the coefficients can change with the realization; think of it right that that is the main difference one main difference between the deterministic signal and the random signal world the mixing coefficient are now random have you just have to keep telling yourself although we do not deal with the mixing coefficients.

And second difference is we look at all realizations collectively, we do not look at the single realization, we are taking a expectation; when we defined auto covariance what it we do? We took an we looked at average property, when we look at mean it is an average property; when we speak of variance it is an average property, always for a random process it is about the average property not of the specific realization, it is of the all average across all realizations and that is exactly what is spectral density here too. The spectral density is also an averaged quantity and the interpretation is that the area under the spectral density gives me the power the contribution of the frequencies to the overall power of the signal a frequencies in that band. When we talk of random process and ultimately if you are able to one important take away from this course for you should be is, when you think of a random process it should always be the collection of signals not of a single signal. If you are able to bring yourself to that imagination, I think you have really understood at least one essential point of this course anyway.

So, this gamma of omega can be thought now as a spectral density, but the question is does this limit exists, how do I guaranty that this limit exists? You cannot simply take limit of it any function and assume it exists correct; we have already learn that a map many times.

So, that is an only thing that is remaining at least in the semi formula approach to establish the condition under which I can think of this spectral density. So, how do we arrive at this condition? All you do is you start with the definition of DFT and plug in. So, what you do is your right mode V n of omega n square.

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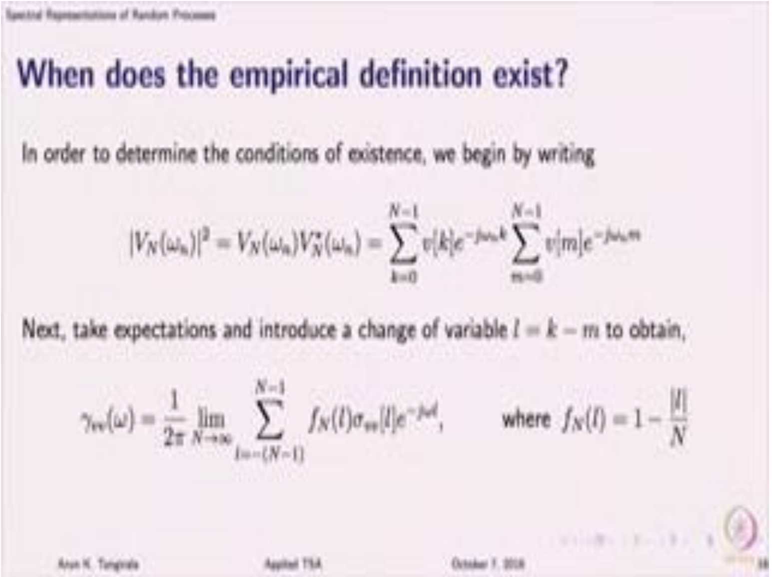

Use your standard DFT definition right that is all I am doing on done for you on the slide that you see right now, I have just taken the DFT n point DFT and mode V n of omega square is a product of the DFT and it is conjugate for that is simple complex variable calculus and what I have written here is rewritten gamma of omega in terms of now do you see the auto covariance appearing, how did the auto covariance appear?

First we wrote the mode V n of omega square in terms of the DFTs and when we do that and then take the expectation that is where the auto covariance appears right that is why the semi formal approaches very nice. It is straight away establishes connection between the spectral density and the auto covariance function, and you can say some school of thought uses approach to even arrive at the Wiener-Khinchin relation and we will do that as well, but the point that you should appreciate is through the semi formula approach not only were we able to construct the spectral density function, not in the very regressive but it is and two we are able to see now straight away the connection between this spectral density and the auto covariance function.

But the story is not complete yet; the stories why because we have not yet established a condition under which we can have gamma of omega, but we are just step away. So, you have this last equation on the screen 1 over 2 pi limit n going to infinity, the summation f of l which is dependent on n, times sigma l times e to them a minus j omega l. So, from here we can derive the conditions under which the gamma of omega exists and I have given the expression for f of l it is 1 minus mode l by n. In fact, if you were to breakup this into two summations, what would be the first summation? Remember f is 1 minus mode l by n correct.

