Applied Time-Series Analysis
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Lecture – 64
Lecture 28B - Fourier Transforms for Deterministic Signals 6

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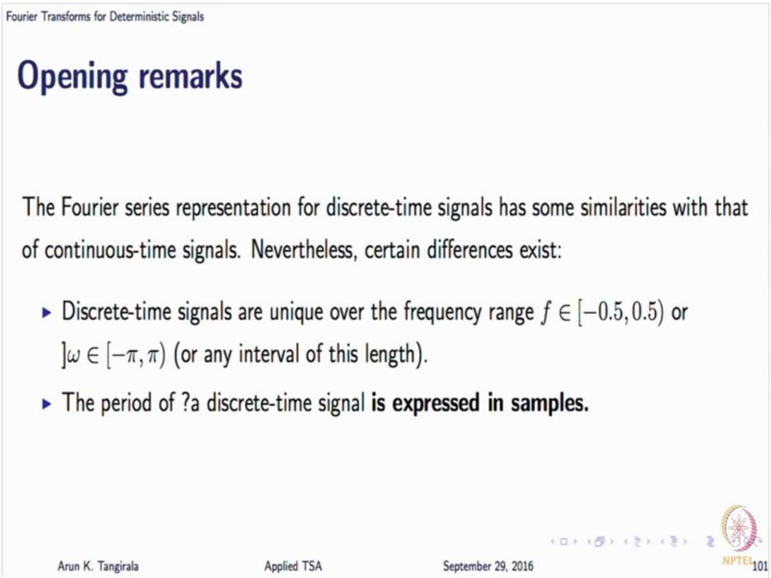

So again in the discrete time world we have two classes of signals; periodic signals and aperiodic signals of finite energy, the story is a same, but how does the discrete time nature of the signal make a difference to the expressions that we have already seen. In a continuous time periodic case, what did we say if a signal is periodic with period tp or t naught then I am going to expand or imagine that signal to be made up of a fundamentals plus harmonics, the idea is a same here, but there is only one difference and that difference has got to do with the nature of the discrete time complex exponentials or discrete time sinusoids.

In the continuous time case our building block were continuous time sinusoids. Here, obviously, the building blocks are going to be discrete time sinusoids; there is no difference, but there is a fundamental difference between how one property of the discrete time sine wave which is not the case with the continuous time sine wave.

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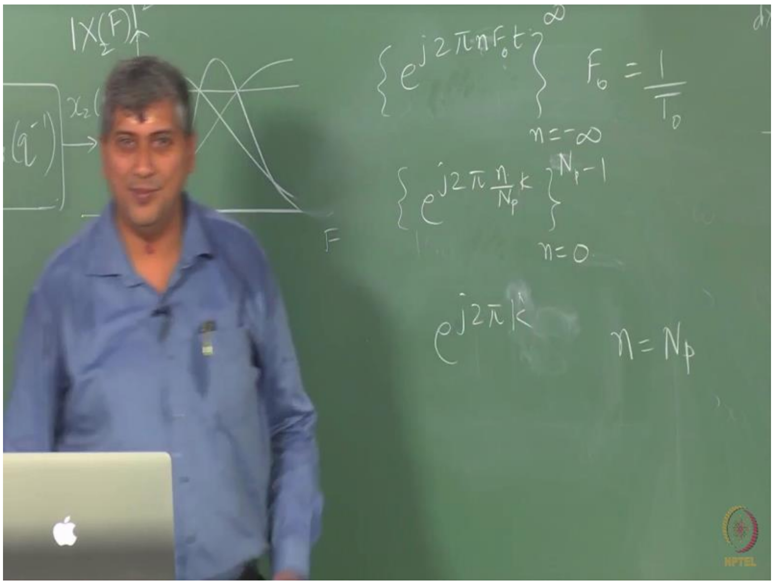

So, I have e to the j 2 pi n f t; as the building blocks for the continuous time periodic case whereas, for the discrete time case; I would have e to the j 2 pi n over n k; what is this n let us say np; np is the period of the discrete time signal, here F naught let me put here F naught is 1 over t p or t naught, here np is the period of the discrete time signal that you are analyzing.

Now, the difference between this continuous time complex expo complex sin and the discrete time complex sin is that this is periodic, it repeats itself after a certain small n you understand. So, here we said all fundamentals and harmonics would be considered when I imagine the signal to be synthesized, but here also I can think of n equals minus infinity to infinity; there is nothing wrong, there is nothing illegal about this or un mathematical about it and incorrect about it; however, these discrete time complex sine waves are unique only in some interval, beyond that they repeat themselves.

When they are going to repeat, they are going to look identical there is no point and actually going from running from minus infinity to infinity and it turns out there they are unique only in a interval 0 to np minus 1, why is that let I go to the small n equals np right. So, I take one of this and say that here at n equals np; this is the case of n equals np, I have this building block or I have this atom; this is 1 of the atoms at now this is 1 and that is exactly at n equal np; what is this here, you have e power j 2 pi k correct, but what is e power e to the j 2 pi k; that is exactly equal to e to the j 2 pi 0, why has this occurred; we do not see this kind of behavior for the continuous time case why?

