Applied Time-Series Analysis
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Lecture – 65
Lecture 29A - Fourier Transforms for Deterministic Signals 7

Good evening, let us begin our discussion if you recall we are in the world of Fourier transforms and in the world of discrete time aperiodic signals. So, if you recall we were discussing DTFT, and just to recall few things let us begin with an example of DTFT; the previous example that we looked at was the DTFT of a discrete time impulse also know as a kronecker and as we know the energy spectral density of this impulse is spread uniformly across all frequencies. The way you should look at it is not just as a mathematical transformation, but some more imagination will help you appreciate this example better.

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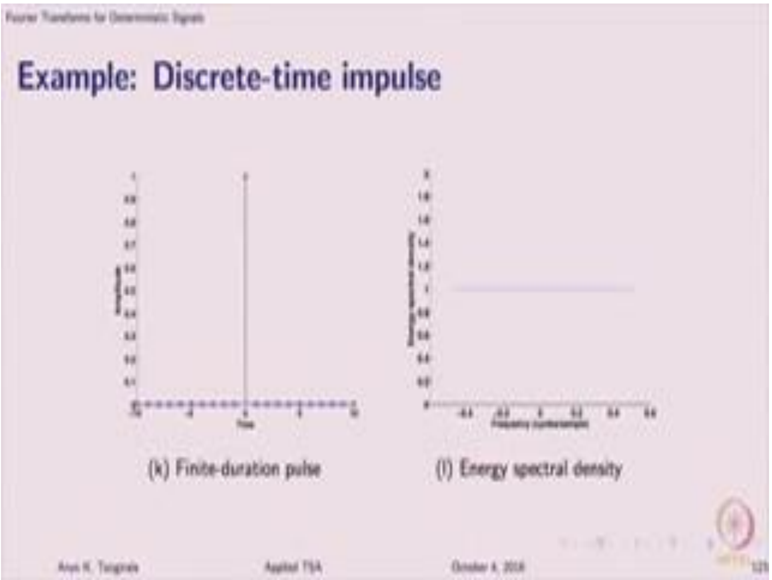

What this example tells us is that it takes all sinusoids, that is sinusoids and when I says sin also imagine cosines, it takes a sinusoids of all frequencies to generate this impulse. Now, I should point out all though we are showing the signal on the left hand side on the spectral density on the right hand side.

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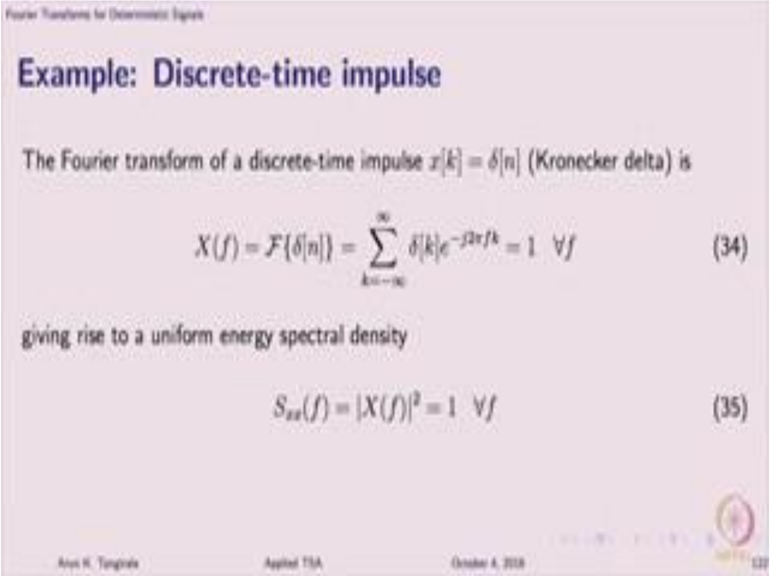

As you know the Fourier transform itself turns out to be one; at all frequencies, what this tells us is to generate this discrete time impulse, one has to really include all the sinusoids. Now that is fine I mean it is a mathematical transformation, but if you look at it, what is this that is actually happening? I mean it does it makes sense if you look at the signal right.

