Applied Time-Series Analysis
Prof. Arun K. Tangirala
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture – 67
Lecture 30A - Fourier Transforms for Deterministic Signals 9

All right. So, very good morning, what we are going to do today is initially for about few minutes we are going to discuss the some useful properties of DTFT and then move on to the practical part of the most practical implementations of Fourier transform which is the DFT. Now of course, what I should also tell you is that I am only giving you an overview an about, I think about 5 to 6 lectures we have kind of covered the most important aspects of Fourier transform theoretical aspects, but there is a lot more and; obviously, this course is not equipped to or we do not have the time to go in to detail, but this entire subject of Fourier transforms is a notion and itself and if there are various applications and then there have been various modifications and variations and so on, there is something called discrete cosine transform and now different kinds of Fourier transforms that are actually taken birth after this.

But we are only discussing the most relevant once, I strongly recommend you go and actually read some literature also not just for learning, but for also appreciating I remember seeing an article in SAIM Journal; which talks about how our human ears are kind of implementing the Fourier transforms because if you look at the beauty of the auditory system, it is able to distinguish between several sounds occurring at the same time and if you look at the signal characterization of auditory signals and so on, we would see that there are many frequencies different frequency content. If you were to stand outside on a road you would hear the sounds of the traffic the sounds of may be dogs barking and humans speaking and so many different sounds. They all belong to different frequency regimes; I mean they have different frequency content.

It is amazing that our human ear is able to really distinguish all of this for us and be able to say yes there are these different sounds occurring out there, not only standing on a road when you are listening to a song they are singers and then there is a orchestra and so on and it just does not take Fourier transform alone to explain those phenomena, you will need more advanced transforms like time frequency analysis, transforms used in time frequency analysis like wavelet transforms and so on, which are able to explain how frequencies change with time. But by enlarge even if you look at the description of our hearing ability, it is given in the in terms of frequency right do we know what is the audible range for us for human beings?

Student: 22.

22?

Student: 20 kilo hertz.

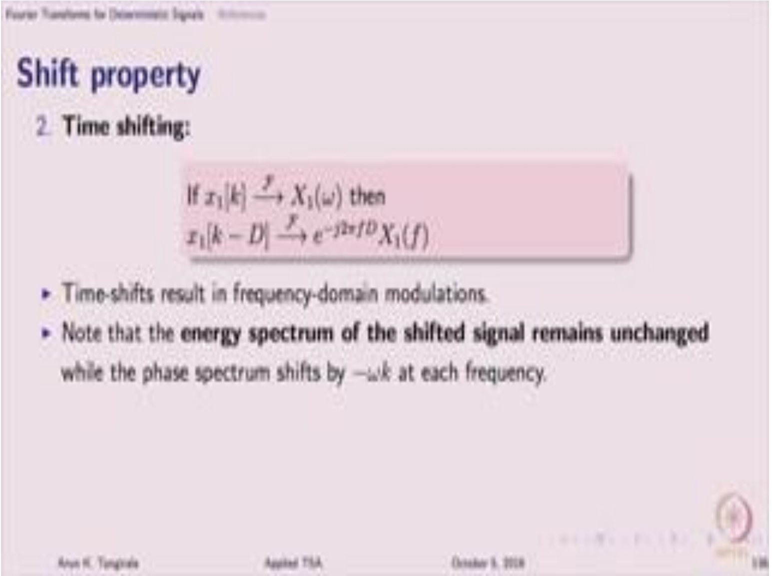
20 kilo hertz and dogs for example, can here lower than 20 hertz right and even elephants for example. So, there different species have been equipped with different kind of filtering characteristics an. so on right. Now the G of f that we talked about, there is a lot of beautiful things out there just read those applications I mean it is not just for learning the theory, but for seeing how you can explain several phenomena using this kind of a frame work like Fourier frame work and so on.

Therefore do not think that frequency domain analysis is a very alien field or an alien subject that I do not see things happening in frequency, I only see things happening in time. Therefore, I am more comfortable with time domain analysis is what many people say, but let me tell you that what you see is not what necessarily is what you want to believe in, and there are so many things happening behind and the these tools help us understand what is happening behind the scenes.

