# Applied Time-Series Analysis

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## Lecture – 71
## Lecture 32A - Spectral Representations of Random Processes 1

Very good morning. So, what we are going to do today is we are going to look at spectral representations of random processes. I use the word spectral representations rather than Fourier analysis or Fourier transforms; although implicitly we are going to base spectral representations on Fourier analysis. The reason we do not use a term Fourier transforms for Fourier analysis is of something that we have already discussed Fourier transforms of random signals do not exist; one has to really tailor it further simply because the random signal is not necessarily a periodic signal in the deterministic sense in which case any way we do not use term Fourier transforms nor is it and a periodic finite energy signal.

So, it belongs to the class of power signals in the sense not all; again random signals is what we are going to look at we are going to restrict ourselves to stationary random signals or stationary random processes and ask a question; how is the power spectral density defined, if it is defined can we come up with power spectral density. This question is natural because we have learnt how this kind of a view point helped does understand deterministic processes in a different manner; in a manner that is different from what we regularly understand in time domain and one of the most important take away is from what we have looked at learnt in the last 5 to 6 lectures is that I can look upon the LTI system as a filter.

So, the question is whether I can carry forward the same interpretation that is can the filtering interpretation to the random process as well and if I need to do that, what do I have to do.

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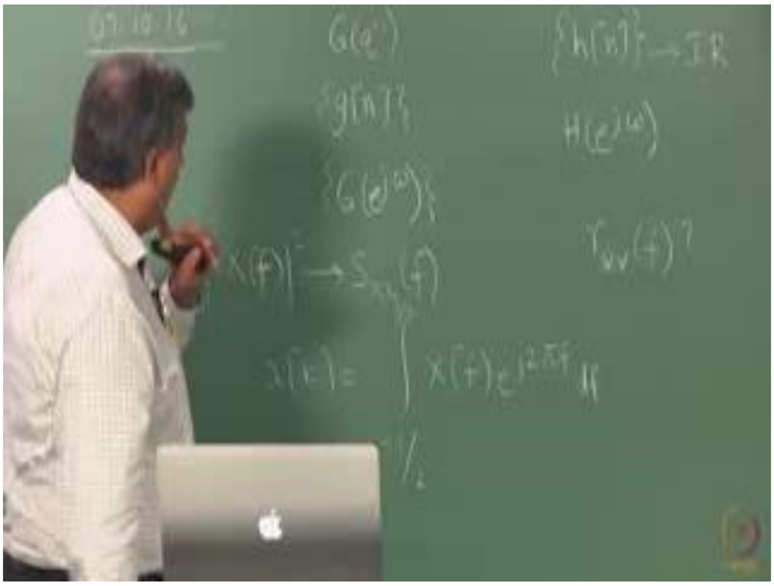

I know at least we know from pervious discussions and pervious lectures that I can give a transfer function operator representation for a linear random process. And I can think of describing the linear random process using these impulse response coefficients for the random process and all of that and for the deterministic process we could give a transfer function operator representation or an impulse representation or a frequency response representation or description.

All of this, each of this representation has its own advantage G of q inverse gives me the transfer function operator representation, allows me to represent the difference equation in a compact way. Then I have the impulse response description, we know that by looking at the impulse response I can draw some inferences about the properties in the such as stability and causality and few other things and the frequency response function which is the object of discussion today, is it gives me a filtering prospective of the deterministic process. It tells me what frequency is the system attenuates amplifies and so on and that is extremely useful not only in communications, but in almost every other applications, even let us say if I want to model the system, I want to conduct experiment to collect data and I want to identify from data not from first principals then I have to excite the system with certain frequencies.

The frequency response function gives me some idea at least in fact, a fairly good idea of what kind of inputs I want to excite the process with. If it is low past filter for example, then there is no point in exciting in the system in high frequencies because I would not get any response at all, pretty much analogous to what we do in many interviews for PhD and MS positions; we ask the student what are the areas that you are comfortable with.

So, essentially you are asking the band width, so that I do not ask any question to which candidate would not respond and so, this f r f is very useful there. Likewise here, can we think of a frequency response function in the random signal world and if we can then what utility does this particular quantity has, frequency response function has. What does it allow me to calculate and as I said early on today, in the deterministic signal world we talked about the energy spectral density for a periodic energy, finite energy signals and of course, power spectrum for periodic signals; can I hear; think of a similar function of course, this time it is going to be power spectral density; is it possible to define such a quantity. And we should not be wondering by we are asking this question because we already know in the case of deterministic signals, the energy spectral density can be obtained by first computing Fourier transform and then taking squad magnitude.