So, if you look at the signal on the left truly speaking there is nothing happening at any non zero time instance, the only activities seen at is at time zero right it where as what the Fourier transform is suggesting is that you have sine waves contributing, that is across all frequencies and they are adding up in a and cancelling out in a particular way so as to produce this impulse. And what I am trying to point out here is the signal itself in some sense is not active, I do not want to use the word nonexistent, but you can use that it is not active at all at non zero times whereas, the Fourier transform seems to suggest that at those times there were these sine waves that were adding and cancelling out each other to produce 0.

Now, when you look at the Fourier transform physically from a physical view point it does not make much sense; on one hand we are saying there is no single activity at all and on the other hand the Fourier transform seems to say no no no there is no activity because there are people who are screaming and there are people who are shutting the mouths of the people of screaming right. It is like trying to express 0 as 1 minus 1, 2 minus 2, 3 minus 3 and so on and why is this happening any idea why is this, is not this actually kind of a spurious thing because on one hand I know the signal is zero valued at non zero likes, on the other hand Fourier transform seems to suggest that no no there are this many signals that are fighting against each other and the net effect is zero, that comes from an imagination right.

At this point one is compelled to ask is this imagination justified? but before we answer that question fundamentally why is it that the Fourier transform is suggesting that during on zero sorry the zero valued instance, which are essentially at non zero instance that there are in fact, signals adding and cancelling out to each other, why is this happening any idea, why do we see with this kind of a result that that is coming out of Fourier transform? You can explain mathematically, but I would like you to look at it from the different angle as well from the angle of the building blocks that we are using, what are we imaging any signal to be made up of in the Fourier world?

Student: (Refer Time: 04:44).

Sins and cosines right, what is the nature of the sins and cosines in terms of their existence over time?

Student: (Refer Time: 04:41).

They exists forever right where as the signal under consideration here it does not exists forever, it only exists for a finite time it could be an single instance or in the next example if you see this is the pulse.

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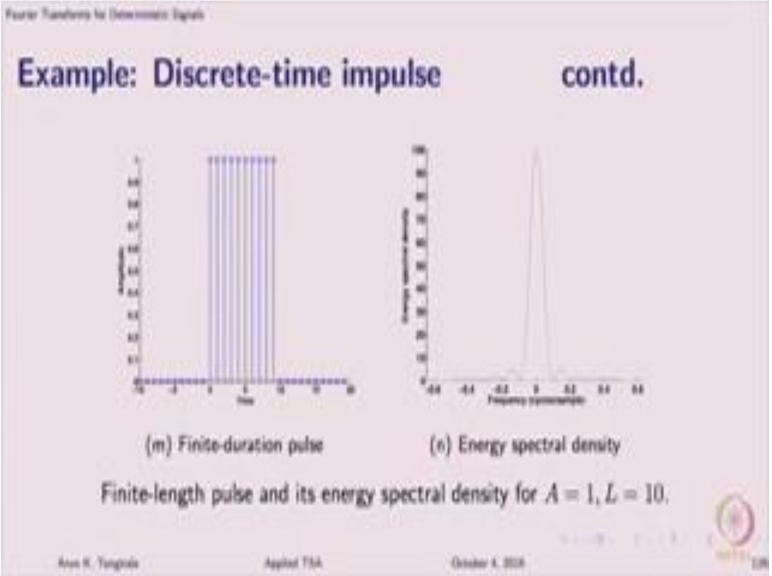

I show you the pulse. So, on the left hand side you have the finite duration pulse even here the signal exists only for a finite time and the rest of the time it is sleeping, here also if you look at the energy spectral density it suggests that or even if you look at the mathematical expression, it suggests that all frequencies are contributing I will beat not uniformly unlike in the previous case.

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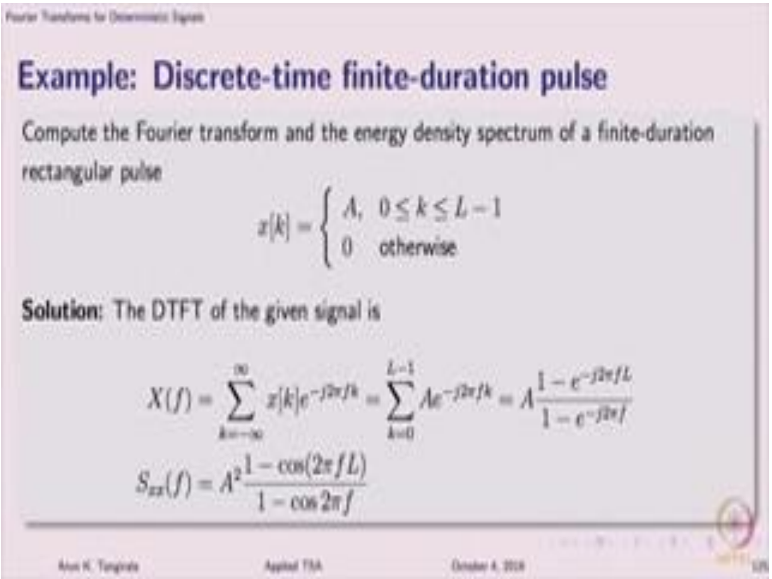