So, if you take the summation forget about the limit and so on if just take the summation, you can write it as two terms what would be the first one.

Student: (Refer Time: 16:02).

That would be simple the DTFT of the auto covariance correct? And then you would have a second term; somewhere if you where to do the reverse engineering which we are generally good at, you should expect the second term to vanish for in order to arrive at the Wiener-Khinchin relation and that is exactly what a turns out; first of all let me write those two terms and then we can straight away spell the conditions.

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So, I am ignoring the limit you have l from minus N minus 1 to N minus 1. The first term is simply sigma of l e to the minus j omega l and what about the second term? The second term would be mode l by N sigma l e to the minus j omega l; clear is explain algebra there is nothing here, I would just taken a expression for f of l and then put in there. Now for the limit to exists first of all in that in the limit as n goes to infinity both the summations should converged; correct? What is the first condition what is it condition for the first summation to converge, what is the condition?

Student: (Refer Time: 17:43).

Sigma is absolutely convergent; auto covariance absolutely convergent correct and that is what we have assume and for the second one to converge of course, you have 1 over n here in the limit as n goes to infinity, what should happens? So, if I take 1 over n here remember that now there is a limit here as well, for this summation to converge already we have required sigma l to dk, when we see sigma l should absolutely convergent sigma l should dk, but we have not said anything about the dk rate, how fast it should dk will require further that it dk is fast enough so that this entire limit exists right? Because what is happening here there is a mode l although sigma dks this can actually dominate, if sigma does not dk slow fast enough correct.

So, what is required is that the auto covariance actually dks fast enough; if you ask enough in the since that it should dk faster than the rate at which l grows that is all right because n as anywhere fall in out of the summation, if this summation should not blow up and for that summation do not blow up we required sigma to dk at a much faster rate than or at a rate faster than the rate at which l grows.

So, there are two requirements one is that the auto covariance should be absolutely convergent that is a must, which means the auto covariance should dk. What is it means? It means that I cannot think of spectral density for any random process just like that; first of all I can think of spectral density only for random period processes that are not periodic. Secondly, it should be stationary one that is a anywhere given; among the stationary processes I can think of spectral density only for those whose auto covariance is absolutely convergent right. Are they stationary processes for which the auto covariance is not absolutely convergent? Yes and those are your periodic random processes.

So, the summary is that the spectral density is now the Fourier transform of I am sorry there is a 1 over 2 pi missing in the expression, I will correct that in the bottom summation, because if you recall there is a 1 over 2 pi write in front of the limit.

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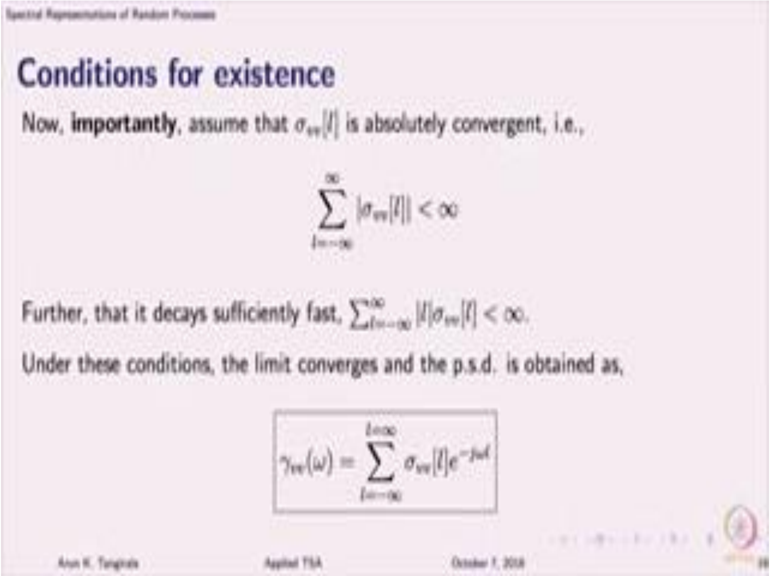

So, your spectral density is 1 over 2 pi times the DTFT of the auto covariance function, subject to the condition that the auto covariance function is absolutely convergent. Even if you do not understand any of this as long as you remember the fact that the auto covariance function and spectral density form of Fourier pair that is more than enough, which is what is Wiener-Khinchin relation.