So, that it was easy to construct and Parseval's relation essentially allowed as to give the energy density interpretation, that is the key. We started off with Fourier transform and then invoked Parseval's relation and then argued that this is energy spectral density. Unfortunately that root is kind of blocked if you just want to take the Fourier transform of the random signal and then think of Parseval's relation; no such root exists. So, that there are two questions, can I think of power spectral density still despite the fact that I cannot Fourier transform and if I can then how do I computed it because I cannot compute in this manner and we already have some hints from the deterministic world which is that; I can start, I can go by the auto co-variances root. So, even if one path is blocked there is another path that is opened to us which is autocovariance root. So, that is what is the (Refer Time: 06:49) of today's discursion.

We are going to link everything, we are going to first talk about power spectral density then ask about how to compute it and then relook at white noise and ARMA processes in the contest of spectral density. In that process we learn the very famous and popular Wiener-Khinchin theorem and then also ask how is the power spectral density related to this and again we have a lot of hints from the deterministic world.

So, let us get going now through the formalities I am going to skip the opening remarks, we have already, I have told you enough number of times that Fourier transforms of random signals do not exist in general and within this random signals you have a periodic signals and also periodic signals right. Now, on the face of it; if you here the term periodic random signal, it sounds bit strange to us. On one hand we say that random signals are not predictable, I mean accurately predictable and on the other hand when we use this qualified periodic; it kind of gives you a connotation that may be itself after a while, but it is not your usual periodic in the periodic sense that we call this as periodic; what I mean by usual is in the for deterministic signal, we say it is periodic if it values repeat themselves after a certain sample or time as a case may be.

Here, that is not going to be the case, for periodic random processes we do not define periodicity based on the values that they take we know that when comes to random processes; it is all about ensemble properties average properties. Therefore, periodic random signals would be defined based on their statistical properties rather than the signal density, but will come to that no worries.

Let us deal with the aperiodic signal first and ask how a power spectral density is a defined, if a power spectral density can be thought of and you have to keep telling yourself one of the main reasons why we are asking this questions is again this root does not exist for sorry; I wrote here exactly (Refer Time: 09:21) this root here does not exists for the random signals. So, what do we do? How do we even think of a power spectral density? As I have said earlier on the we were able call this as the energy spectral density because of the Parseval’s relation that we have and before we headed to Parseval’s relation; we saw that the Fourier transforms of the signal that has to be taken and then you could apply the Parseval’s relation.

So, now what do we do we do not have such an option, it turns out because this problem was studied a long ago; it turns out that there on paper at least three different ways of arriving at the power spectral density of which one is the most regress one which proves that you can think of a power spectral density provided some conditions are satisfied; one condition is stationarity; you will have that actually assure that the process is stationary. And secondly, we already know the condition at least through the Wiener Khintchine relation we have been seeing that if there is power spectral density it is going to be the Fourier transforms of auto covariance function.

So, we can think of a power spectral density only when the auto covariance function is absolutely convergence. So, under these two conditions we can arrive at power spectral density. So, of the three different methods to arrive at a spectral density one is most riggers and perhaps the most well known, but required some advanced (Refer Time: 11:13) such as stochastic integrals and so on and that is your Wiener's generalized harmonic analysis.

Basically what Wiener did was he said look you cannot define a Fourier transform or you cannot give a Fourier representation, what you mean by Fourier representation is here in the deterministic world if you recall; we have this synthesis equation, this is called the Fourier representation of the signal the deterministic world, whereas we do not have such a luxury here. So, what Wiener proposed if you recall in this deterministic world, we introduce what is known Fourier-Stieltjes integrals which combined both the periodic and aperiodic case.

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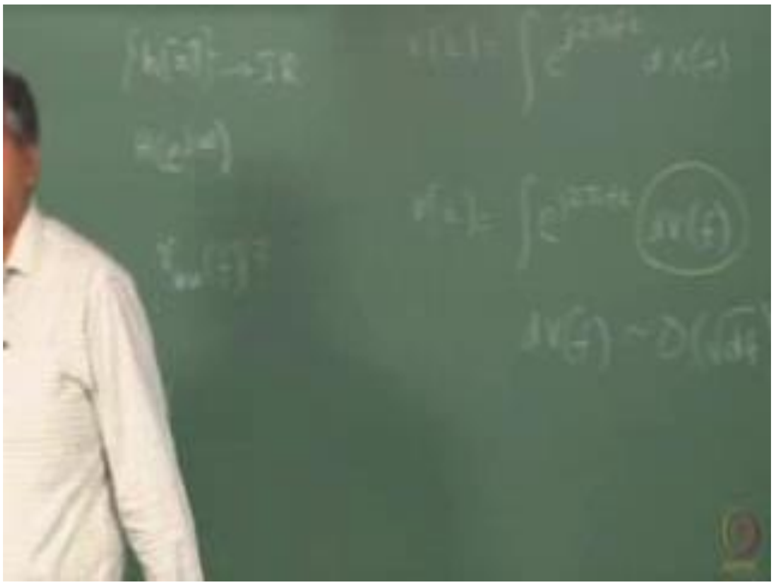