Here the contributions of frequencies in the low frequencies are more as compare to high frequencies all right, but the fact remains common in both examples that all the sine waves are participating in explaining a signal and we know that the signal in time is active only for a finite duration. So, that is the disparity. So, the point now you should observe is the sine waves which are of building blocks exists forever, where as the signal that I am trying to synthesize does not exists forever, it exists for a short period of time for a finite amount of time and then there after goes to zero.

So, here is where one questions the appropriateness of using these kind of building blocks for such signals; it straight away tells us that if this may not be the best way to break up a signal of this kind. In other words you want the building blocks to be commensurate with the feature of the signals and the feature of the signal here is not really commensurate with the feature of the building blocks. So, do I still go ahead and use the Fourier analysis for such signals, yes not primarily for detecting the features of the signal per say.

But mostly in these kinds of cases we use Fourier analysis for understanding the system that generate this, that is there may be a system which has an input and it is producing this kind of an output, in such cases that is when you are dealing with signals like this, it is perhaps more appropriate to use the Fourier analysis for the understanding of the systems rather than the signals. When it comes to periodic signals may be Fourier analysis is really good that is one way of looking that that is one important point that on one hand we use Fourier analysis for analyzing the signals and on the other hand there exits this huge set of applications where we use Fourier transforms for analyzing systems their characteristic filtering nature and so on.

The other way of putting it if you were to be reading the time frequency literature is that Fourier analysis is not ideally suited to what to figure out what frequencies are present at what times. So, for example, if you look at the signal or even the previous example one of the things that you can actually answer is what frequencies have contributed, but if I were to if I want to know what frequencies are contributed over what time interval, there is no such information here at least in the spectral density, you may argue that the phase information contains that. But if you look at the spectral density the only information that it gives you is what is a contribution of the frequency component to the overall energy, it does not tell you when a particular frequency component was present or what time interval and in that is the limitation of working with spectral densities that limitation would not hit us in this course.

When we go into an advanced time kind of time frequency analysis course joined type frequency analysis course, then the limitation does hit us where we turn to tools like wavelet transform and so on. So, it is just to give you some over view. So, now, let us proceed and now understand how DTFT is useful in analyzing systems not just signals, but before we do that we go through this standard result that we have learnt earlier as well for the discrete time periodic case, where we did although we did not prove it and I asked you to just prove it by yourself fairly easy proof; that the energy there it was a power spectrum and the auto covariance function, here it is energy spectral density we can think of a density here and the auto covariance function they both form a Fourier pair once again.

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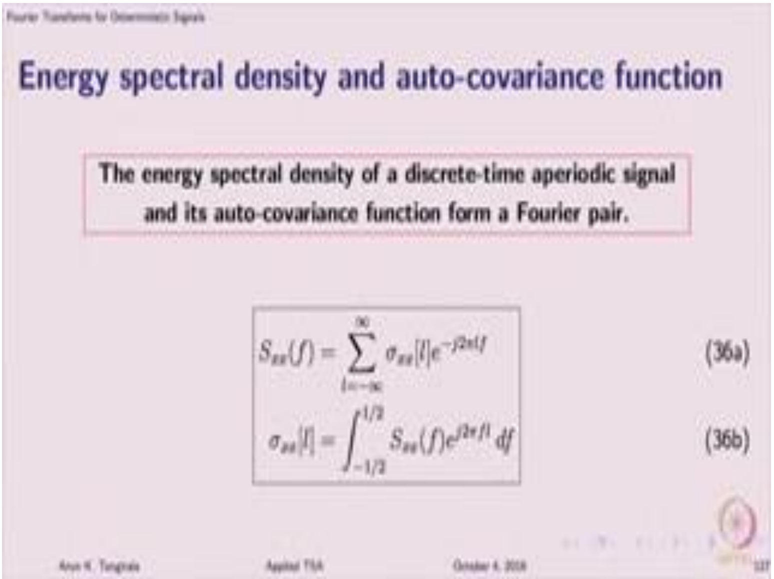

