Applied Time-Series Analysis
Prof. Arun K. Tangirala
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture – 68
Lecture 30 B - DFT and Periodogram 1

Alright, so now we move on to one of the most fundamental properties of Fourier transforms which is perhaps the most widely used in the entire signal analysis and signal processing field, which is that the Fourier transform of a convolution operation in time. You should start understanding now, when I am operating, performing some operation in time domain; behind the scene some operation is happening the frequency domain and we have no say on it. There is are invisible thread that is connecting the time domain and the frequency domain and these properties are telling us what connections these threads have and what kind of repercussions you have in the frequency domain.

So, these properties specifically says if I convolve two signals in time domain; when do I run into this kind of an operation, when do I run into convolution operations; in all linear time in variance systems, when I excite the system with an input; what is the system doing, it is actually convolving the input with its impulse response and producing the output. So, any linear time in variance system, any linear filter when it is exited by an input, it actually performs this convolution operation and this property tells me that in frequency domain what is happening is a product operation. So, convolution operation in time domain translates to product in frequency domain.

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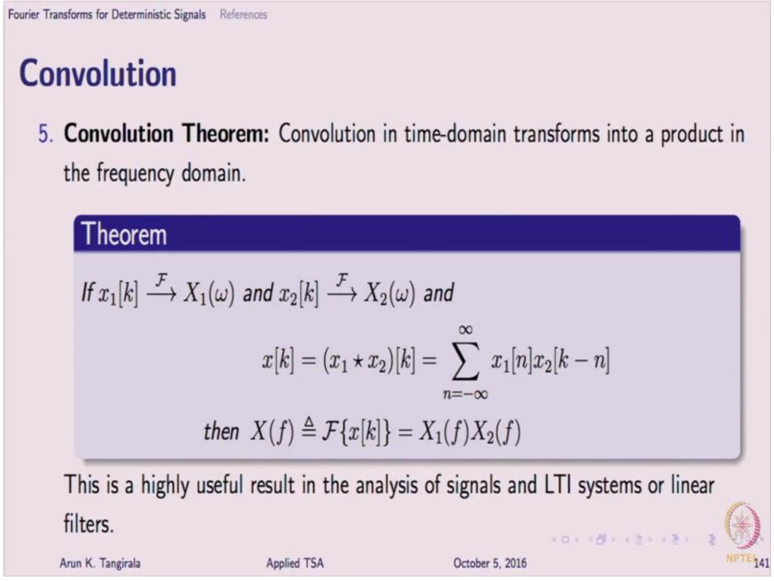

Now, the beauty of this result cannot be explained in 5 minutes, but at least you should appreciate the fact that convolution operation is somewhat complicated operation in time domain. You can see it is not a straight forward product, where as the equivalent in frequency domain is a very simple algebraic operations, which is simply a product alright and the other thing before again we dwell on this other thing that you should understand is what Fourier transform is essentially doing for you is, it is collecting all the information in the signal over the entire time from minus infinity to infinity and shrinking it to one point and frequency domain which is at f.

We had a expression earlier for d t f t you seen it before what is doing; it is summing up the signal over the entire time. So, that is another way of looking at Fourier transform that your really collecting all the features of the signal; over the entire existence and shrinking it to a single point and frequency domain and that perspective helps us in understanding this of course, proving this is pretty straight forward you can actually just sit down in 2 3 steps you can prove that this result holds, but getting a perspective really helps.

So, what is the convolution operating operation doing; it is multiplying two signals not in a straight forward way, it is actually reflecting one of the signals there are in fact, four operations involved hidden in a convolution operation and what are those four operations you can see at least you can name one is reflection of the signal right and then shifting it and then multiplying those two signals and then summing up.

So, those are the four operations involved in a convolution operation and therefore, from a computational view point, it can be heavy on the computer whereas, in the frequency domain it is simple product of course, you may argue who will give me x 1 of f; to go from x 1 k 2 x 1 of f I have to again do a computation, but computationally efficient algorithms exist for computing Fourier transform. So, you can make use of that in fact, towards the end of today's lecture or let us hope that we reach that I will list the commands in our that do that are relevant to what we have discussed until now, what we have learnt until now and one of them is going to be convolution and this convolved routine in our implements calculates is convolution by going in to the Fourier domain, that is computes the Fourier transforms of those two signals, multiplies them and then does not involves Fourier transform.