So, we are striating think formally; we are saying that any stationary process, which has an absolutely convergent acbf can be now given a spectral representation; why do not we call it as a Fourier representation?

Student: Fourier representation.

It is still based on Fourier transforms right I mean Fourier analysis; typically we talk of Fourier representations of course, you can talk for any sequence you can say it is a Fourier representation of the auto covariance sequence. But we use the term spectral to explicitly say that the Fourier transform of the auto covariance function yields the spectral density, when I take the Fourier transform of any signal it is some quantity, it is mean it mean not be the spectrum, but here specifically the Fourier transform of the auto covariance sequence is a spectral density function and therefore, we call this as a spectral representation right.

So, if you remember that auto covariance function and the spectral density constitute of Fourier pair that is more than enough, at least you know to compute and the spectral density and so on. Now using this Wiener-Khinchin relation, we can construct spectral density for some well known processes; what I mean by well known is? What is the most well known frequently encounter stationary random process the white noise correct? The inevitable white noise process; whether there is correlation or not in a process defiantly you have the white noise components sitting there. So, straight away using this relation you can arrive at the spectral representation of white noise process.

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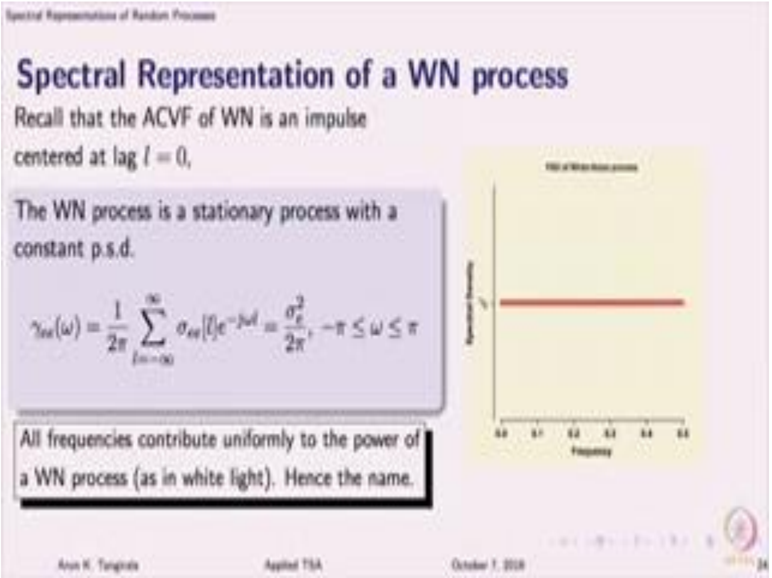

Simply apply the Wiener-Khinchin relation, remember we did not use Wiener-Khinchin relation to define the spectral density, we started with the fairly generic approach and then arrived at Wiener-Khinchin relation.

And now the Wiener-Khinchin relation can be used to compute the spectral density, think of it this way until now we have describe random process in terms of auto covariance. Now we are describing the same processes in terms of the spectral density, that is the - you know real natural things natural for you; straight away now you see what you see here on the left is the theoretical expression for the spectral density, what is it that we have?