The only difference is in the periodic signal case the auto covariance function was periodic and we could think of a power spectrum and they formed a Fourier pair very much like how aperiodic signal and it is Fourier series form a pair. Here the auto covariance function is not periodic because the signal itself is not periodic and we are looking at energy spectral densities, they form a Fourier pair exactly in the way the signal and it is Fourier transform from a Fourier pair.

Now, pretty much like what we saw earlier auto covariance function, for the random signals satisfy the same different equation as there that of the signal in the auto regressive case, here also we can say the auto covariance function and the energy spectral density from a Fourier pair. So, there are two ways of computing energy spectral density or even the power spectrum; let us look at the energy spectral density one way is to compute the Fourier transform of the signal right and then construct your spectral density using this expression.

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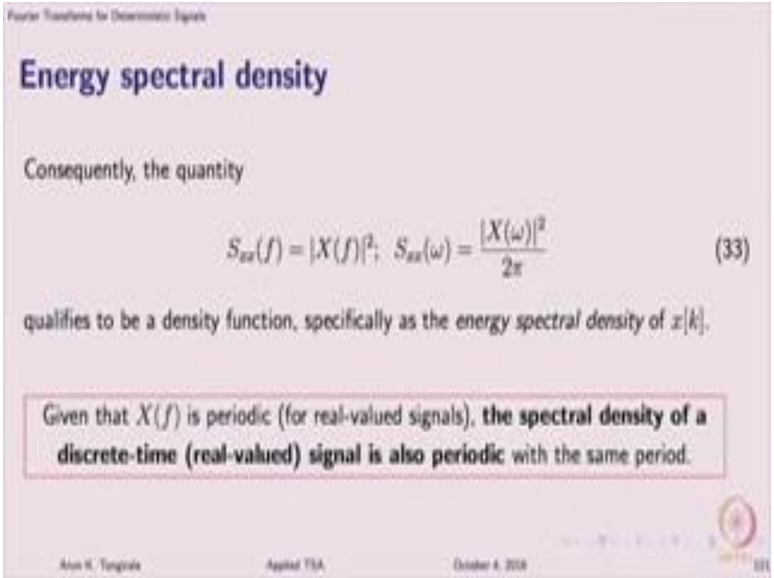

So, if you are given the signal you can compute the discrete time Fourier transform and then either use if you are looking at energy density in cycling frequencies: simple use mod x of f whole square or angular frequency: just scale it by 2 pi and arrive at the energy spectral density that is 1 root.

The other root is to construct to auto covariance function and then simply use the Fourier transform; that root is particularly attractive as we will learn later on because for random signals the Fourier transform does not exists. Any idea why? Why do not why cannot we think of a Fourier transform for random signals?

Student: (Refer Time: 11:53).

It would not be what would be do not know.

Student: (Refer Time: 12:00).

For the Fourier transform.

Student: (Refer Time: 12:03).

What kind of convergence?

Student: Absolute convergence.

Absolute convergence are random signals absolutely convergent they are not because they exists forever, any signal that exists forever right and it does not decay when we when we say exists, forever it should not go to zero asymptotically at all for such signals; obviously, you cannot expect absolute convergence. So, for random signals I cannot think of a discrete time Fourier transform therefore, I should ask if I can think of a spectral density at all right we know already that random signals are not energy signals we know they are power signals. So, I would like to think of a power spectral density, but unfortunately I do not know how to construct it, it turns out later on as we shall learn the Wiener-Khinchin theorem allows such to compute although it does not is not a definition per say, but it allows us to be compute the power spectral density given the auto covariance function.

So, the route of the method of arriving at spectral densities from auto covariance functions is in some sense unified for both deterministic and random that is a very important point to remember all right. Now let us move on to the analysis of systems using discrete time Fourier transforms you want go much in detail typically this is dealt with the lot more in detail in a proper course on linear systems theory. In linear systems theory you would actually defined what is known as a frequency response function and so on, here also we will come across shortly.