So, with those words let me now get started on the properties of DTFT, which are quite useful not only once again in the analysis of signals, but also in analysis of systems. The first in order is a linearity property it is a very useful property it says that the DTFT of the sum of two signals. So, super position is the super position of the respective DTFTs it is a kind of a straight forward thing to prove and we use this as we know very extensively when you are looking at mixture of signals.

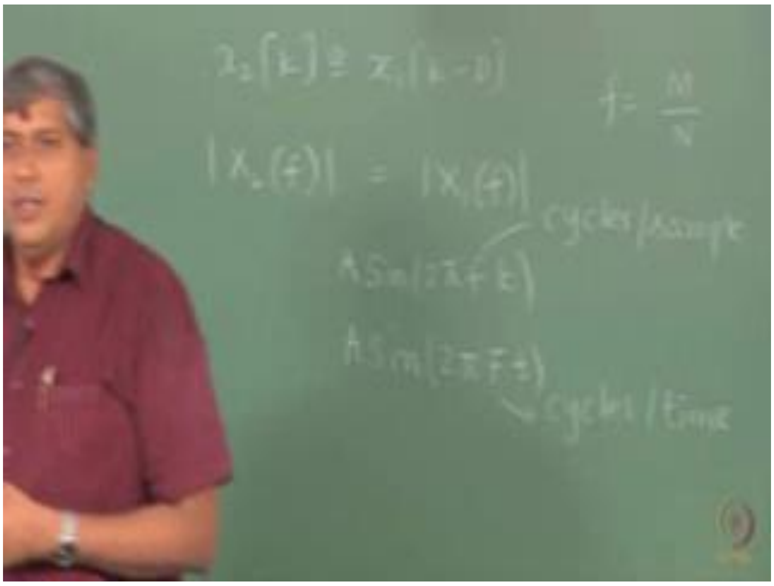
The next property which is very useful is that of time shifting; now all these are remember based on this theoretical DTFT that is the DTFT that we are talking about. So, it says that if a signal as a Fourier transform x of omega then it is shifted version. Is simply the Fourier transform of the original signal multiplied by e to the minus j 2pi f D.

(Refer Slide Time: 05:19)



What it tells us is that only the phase of the shifted signal is altered right if I look at the magnitude of the Fourier transform of the shifted signal. So, let say X 1 k is the original signal, the delayed signal is x 2, which is nothing but X 1 k minus D then from the given expression it is kind of obvious.

(Refer Slide Time: 05:51)



By denote X 2 k X 1 k minus D by X 2 k; kind of obvious that the magnitudes of the Fourier transforms of the shifted and the original signals are identical. By the way I should tell you that the small f that we are being using has different units from that of the big f; do you aware of that what are the units of small f? Hertz is the very generic thing just do not go by hertz, cycles that is a.

Student: (Refer Time: 06:30).

Not by cycles know cycles per.

Student: (Refer Time: 06:32).

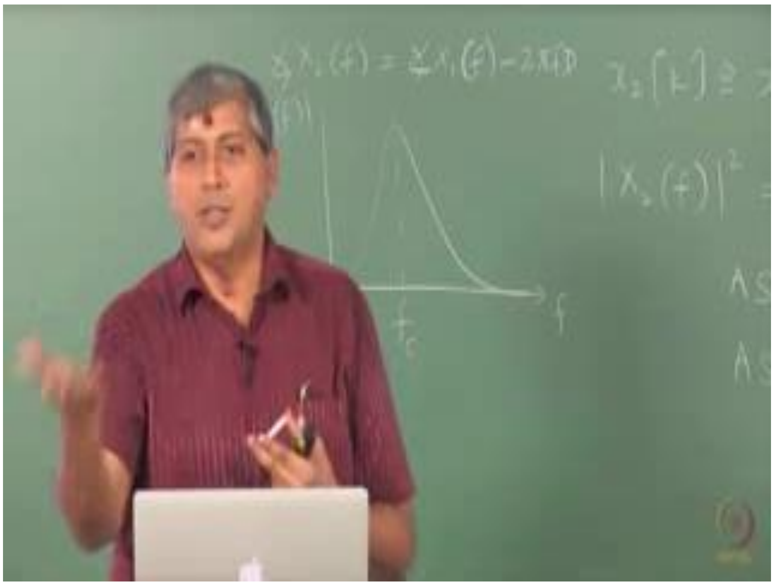
No go back that is why asked you this question I am I am sure many of you are not familiar with it. So, compare these two signals: this is a discrete time sin wave and here is a continuous time sin wave right. So, now, can you be correct in specifying the units of small f? Cycles per sample very very important right where as big F as the units of cycles per unit time if you whatever time if time is seconds and so on both are called hertz, please keep that in mind. Hertz is a very generic name it is even use for car ankles, but it is a as generic as that. So, simply do not say hertz for example, hertz is used also for specifying sampling frequencies you know what are sampling frequencies that is how fast you are sampling for example, a continuous time signal to produce a discrete time signal all right.