That is suppose to be a lot more efficient then straight forward convolution operation in time domain and in fact, almost all the routines, in all software packages even if you take MATLAB and so on, the convolution operation is implemented using this property, but that is as for as computation of convolution is concerned, but the use of this property is a lot more particularly in theoretical analysis of linear time in variance systems. We have seen yesterday, we have used this property to understand what the LTI system does to an input in the frequency domain; that means, it gives us insides into filtering characteristics, when I look at this result in a context of linear time in variance systems; I can think of x 2 as the impulse response and x 1 as the input or vice versa does not matter and x being the output.

This result tells me that that linear time invariance system is somehow altering the frequency content of the input and what is responsible for altering the frequency content g of f and plotting mod of g of f versus f tells me, gives me a lot of valuable information about the filtering nature of the system and we will also discuss is briefly in the context random processes and then you will understand what is mean by white noise and coloured noise and so on right. So, we will move on and look at the duel of convolution you should expect product in time domain, corresponds to convolution in frequency domain that see beauty again of the duality and finally, we look at this correlation theorem which is nothing, but the equivalent of Wiener Khinchin theorem that we will learn in the stochastic world, this is the Wiener Khinchin theorem version for the deterministic world, it says that the Fourier transform of the cross variance; a signal process in people would like to call is a cross correlation, I have already caution you on that; that is why it is called correlation theorem.

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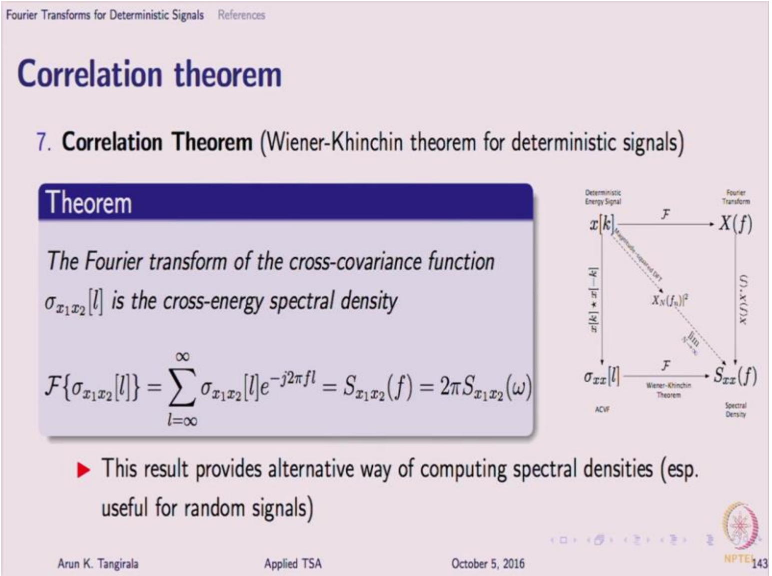

The Fourier transform of the cross covariance again assuming x 1 and x 2 are finite energy periodic signals is nothing, but your energy spectral cross energy spectral density. So, the figure on the right shows how you can obtain the cross energy spectral density in different ways for example, I am only showing this for auto energy spectral density not the cross energy spectral density, but the same applies to the cross 1 as well. To arrive at the auto energy spectral density for example, I can actually take the Fourier transform and simply take the squared magnitude that is 1 route..

And the other route is to take the auto covariance and take the Fourier transform right and then of course, there is another route which will talk about later on, but those are the two different routes that you can take, the one that you see here which is the Fourier transform route to arrive at the energy spectral density is fine for deterministic signals and I have been saying this earlier also; for random signals this route is closed if I mean ah if you think of this as a power spectral density of a random signal, this route is not as straight forward as it appears and in fact, to adopt this route one has to turn to what is known as generalized harmonic analysis that was introduced by wiener if you want discuss it at all, I will just briefly mention; we will not use the generalized harmonic analysis route. We will primarily use this route that is we compute the auto-covariance and take the Fourier transform and arrive at the spectral density there.