If you recall we had return for the deterministic world; x k we remember what we wrote for Fourier-Stieltjes integrals which fused both the periodic and aperiodic cases; e to the j 2 phi f k times d x of f that is it right and d x of f takes two different forms depending different form depending on whether x case periodic or aperiodic; if x k is periodic then d x of f has an impulse like structure.

And if x k is aperiodic finite energy then d x of f can be written as x of times d f; what Wiener did is essentially extended that idea to the random signal world, but it is not so easy right I mean looks pretty straight forward. Now what he said is take a random signal and then you could represent this also in the same way as the deterministic signal. So, this is Wiener's idea, where the notation is kind of obvious, but there is a huge difference in the top signal which is for the deterministic signal world and there is top equation which is for the deterministic signal and bottom one which is for the random signal and that difference has got to do with how this quality behaves.

Now, we just now said if x k is a periodic, we can write d x of f as x of time d f which is what brings us to this equation, what Wiener showed is essentially for the stationary random signal d v of f is proportional to and perhaps beyond this will not proceed, we showed that proportional to order of square root of d f; it is of order of square root of d f. Now it is hard to interpret beyond this and there is yet another fundamental difference and that fundamental difference is that since x k is deterministic; d x of f also deterministic quantity, but here since v k is random, d v of f which is increment in the Fourier transform if you think of Fourier that is also stochastic.

So, here d x of f this d x of f that is it; it is a increment in the Fourier transform when in the frequency domain. Since x k is deterministic that increment is also deterministic, it is a deterministic quantity where as v k being random d v of f is also a random quantity and therefore, although symbolically these two are integrals, this is a deterministic integral that is your regular integral and where as this is what is known as stochastic integral and that is why I said the Wiener's generalized harmonic analysis, why does a call generalized harmonic analysis is because it is fairly generic and also he uses the term harmonic.

The reason for which will be become obvious bit later on, but what Wiener had contributed through g h a is essentially a Fourier kind of representation for random signals stationary random signals and beyond this, I would not go further we will take a different route to arrived a spectral density. There are you can look up either my book or any times series analysis book to see the full details of Wiener's generalized harmonic analysis here where further properties of d v f are given and then ultimately how do you arrive at the spectral density starting from this integral.

My book does not carry the full details, but it gives you essentially the salient once, but if you look at other text books like such as the time series theory and modeling by Brooklyn Davis or even by the book by Presley and so on. There is much more, there are many more details that show you how to arrive at the spectral density, gamma of f; starting from here and also what are further properties of this increment in the Fourier transformation.

So, if you do not understand any of this; think of it this way, when we started of furrier series; we said essentially I take a periodic signal and I express it as a linier combination of sines and cosines correct; that is something that we understand very well now. The coefficients of your expansion, so if you recall in a periodic signal we had a n and b n or we had c n.

Now in the random those coefficients told us how much in the sign and co sign you will require to synthesis the given signal. When you move to the random signal, you can think of the similar situation but with the change; the change is now the coefficients of expansional no longer deterministic, they are random variables, that is how you can think of; that is what Wiener's idea essentially leads to. Basically what he did was he said look I can still use the sines and cosines, you still see e to the j 2 pi f k, but if you think of this as some kind of a coefficient here it is deterministic whereas here it is random.

So, what you are doing is; you are taking sines and cosines, but for one realization you use one values of a's and b's or c's; another realization you use another and so on and for the entire ensemble you need an ensemble of your mixing coefficients and therefore, they are random variables and so on and that is the basic idea that you can keep in mind in always when you move from deterministic to random world. I will repeat in the deterministic world we think of the signals as being mixture of sines and cosines with a mixing coefficients being deterministic.

In the random signal world also I can think of the same thing, but now the mixing coefficients are random variables that is the prime difference does not matter whichever root you take; ultimately you will be led to this interpretation and that interpretation will become a lot more clearer; when we go to the world of periodic random process at this movement you may not appreciated that much.

So, we will skip this Wiener's g h a root and take what is known as a semi formal approach to arrived at spectral density. The semi formal approach is a lot more appealing, intuitive, it is not its rigorous, it does not have the mathematical rigor that you want; however, it has set of appealing elements in it so that you can immediately relate to the power spectral density based on what we have learn until now. So, that is a first approach and we will take this approach.