So, just saying hertz is not enough you have to have an accurate understanding of the units of f. So, this as a units of cycles per sample whereas, the big F as the units of cycles per unit time right both are continuous valued there is no confusion with respect to that and of course, you know the difference that this sin wave is not necessarily periodic unless f can be expressed as a rational number right, if you are able to express f as some M over N; where M and N are co primes then N happens to be the period of the discrete time sin wave. Whereas, the continuous time sinusoid does not have any of these restrictions why is that because the period of a continuous time signal is on the real number on the real axis right it belongs to the real number said.

Whereas the period of a discrete time sin wave belongs to the whole number set well natural number set I should say; that means, non positive integers therefore, not all discrete time sinusoids are necessarily periodic. It should keep telling yourself that until you know by hard. So, what we observe here is that the magnitude of the delayed signal is the same as the magnitude of the original version, which means energy densities would also be identical. In other words using this is what exactly I meant yesterday and may be lecture before as well that this spectral density plots do not allow us to figure when a particular frequency existed in over a time interval, it this is that is another that is a consequence of this property here, that is the energy spectral density is blind to time shifts which is both good and bad.

We will not discuss the good part of it and so on or not even the bad part in detail, but I am just telling you that it has both it is merits and demerits; on the other hand the phase of X 2 f right.

(Refer Slide Time: 10:08)



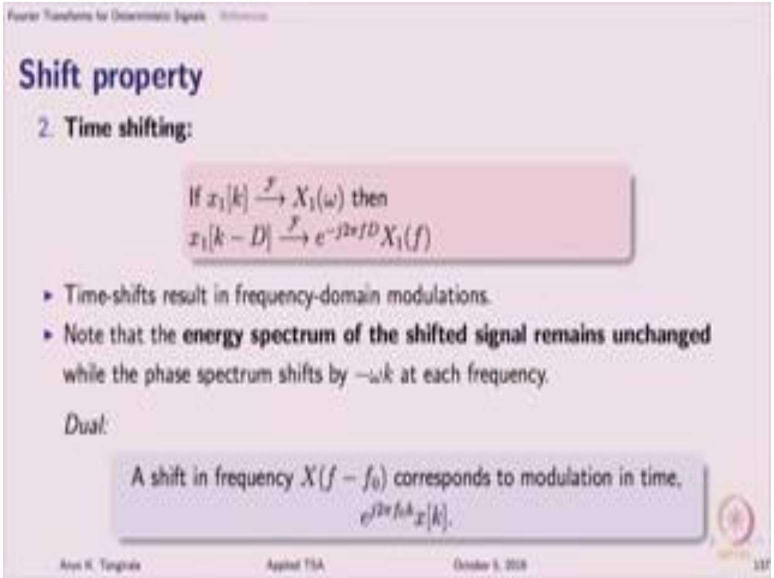
The phase of X 2 f differs from the phase of X 1 f by how much?

Student: 2 pi.

2 pi very good. So, it differs by two pi two pi f D; typically I mean if you assume D to be a positive quantity then that is the phase difference between the delayed version of the original (Refer Time: 10:42). In fact, this relation is used extensively in estimating delays. So, what you do is I take as I said in radar signal processing, I would like to know for example, how far the object is right or input output systems in many systems I want to know the delay between the input and output.