So, you should now again appreciate that this result unifies the world of deterministic and stochastic signals, but you should be cautious in the world of deterministic signals we are only talking of energy spectral density, we have never talked about power spectral density because in the deterministic world periodic signals are what we are looked at for power signals and periodic signals do not have a density they only have a distribution and the moment we move to a periodic signals, we say for the Fourier transform to exist it needs to be finite energy therefore, you have only talk of energy spectral density.

So, there has been no scope for discussing power spectral density at all in the deterministic world, the random signal world will offer that opportunity. So, that kind of concludes the most important properties that is the huge list other list of properties; we do not have to worry about those, these where I thought the most important once for this course usually one finds the table of properties you can find them everywhere; almost everywhere right except may be at Amazon and so on, but everywhere else you will find. So, now, we turn to the practical implementation of d t; f t, how having learnt so much we have also learnt that d t; f t is extremely useful in theoretical analysis, but what about practical signal analysis and we ask two questions in this regard yesterday.

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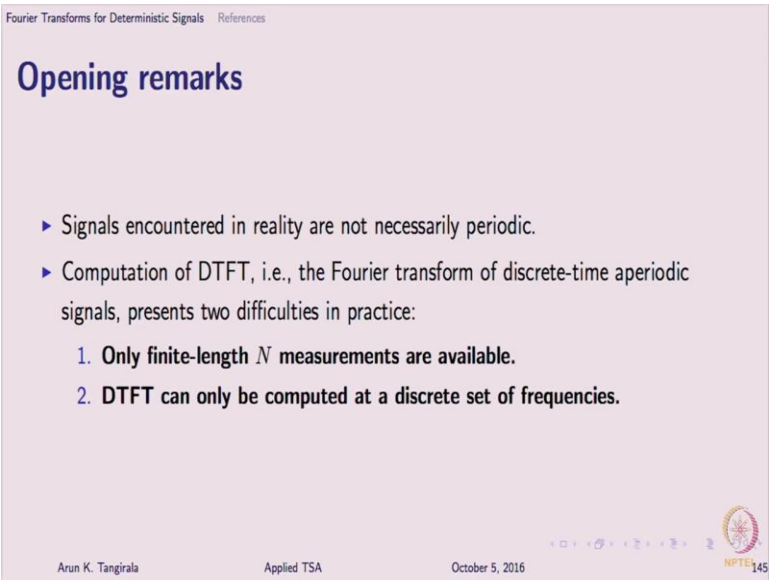

One is that the signals encountered in reality are first of all finite in length and they are not necessarily periodic of course, right that is the first issue; I have only finite length measurements and secondly I have a computation issue, I can only compute or a grid of frequencies. So, these are the two issues that we have, so keep the d t; f t in mind and ask this two questions.

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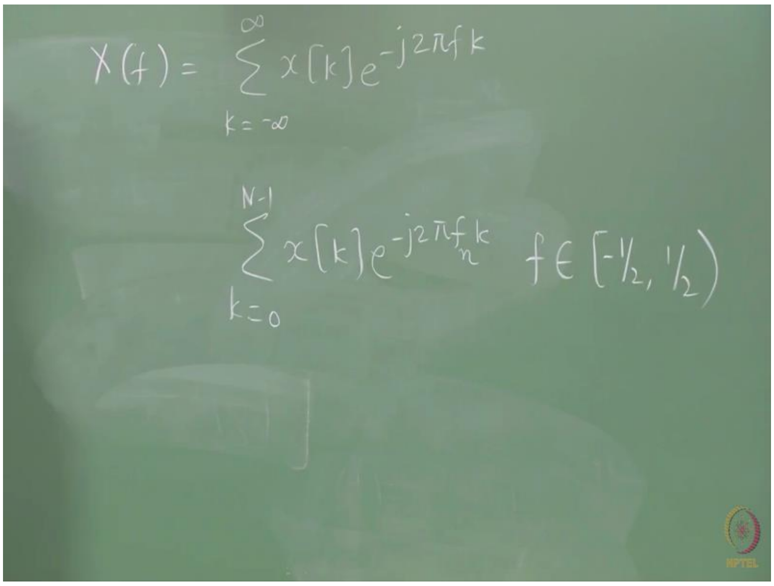

So, x of f theoretically is x k e to the minus j 2 pi f k; k running from minus infinity to infinity. Now I want to compute x of f for some signal; at the moment do not worry whether let us assume that the signal is still deterministic, let us not worry about the randomness in the measurements and so on. So, I am presented with those two issues; how do I handle these 2 issues. Can I for example, only evaluate the Fourier transform over the length of the signal that I have that is a natural idea that comes to mind right. In other words can I truncate this summation to the duration of the signal that I have; is it to do that or not will a get me exactly x of f; it will not right.