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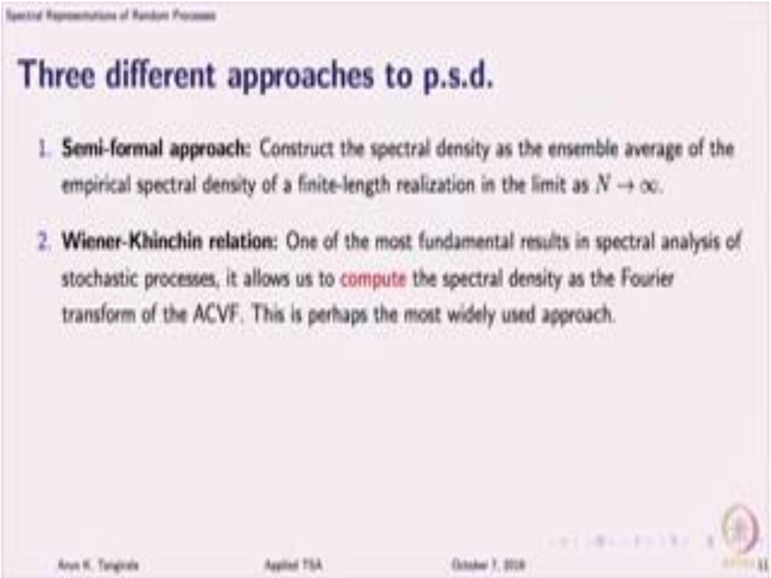


The second approach that is normally presented in many text books to arrive at the spectral density is a Wiener-Khinchin relation and many text books go a step further to say that spectral density itself is defined through this Wiener-Khinchin relation and that should be taken with a pinch of salt or sometimes with the truck of salt, it need not the spectral density is not necessarily defined through the Wiener-Khinchin relation, but there are some who may argue know that is how the definition originated. Strictly speaking, if you want a rigorous definition of power spectral density; you should take the Wiener's g h a root why? Because that is what we did for the deterministic world, we took the Fourier transform and then we went through the Parseval's relation.

So, the equivalent of Fourier transforms; a Fourier representation in the random world is the through the g h a through the Wiener's integral there and then you should proceed to arrive at the spectral density, it does not matter whether you do not want to debate all those things. Ultimately in practice you would use Wiener-Khinchin relation to arrive at the spectral density. At a later stage we will realize there is yet another way of computing the spectral density, which is through this right.

Essentially remember when it comes to description of random processes, I can be given the auto covariance function which tells me what is the correlation structure or I can be given the time series model, depending on what you have in hand typically in practice you will have data. From data if you want to compute power spectral density, you will realize soon there are two roots; one root is to compute the auto covariance function and they take the furrier transform and the other root would be to build a time series model and then use the time series model to compute the spectral density.

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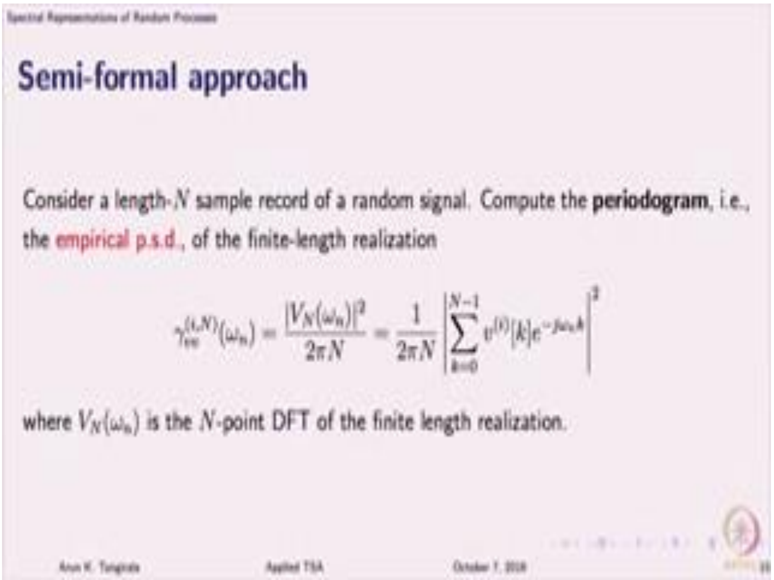


So, all of this will fall in place once we go through the discussion, we going to skip the third one which is a g h a and now turn to the Semi-formal approach to arrive at the notion of a power spectral density.