So, imagine that X 1 is the input and X 2 is the output of system then all I need to do is look at the phase difference between the input and output and of course, you know I can just use a single sin wave of some known frequency and then since I know f since I know the phase of X 2 which is output and or the phase difference you can say I know f and therefore, from there I can figure out what is d the delay; this is the standard and very well known way of estimating delays in not only engineering systems. But any input output system that is linear and time in variant; you can find numerous variants of this method. So, this is a very important and useful property in the analysis of linear systems and likewise the dual of the statement. The nice thing about the properties of this Fourier transform is that there is always a dual.

(Refer Slide Time: 12:05)



What we mean by dual is just now we talked about free time shifts and we said that results in the multiplication by e 2 the minus j 2 pi f D, you can also ask this question if I were to shift the frequency of x of f; what do we mean by shift? Imagine that your X 1 of f, I am just going to plot the magnitude let us say for some signal it looks like this and it has some central frequency f c, what we mean by shifting? The frequencies you are going to shift this entire X 1 of f magnitude of X 1 of f by a certain amount in frequency domain right essentially shifting this centre frequency to something else. So, when you do that I am when do you have to do that in all communications and signal basically when you are communicating signals this is a standard thing, what happens is when we are speaking over the phone or when a signal is being transmitted.

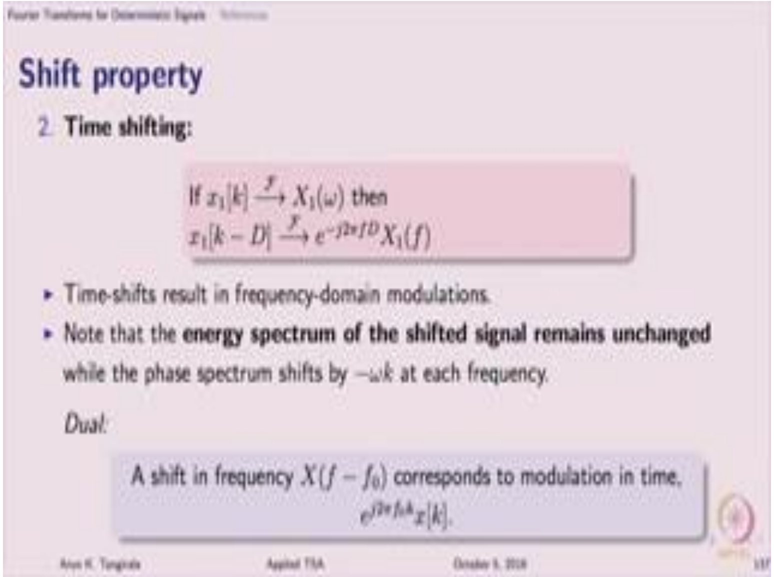
Let say let us take the human signal; it does not have enough strength at that frequency to be carried over long distances. So, what is done in communications is this signal that we speak is actually rapped in a high frequency carrier signal as we called messenger signal it is like the postman. So, the postman has a different has the ability to carry your message. So, what is done is there is a shift of frequency of whatever speech signal that is being that has to be transmitted and then of course, at the receiving end you will kind of d shift you can say, that is essentially the principle essential principle in communications where you wrap the carry the main message in messenger signal, which is of higher frequency.

Why do we want to wrap the message into a high frequency signal? because the losses that occur at high frequencies are lower compared to this low frequency signal right that is if I speak for example, my speech signal has the ability to reach may be the last bench or may be a bit that is all, it dies it is death because there is loss whereas, when I am speaking over the phone it can actually travel thousands of kilo meters right.

So, how is that happening? that is because of this raping of the message signal and that is also true in teaching when a if you take the raw subject raw concept, it becomes very difficult to understand it raw form, when it is wrapped in a certain con context or when it is explained in terms with a story wrapped around it. Then it becomes easy for the student to understand, the only point is you should not confuse the story for the message, you should be able to unwrap and say yeah you know that is a rapper, the actual chocolate of the concept is inside.