Let us pose the question other way round; given the finite length signal that we have can we somehow reconstruct the infinitely long signal; is it possible, it may be an approximation that is, but is there a way to do it; what are the different ways in which I can reconstruct. See essentially what we are asking is; I have only observed for n time instance, I do not know what was happening in the signal before I started observing and after my experiment. Now can I do some kind of imagination and reconstruct the infinitely long signal.

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That is one, so essentially now what we are getting in to is extensions of signals beyond the observation period; one is a periodic extension. So, you assume that the signal is periodic, is it a fair assumption very bad isn't it, what is the other assumption that we can made that it was 0 before and after; is that a good one that is also not good, what do we do; some extension you can. So, you can think of this problem as a problem of reconstructing the infinitely long signal with the given finite length that is one way of looking at it, the other way of looking at it is you say that I am only going to work I do not know what happened outside well this is essentially what we are going to do, but I am only going to truncate this summation to the length of the duration.

Effectively you are saying the signal is 0 correct, but then we have this another issue which is computing this or a calculator or computer and we can only do this over a grid right. So, now, we say that I cannot compute at all f in this interval, but I can only compute at discrete points enough therefore, we now introduce discretization of f.

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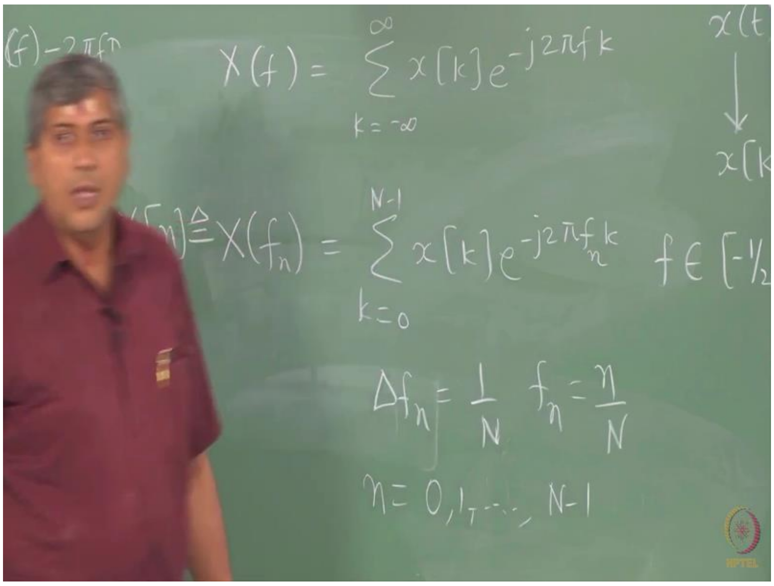

And now call let us say this as x subscript f n; do you expect x of f at n th point in frequency to be identical to this it will not right. So, I may need some reconstruction if I am interested in the x of f the continuous function, this is not a continuous function in the sense it is the domain here is now discrete. In fact, we will use a notation x of n like we have used x of k for discrete time signal, now you should remember that k denotes time instant and n denotes the frequency point grid point.

So, we introduce x of n which is nothing, but this Fourier transform; this is the finite length Fourier transform evaluated over a grid. We have not made any decision on what should be the grid size, so what we have done is two things we have truncated this summation and we have discretize the frequency axis. Now let us see what is the consequence and by the way this is what is nothing, but is called the d f t; the discrete Fourier transform.