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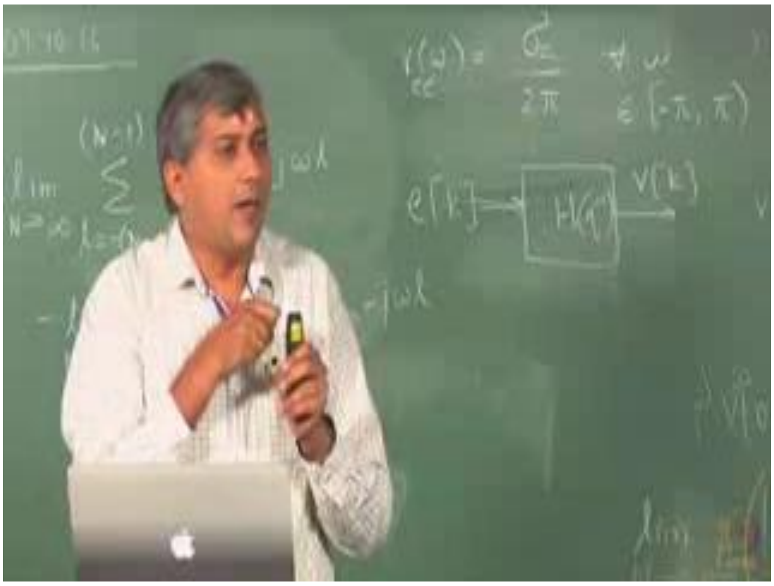

We have gamma of omega for the spectral for the white noise as sigma square e over 2 pi; remember we are writing in terms of cyclic frequency, is the spectral density sorry angular frequency. If the spectral density were to be expressed in cyclic frequency you would only have sigma square, but that is a minor point most important thing to observe is this is fixed for all omega, what is the interval for omega? Minus pi 2 pi right. What is it mean? It means all frequencies are contributing uniformly to the power of the random process, remember I said something long ago when we talked of energy densities and power densities energy and power, I said the energy or a power of a signal has to be related to the process that is generating the signal.

So, if I say a signal as high power, it automatically implies that the process has that power to generate the signal. So, here what we are saying is all frequencies are uniformly contributing to the power of the process that is generating white noise and in analysis with white light, which as uniform contribution from all frequencies; engineers call this as a white noise process. Statistian did not gives that name for the original name for this was the uncorrelated process ideal uncorrelated process, but then engineers said well will give it a name and called is at y as white noise and it made sense, because when you turn to AR and MA processes

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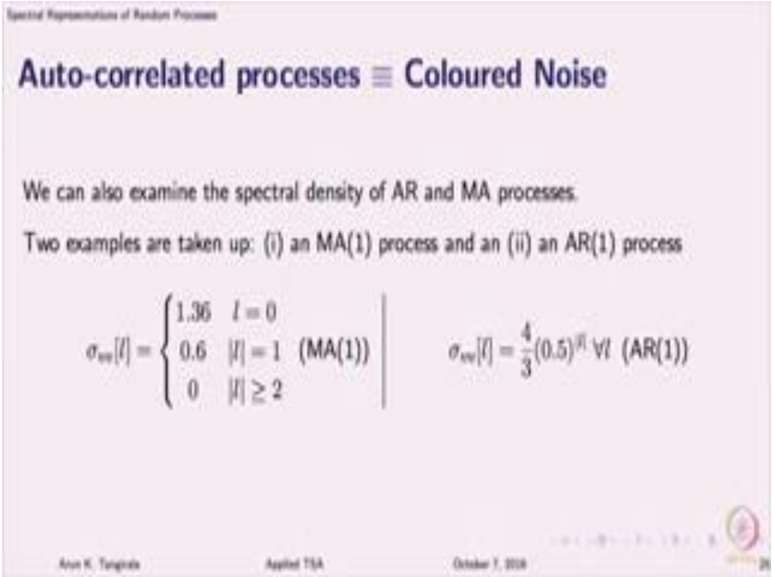

For example, if you take an MA 1 with some values for the coefficients on, you can get this auto covariance function we are now very familiar with that or I look at an AR 1 with coefficient 0.5, we know the auto covariance function is given by this when I look at the spectral density, what do you expect to see? Do you expect to see a flat one like this? no right what do you expect to see what kind of spectral density do you expect to see for these processes like this MA 1 AR 1, whose auto covariance is not an impulse? It will be anything, but not flat correct. In fact, it turns out that they look like this.