That is exactly that is what is done in communication devices as well; there is a coding happening at the transmitting device and then there is a decoding happening at the receiving l and there are some industries standard for that and so on. So, here we are saying that a shift in frequency corresponds to what we call as modulation in time; when you that you can see this duality here, when we shifted the signal in time it was amount it amounted to multiplication of the Fourier transform by e to the minus j 2 pi f D. Now when we are looking at shifts in frequency, then it amounts to multiplying the time domain signal by e to the j what you see their 2 pi f k right.

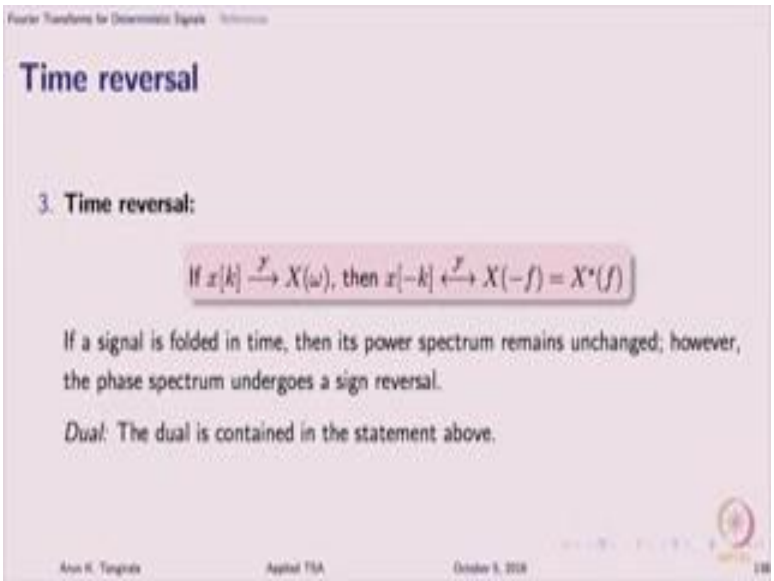
(Refer Slide Time: 16:08)



Your f 1 k whatever that you are shifting by.

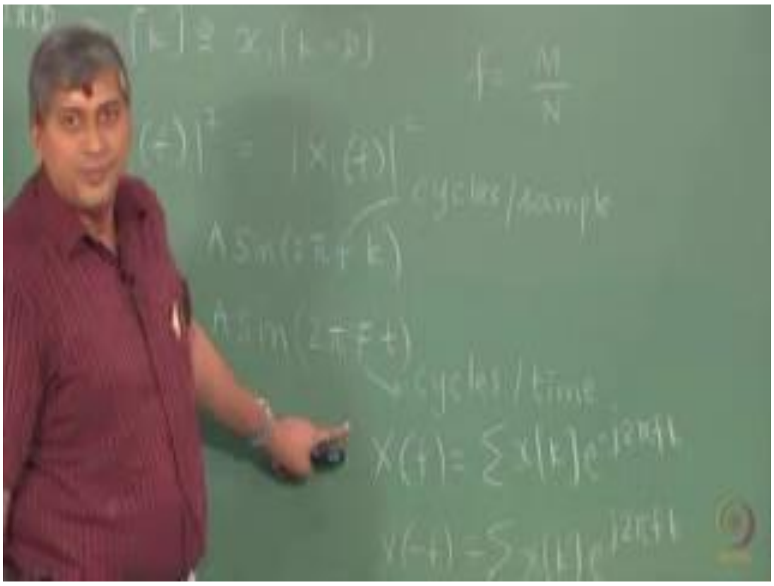
So, there is a similarity and you will see this kind of a duality in almost all the properties of Fourier transforms and that makes it easy to remember; if you remember what is how the Fourier transform changes when you perform in operation in time, you should be able to then remember what happens to the time domain signal, when you operate in frequency domain that that is the beauty of this.