Now we drop the tea in d t f t because now both time and frequency axis are discrete; in d f t only the time is discrete and this is what your f f t algorithms implement; this d f t. This what people use widely, you may wonder how can I use this what warrants usage of this, what does it mean to work with this kind of a transform this called by I mean we will come to the n point d f t shortly, but essentially now we have to ask before we start using this; what is the meaning of using this transform as against working with this.

I want this, but I am working with this and more over I do not even know if I should use d t; f t or d t; f s because I do not know a priori whether the given signal is periodic or aperiodic. Until now we have studies different classes of Fourier transform, Fourier series and so on, assuming that I know a priory whether the signal is periodic or otherwise, but know we are really touching you know where coming face to face with reality and saying I do not know a priory with the signal is periodic, I do not know if I do not have infinitely long signal and I cannot compute over continuum of frequencies and so on.

Obviously, this is the most practical think that you can think of, but before we use this, we have to ask what does it mean with respect to the correct one that I should be using, if the underline signal is periodic; what is the kind of transform or analysis that I should be doing; Fourier series, I should be computing Fourier series coefficients and in the (Refer Time: 18:03) signature a periodic I should be working with d t; f t, but I do not know any of that.

All I know is I can compute this and this is what my a (Refer Time: 18:13) does for me will briefly ask two questions; one how should I choose the grid spacing and two; what is the consequence of working with this d f t with respect to the actual one that I should be using. So, let us ask actually the first answer the first question which is what should be the grid spacing. Now before we discuss the result, let me draw parallels of this question with the question that was raised by people long ago in the context of sampling a continuous time signal, see you should see that x of f is the continuous function and I am asking; how this continuous function should be sampled in frequency.

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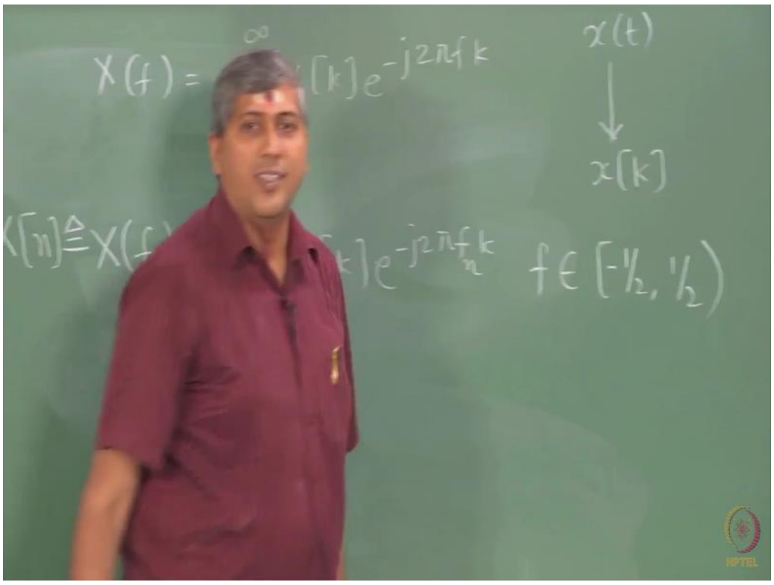

A similar question was asked long ago in the context of sampling where I need to figure out how fast should x of t be sampled so as to produce the discrete time signal. How frequently should I sample a continuous time signal assuming that I am going to sample uniformly that is what gave birth to the sampling theorem; the celebrated sampling theorem (Refer Time: 19:32) and so on, which says that if the signal has a frequency maximum frequency f that is the continuous time signal then the minimum sampling rate that you should choose is twice, later on people said; it is all common sense and then even if you look at Newton's law anything now will (Refer Time: 19:53) common sense matter it was it must have been. So, easy to device a smart phone for example, you can say few years later what is this anything in (Refer Time: 20:02) looks easy even our ten standard questions and so on, but when you are at it at that time, it is a great invention.

So, it is obvious now that sampling minimum sampling rate has to be twice a maximum frequency, but how did people arrive at that result.