But first let us being with any two signals like we have defined cross covariance function for any two any pair of signals, here if I have a pair of signals X 1 and X 2 that are finite energies signals then I can define across.

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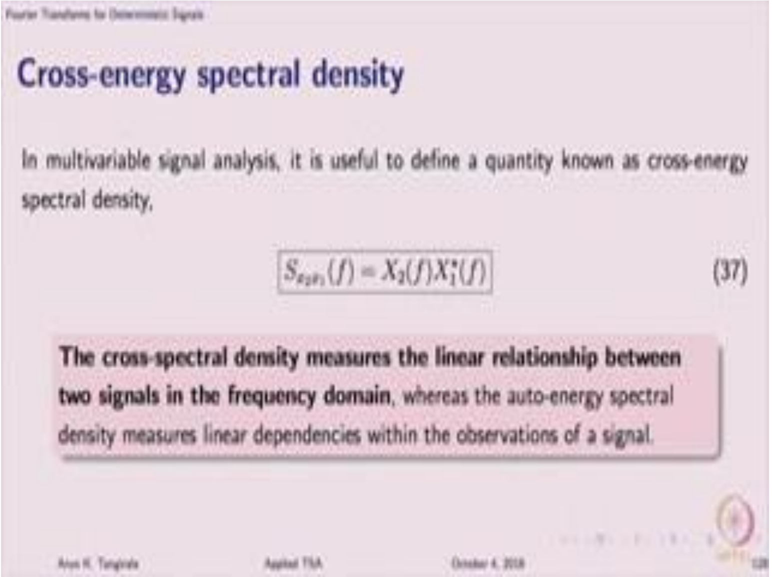

Spectral density cross energy spectral density as X 1 times f, times X 2 star of f or y series. In fact, depending on the order here, if you are looking at a S X 2, X 1 as you see on the slide.

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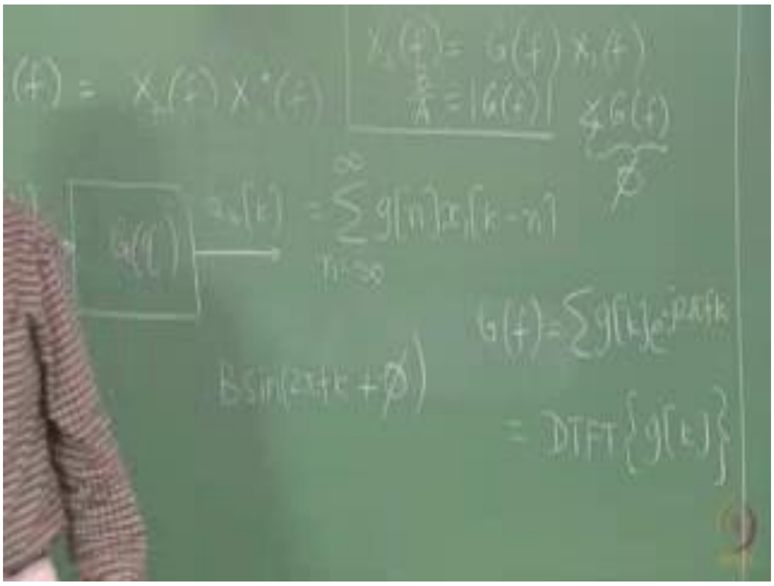

The cross energy spectral density simply defined as a product of the Fourier transform of the first signal times the conjugate of the second with independent I mean what is first and second, but the order here matters which means if I swap the order here the expression is changed according; you would have X 1 of f times X 2 star of f right this is how the cross on the spectral density is defined and you can immediately check if I set X 2 equals X 1; that means, if I am looking at the same signal then what do I recover it is specializes to your auto energy spectral density that we have been studying right.

Now, it turns out that this cross energy spectral density is nothing, but the Fourier transform of the; what do you expected to be?

Student: Cross covariance.

Cross covariance function right it is extension of the pervious result, but we will come to that result a bit later we will study a more important result which is used in the analysis of linear time invariant systems and this result says if there is a system which is being driven by a signal we have looking at the deterministic systems. So, there is a system G that is being driven by X 1 and is generating X 2. In other words X 1 is input and X 2 is the output that is imagination you can get and think of this G as an LTI system, it as a transfer function operator representation, we already know all linear almost all linear time invariant systems are describe by the convolution equation right.