(Refer Slide Time: 16:36)



So, for example, here time reversal that is what we mean by time reversal is, you want to flip the signal that is what we mean by time reversal. You are just going to mirror the signal and that amounts to actually taking the conjugate, if you can see their x of minus k results in X of minus f, but that is nothing, but x star of f right how do you prove that it is very a simple x of f is sigma.

(Refer Slide Time: 17:06)

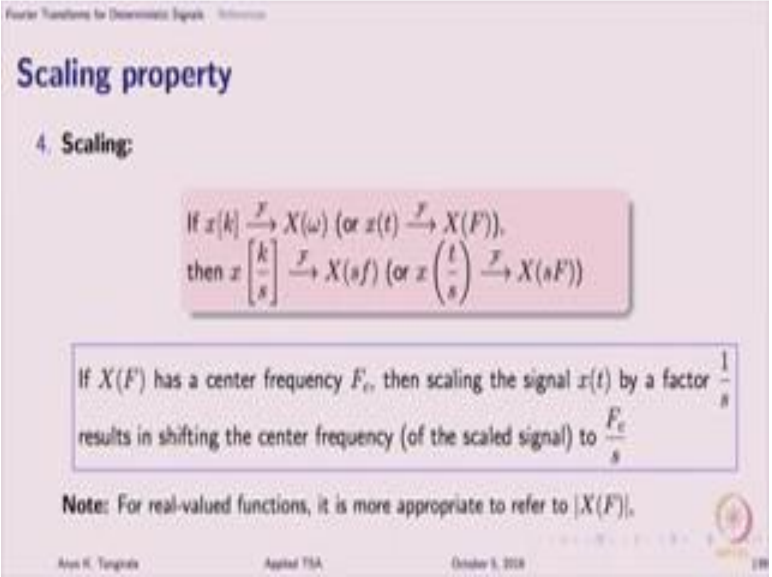


X k e to the minus j 2 pi f k and x of minus f is therefore, sigma x k e to the j 2 pi f k which is nothing, but the conjugate of this.

So, flipping this signal in time amounts to taking a conjugate in frequency domain and likewise you can see here now flipping in frequency amounts to actually flipping the time domain signal, so the dual is contained in the statement. Now this is the scaling property with the third the forth on that we are looking at is a scaling property, which is not so useful in this course, it is lot more useful in time frequency analysis.

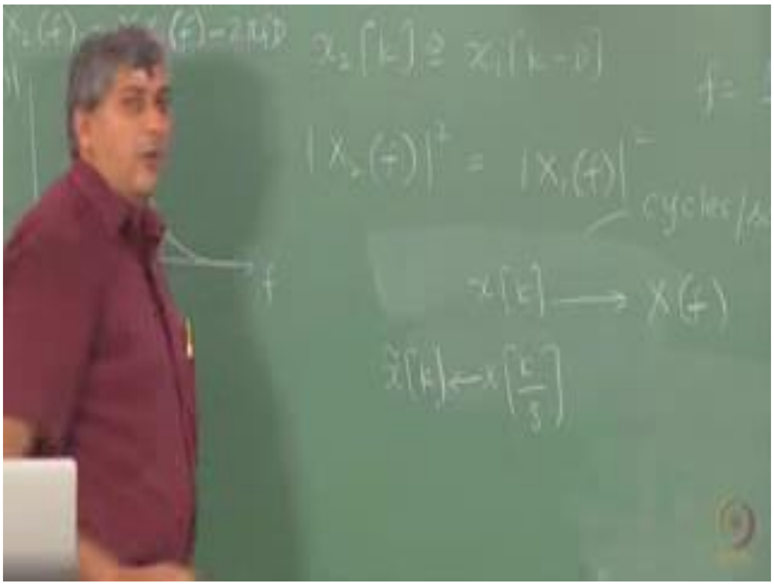
What this property says is if I have a time if I have a signal in time, whose Fourier transform let say is x of omega or x of f, when I scale it and here we are not talking of amplitude scaling, you should not confuse the scaling with amplitude scaling; when I what I am looking at here is scaling in time. So, you are actually shrinking or dilating the time axis, if you look at that carefully you are saying x of k by s, it says that the Fourier transform accordingly is dilated or compressed.