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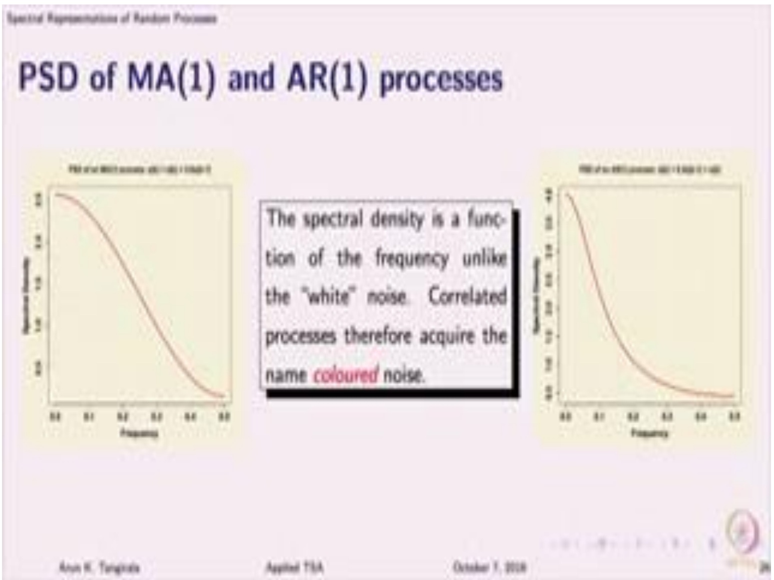

Of course this is specific for this ma one process, if the coefficient changes of the sin of the coefficient is different then you make get a different value correct, but what you should understand is any deviation from whiteness, what does it cause to the spectral density? A non flat one correct; which means now what is it tell you about the process itself, when you look at the spectral density what does it tell you about the process nature of the process? Not all frequencies are contributing uniformly, there not participating only a set of frequencies are participating in the generation of this processes and once again engineers called this as colored noise, why? Because unlike white light if I take a color red, green, blue whatever color I take we know from fix that the colors exists are associated only with the band of frequencies.

So, they said all correlated processes are colored process, where as uncorrelated once are white noise; based on this spectral density description. So, when I tell you it is a colored process, you should understand correlated process of course, further I may have to tell you whether it is a MA 1 or ARMA and so on, but the fact is any correlated process will have spectral density that is different from flat spectral densities, there is a function an R I will show you in the next class ARMA spec. If you recall we talked about ARMA ACF, what is ARMA ACF and R do? It gives you theoretical ACF; ARMA spec go back it is not an states packages in the TSA package, try out ARMA spec and of course it is a good habit to develop the expression for the spectral density by hand and then use R to conform that indeed you get that plot.

So, now what we will leave with today's class is, on one hand you have the spectral density of the white noise process which looks flat and then the other hand you have the spectral density of the correlated process, which gives as a you know to paint in the process, we said colored process; when do you produced colored light from white light how can you produce? Apply, right people do that in many hotel and parties I want to actually produced coloured one, either you have white light and you apply a color paper, stick a color paper on top of the bulb or within the bulb itself I mean with they there is mechanism of generating the colored light. Exactly the same interpretation now comes fourth; what we have learnt and we have come back in full circle to say that the any correlated process stationary correlated process, that has absolutely convergent auto covariance at least in this sense, can be given a filter representation right.

So, when I filter white noise, here I have written the signals but behind the seen what is happening filtering. So, the white light is actually passing through a color paper and what kind of color paper is it we do not know it depends on the process, ultimately produces as spectral density; that tells you what frequencies are present are contributing let us say not present contributing or participating in the generating process. So, when we meet again in the next class, what will do is will complete this discussion, will also talk of the spectral factorization and then put a close to the spectral analysis. Hopefully, you enjoyed the class.