What do we mean by that? The output X 2 can be written as the convolution of X 1 with the input; do not get confused with this n and n that we have been using in the power spectrum. It is a dummy variable right and in symbolic form we write this to be convolution of sorry of the impulse response with the input this star is a special one; it is not product regular product it is stands for convolution.

Now, the result is that first of all as we will learn also later on; the Fourier transforms of the output and input are related in a very nice fashion and that is the beauty of working in frequency domain. So, you can see that X 2 and X 1 in time are related through convolution, the moment you study the relation in the frequency domain, you can show that X 2 f is nothing, but G of f times X 1 of f. what is X 2 of f? It is a discrete time Fourier transform of X 2 and likewise X 1 is of f is the discrete time Fourier transform of the time input. G of f is called the frequency response function; we know now already that there is a name to this small g, what is that? Impulse response sequence.

Why is it called impulse response sequence? Because it tells me it is exactly the response that the system will produce to an impulse input; what about G of f, why is it called frequency response function? Because of this very important reason that if I look at the magnitude and the phase of G of f, it tells me when I feed when I excite the system with the sinusoid of frequency f pure sin, mod G the first result is assuming the system is stable. The output is also going to be a sinusoid when the after the transients are settled down, the output is also going to be a sinusoid of the same frequency, that is the beauty of a linear invariant system, it is a trade mark characteristic of a of an LTI system; all of this is useful to us later on when we move to the random signal world.

So, if I feed in a sine wave here out comes the sine wave of the same frequency, but of a different amplitude and phase, otherwise a frequency remains the same right in other words if X 1 of k is some A sin 2 pi f k out comes here B sin 2 pi f k plus phi and B over A is given by the magnitude that is the called the magnitude ratio or amplitude ratio and phi is given by the argument of G; remember your G of F is a complex valued number.

So, by looking at the magnitude of G of F, I can actually comment on whether the system is amplifying or attenuating the sine wave. why do you think all of this is useful, do you think can you think of any application where this kind of analysis is useful, how do I care what an LTI system does to a sine wave? So, what do you mean by filter? So, name at least one application where you think this is useful, all communication devices that we are looking at right you tune into a radio station they are characterized by frequencies; any remotes they are in a certain operating in a certain frequency band, you have 2 g, 3 g, 4 g and all of those right. So, what is this 2 g, 3 g and all they are all corresponding to some frequency bands and your devices that you are using cell phones and so on they all have certain filters that are tuned to receive frequencies only in certain frequency bands.

So, whether you like it or not this finds extensive applications in all the technology that we use today. Essentially that is in technology, in signal analysis it is used in designing filters; the actual signal may not be a sine wave that is one thing that you should remember, in reality the true signal that is exciting the system may not be sine wave. But Fourier analysis allows us to imagine that signal that is coming in to be made up of different sine waves and G of f tells me how the system treats each of those elementary sine waves that are making up the signal, and because of linearity I can think of the output signal being again resynthesized only after the signal the system has alter the amplitude and the phase of the individual components of the signal.

So, the truth may be that the signal is not a sine wave, but we are imagining the real's input to be made up of many sine waves and that the system is actually responding to each of those sine waves based on G of f and then the at the output side there is a re-synthesis; all of this is imagination. But this imagination as helped us enormously that is the point and that is if you are appreciate that then you will have a better grander respect for Fourier analysis.

So, this is the first result that is X 2 of f is G of f times X 1 of f, but a more useful result is in terms of spectral densities. Why do I keep saying this? because when we move to the random signal world, I would first of all we have already said this Fourier transforms of random signals do not exists, but spectral densities may exists they are not yet talked about it, but spectral densities can be thought of and therefore, I would like to have a result that tells me how this spectral densities are changed by the system rather than the Fourier transform.