(Refer Slide Time: 18:44)



For example you take x of k over s; the property says that the Fourier transform of x of k over s is x of s f. So, the reverse is happening in the frequency domain, let me ask you a simple question here now.

(Refer Slide Time: 19:04)



If I have x of k whose Fourier transform is x of f; let say all right I have used a big F I will correct that and now we are looking at the Fourier transform of the scaled signal and here scaling is in time.

So, let me ask you a question now if s is some scaling parameter, suppose s is 2 now does it amount to stretching the signal in time or compressing? As usually I will always have two different answers, what you think? Sure how do figure that out suppose I have to ask you to simple question how do you figure that output? think of this as assign this to some x tilde of k call this as x tilde of k, this new signal as x tilde of k; will x tilde look like a bloated version of x k or a compressed version of x k?

Student :(Refer Time: 20:18).

Sorry.

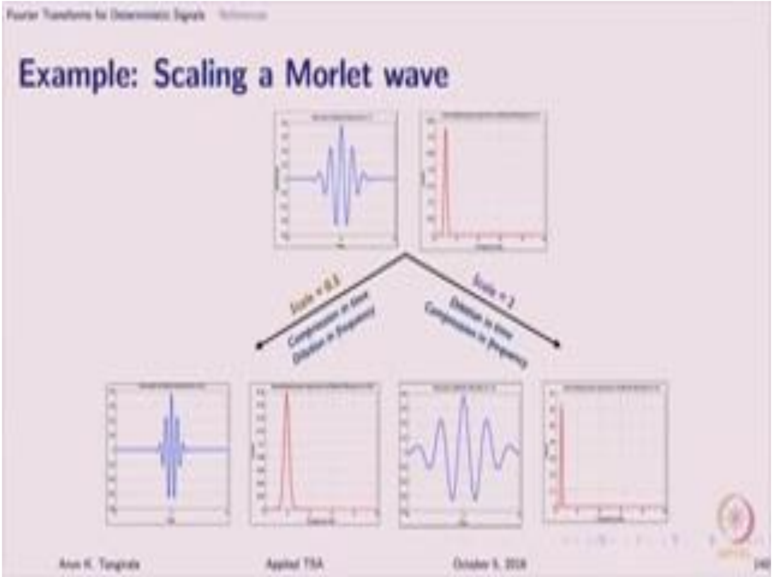
Student :(Refer Time: 20:27).

Hm.

Student: (Refer Time: 20:30).

Correct. So, you can look at this way right x tilde you are looking at k here x tilde at 1 would be suppose s is 2, x tilde it 1 would be at what x is that half right and x tilde at 2 would be what is except 1. So, which means you are really stretching the signal now by a factor of two. So, all values of great s greater than one, will result in stretching of the signal we call this as a dilation and what about; obviously, now values of s less than 1 we will result in compression of course, s equals 1 is your original signal. Now you should understand what happens in the Fourier domain in the frequency domain, where because we said when s is greater than 2, when s greater than 1 sorry, results in dilation of x, you should expect the Fourier transform to compress right and let me show you by an illustration here.

(Refer Slide Time: 21:40)



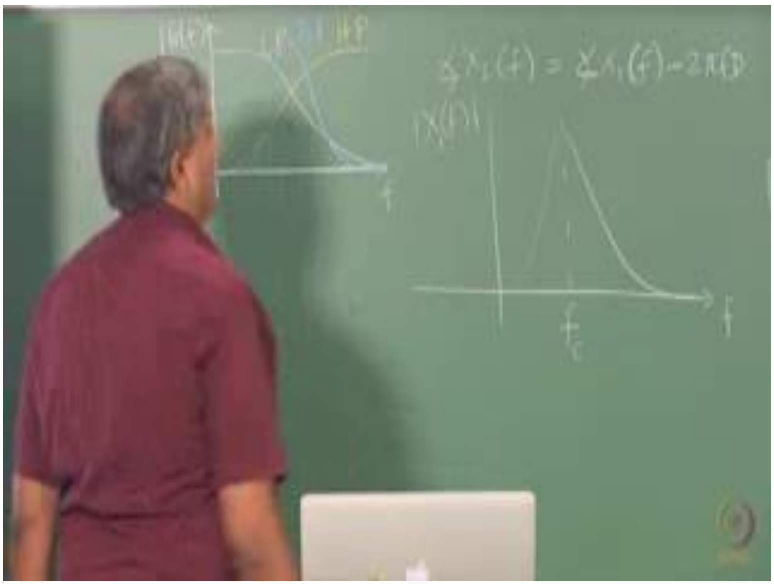
So, on the top you have some signal that is your x and on the right hand I am showing, you can say the Fourier transform and you can even think of it as a magnitude of Fourier transform. Now I am showing you what happens when you scale right. So, on the left hand side you have the signal shown for s equals 0.5, it conforms what we just discussed for values of s less than 1, you generate compressed versions of the signal and that is what you see here; here is your original signal on the top and on the on the left bottom you see the signal when s 0.5, which is the compressed version what does happen to the Fourier transform? It has stretched not only as it is stretched something else is happened.