One of the criteria that was used is; I should not have a loss of information when I sample. What is it mean by no loss of information; what it means is if I were to be required to go back to this x of t; that means, if I have to reconstruct this x of t from the discrete time in principle, I should be able to do it; whether I will do it or not it is a second thing, but they should be sufficient information in x of k to be able to recover x of t. The same question can be asked here, the same criteria can be imposed here; if I were to recover x of f from its discretized version then I should be able to do it, we do not do it let me tell you we do not recover x of f practically, but theoretically you should guarantee that you have chosen the grid spacing in such a way that there is no loss of information and it turns out that here in the d f t the grid spacing in frequency domain should be 1 over the length of the signal.

That is the minimum grid spacing you should have; you can have more than that like your minimum sampling rate. So, it says the maximum grid spacing is 1 over n t. So, delta f n should be 1 over n; where n is the length of the signal or I have used n subscript l in the slide n l is the length of the signal. So, x of f is perfectly recoverable from x of f n.

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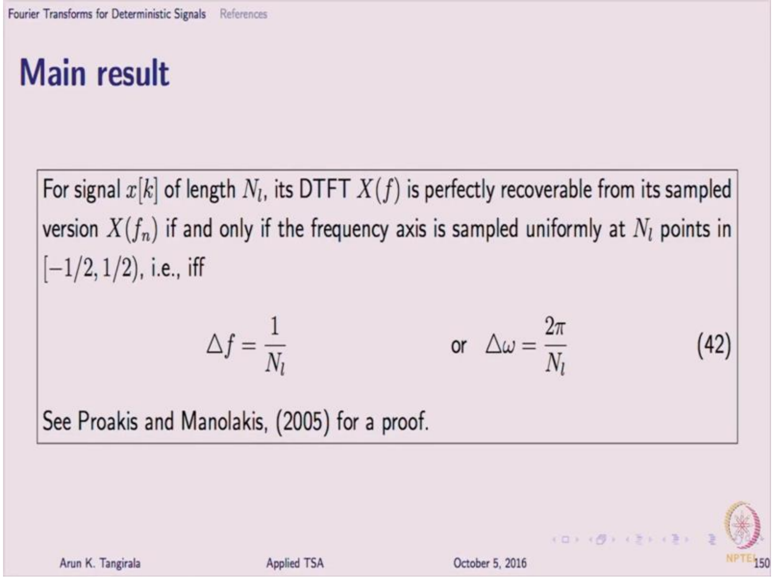

When delta f n is 1 over n l; that means, if I have 1000 points that is 1000 observations of a signal, the frequency grid is of spacing 0.001 and what this also tells me is that I will compute by the way I can notice from the property of by the way is f n let me first tell you what is the f n; f n is now n by m, I will throw away the subscript now.

So, the n th frequency is simply the small n over big n and n runs from now n should run from 0 up to n minus 1; why not n will repeat. So, same story what we have learnt in discrete time Fourier series. So, now, you see there is some similarity of this d f t with some Fourier series when do you use discrete time Fourier series when the signal is periodic. So, something is now hiding behind the bushes there; that is some interpretation is waiting; it is a question that we asked earlier what is the consequence of working with this kind of a transform. Earlier we thought we are assuming the signal is 0 outside, but now we turns out that the assumption is that the signal is actually periodic with what period the length of the signal, that is what your said earlier you can assume a periodic extension.

We started off by truncating the signal; by assuming the signal to be 0 outside, but then we did 1 more thing which is discretizing the frequency axis; had we not discretized then that 0 outside the observation interval assumption is correct, but the movement we sampled in the frequency domain, we have introduced periodicity assumption in the time domain that is what we mean by sampling in time introduces periodicity in frequency; we have already seen that when we event from continuous time signal to discrete time signal; we said the Fourier transform becomes periodic. Now we are observing the duel which is sampling in frequency results in periodic extension periodicity; periodizing in time domain.

So, the duality exist everywhere and that is a beauty of this time and frequency domain analysis; is just enormous applications of this duel properties and so on. So, what will do tomorrow is will duel a bit more on this and I will talk bit more about d f t and conclude the talk with periodogram, show you how to do things in r and then will start off with the spectral densities for random process that is not much there, now that your understood all this basics; all that remains is to understand how spectral density is defined first of all a for a random signal and then the Wiener Khinchin relation will come in and help us in computing the spectral density then we look at spectral densities of white noise and ARMA processes and so on. So, see you tomorrow.