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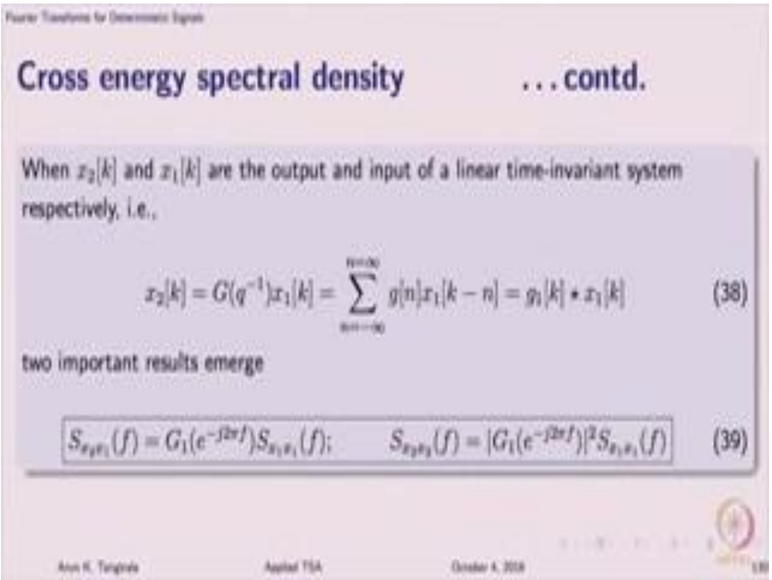

So, the first results says that the cross energy spectral density here we are talking of energy spectral densities, it says the cross energy spectral density is nothing, but the auto energy spectral density of the input multiplied by G of f; I have written G of f on the board, but on slide you have the more correct one G of e to the minus j 2 pi f, how does one obtain G of f by that they have not given that definition here.

But by now you know I suppose it is nothing but the Fourier transform of the, I have reached my limit. So, it is at discrete time Fourier transform let me put it this way of the impulse response sequence, so ever where the DTFT is coming handy.

Now, you see more than the signal analysis part, the DTFT is useful in describing the systems behavior right and this should tell you we already know DTFT of any sequence exists only if that sequence as an important property, which is that the sequence should be absolutely convergent, what us it tell us about the kind of systems for which I can think of G of f? It is impulse response should be absolutely convergent correct; what is it tell us about the nature of the system?

Student: Deterministic (Refer Time: 25:21).

Deterministic world stable random world stationarity we will come to correct good very good. So, only for stable systems you can think of a frequency response function why? because we have go back to what I said earlier, I said LTI systems have a trade mark characteristic which is if I feed in a sine wave and you allow the transients to died on, you will see the sine wave of same frequency, but when do the transients died on only if it is stable correct. So, all of it is inter connected there everything is consistent here. So, you should think and speak of frequency response functions only for systems that are stable in the deterministic world.

So, now the result on the screen says that the cross energy spectral density is the frequency response function, times the auto spectral density of the input, that is how it is shaped; you should check that everything is consistent on both sides, what I mean by that is on the right hand side you have a product of G of f times the auto energy spectral density, we are looking at the left bottom relation. The energy auto energy spectral density is it real valued or complex valued? Real valued right because it is a squared magnitude squared quantity; what about G of f?

Student: (Refer Time: 26:56).

Its complex valued. So, we know already that the cross energy spectral density in general is complex valued so things make sense. On the other hand you look at the second result they said G 1 do not worry about G 1 there that is you should have not appeared there anyway. So, the auto energy spectral density of the output that is the second result tells me, how to compute the energies spectral density of X 2 given the energy contain spectral density here of the input and the frequency response function.

What does this tell me how to what is a Fourier transform of X 2, given the Fourier transform of X 1 and G of f? But now the result tells me how to compute the energy spectral density of the output, given energy spectral density of the output and G of f why is it useful? It is useful because now this result will tell me what the system is doing to the input right because the energy spectral density of the input being shaped by magnitude square of the frequency response function. Look at the difference in the two relations that we have at the bottom: on the left hand side the relation tells me how the cross energy spectral density is shaped by the system and on the right hand side, how the energy spectral density of the output is shaped by this system and in both cases the frequency response function plays a critical role, but the only difference is in the second case only the magnitude square is responsible for the energy spectral density the phase as no role to play.

So, therefore, this magnitude square of the frequency response function is a very important thing that we want to look at. In fact, when we move to the world of random signals we will see a similar result; where what do you expect X 1 to be replaced by white noise very good right and X 2 would be the stationary process under consideration and G would be replaced by h right. And there is one more difference which is the energy spectral density being replaced by power spectral density; so, this is all a curtain raiser for you so that you are not in some for surprise later on and kind of developing the connections now itself. So, you can think of this as your know home and that as your in loss place the random world. I am just showing you what you are going to experience there, fine.