The central frequency as shifted to the right it make sense; on the other hand of I dilated the signal, the Fourier transform as actually become narrower that is a let me say the energies spread as become narrower and the central frequency has shifted to the left. This is a very beautiful and fundamental result which is used in wavelet transforms; the entire theory of wavelet transforms the rests on the single property.

What it says is imagine for example, the top signal to be the impulse response of some filter, let us say it is impulse response of some of some filter and therefore, it is Fourier transform is a frequency response function, it explains a filtering characteristics and what kind of a filter would be the top one that is if a system as frequency response like that I have shown on the top, what kind of a filter is in?

Is not low pass, it is not high pass, what is a how does the frequency response look like for a low pass filter? It should be necessarily non zero at low frequencies; at zero frequency it has to be non zero right.

(Refer Slide Time: 24:06)



This is on the other hand here for a high pass filter the frequency response would look like this and for a band pass filter, you would see something like what you see on the screen like a notch I mean you can have band reject also. So, bands pass filters. So, this is a typical band pass filter that is what you would see. So, what you see on the top if were to imagine the blue one to be the impulse response of a filter, then it as a characteristics of a band pass filter very good.

Now, what this result tells me is all I have to do is if I want to generate another band pass filter, which focuses on the lower frequencies what do I have to do? I have to do design a filter whose filter is simply whose impulse response is simply the dilated version of this filter at the top and the left hand side figure tells me that if I want band pass filter with centre frequency shifted to the right, all I have to do is compress design a filter which as a compressed version of the impulse response now the original filter. In fact, in wavelet transforms that is why I call this as a Morlet wave, this is one of the waves or wavelets that is widely used in wavelet transforms, on the top what you have is known as the morlet mother wave.

So, everywhere mothers are worshipped which is great. So, here what you have on the bottom are the children, they are born by simply scaling right. So, the left on at the top can say mother you look fatter than me in time domain right because children do say this these days, but then the mother can respond saying in the frequency domain and slimmer all right see it all depends on which domain you are looking at. Likewise you know the mother can tell the other dilated version do not worry child you may be fat in this domain, but you are much slimmer than then any of the other things in the frequency domain, right.

So, there is always is trade of now and this is what is the as I said the central property on which the wavelet transforms rests, in the using the simple slide really you can understand how wavelet transforms are used essentially you can think of wavelets as functions or filters. Even in Fourier transforms you can think of the Fourier transform as a decomposition of the signal on to sinusoidal functions, space of sinusoidal functions or you can think of it as a filtering.

What you are doing is you are taking the signal and passing it through a filter, this is not the typical the Fourier atom; how would the Fourier atom look like? Is a sin wave and how would it is frequency response look like peak right. So, Fourier transform in a sense is like filtering, what you are doing is you have a filter which as a very fine bandwidth and exists exact extracting in theory that frequency component for you. So, that is a good perspective to have as well think of always transforms as filters you can also think of transforms as approximations and so on as projection and so on.

Anyway, so hopefully you enjoyed this scaling property here and for those of you are working a plan to working in wavelets, hopefully this as done some eyes breaking for you.