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t can rather than saying t is a fraction t can be; t is a real valued number whereas, here k is an integer and why did we running this integer because of sampling, because of sampling we have running into this situation what this tells us is, there is no point here in going beyond this because you will find (Refer Time: 04:40) exactly the same ones here. So, it is sufficient only to consider this set of atoms when it comes to discrete time periodic case and that is the big boon to us because I do not have to really sum up over some of infinite terms alright, but this observation here that e to the j 2 pi n over n np k repeats itself with the period np.

Whereas, you do not see that here is generally summarized as sampling in time introduces periodicity in frequency is a very profound statement, the moment you sample in time you are introducing periodicity in frequency and we will see this manifesting very soon. So, to summarize the difference between a continuous time periodic and the discrete time periodic case is only in the set to in the family or the members of the family that you are considering.

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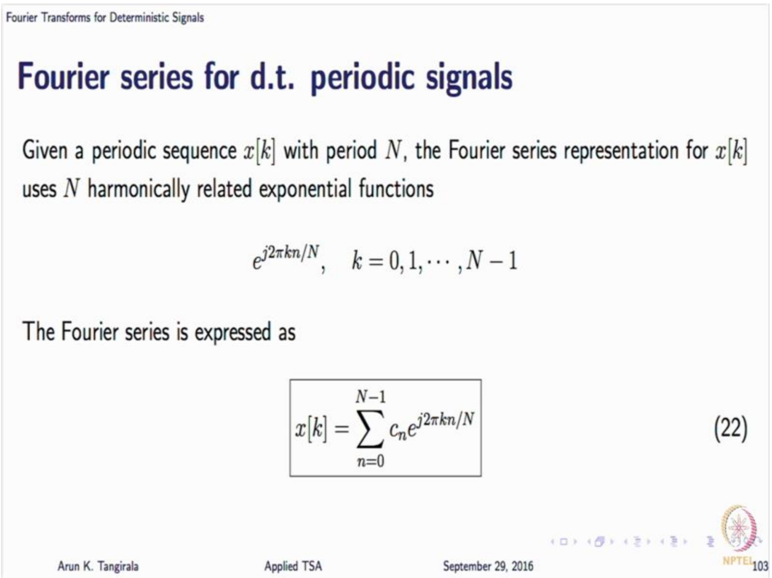

And therefore, the discrete time periodic synthesis equation of the discrete time Fourier's series runs only from n equals 0 to of course, here I use n not np, but h please do understand that here n is a period of the signal, that is only difference; otherwise it looks strikingly similar. Here also we are considering a fundamental frequency a dc component and harmonics, but we are not considering infinite number of harmonics, we are only considering harmonics up to a certain value after that we say they repeat themselves, so there is no point in including that is all.

The rest of the story is the same, in the sense you have the same interpretation for cn; it is a Fourier coefficient and you have a Parsevals relation and so on.

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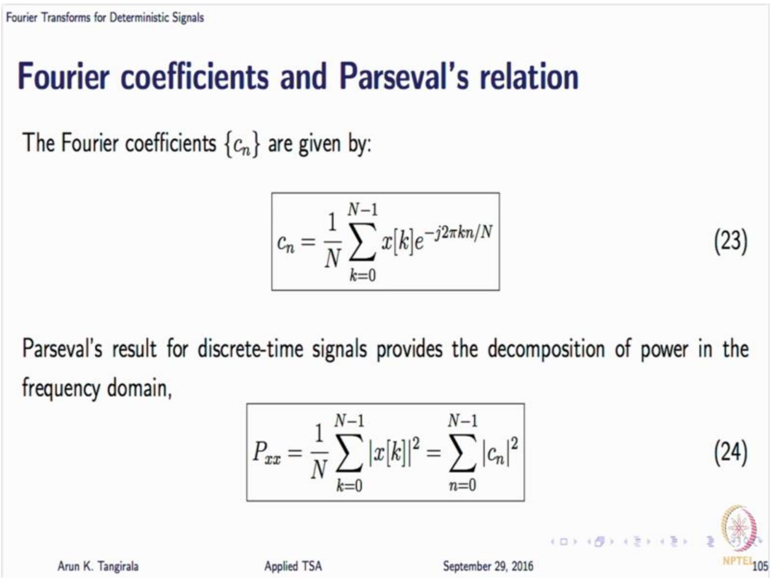

So, let us look at the expression it looks quite similar to what we saw for the continuous time periodic as the Fourier, the expressions for the Fourier coefficient had an integral earlier because we were dealing with continuous time periodic signals. Now we are dealing with discrete time periodic signals, the procedure to arrive at this expression is the same as I had explained for the continuous time case, all you have to do is multiply both sides with the conjugates, use the orthogonality property of the complex sin waves and then you will left be only with one term and that is what leads us to this result.

Though in the continuous time case we had 1 over tp, you recall here we have n over n or np and the integral there ran from minus tp by 2 to tp by 2 which is over 1 period here also we are summing up over the over 1 period. So, you do not have to get confused at all; all you have to remember is in the discrete time case, the complex the your atoms repeats themselves after certain harmonic index and that is it.

So, the rest of the story is the same, you have a power decomposition equation due to parseval; same story right because we are talking of periodic signals, we should be talking about power and this result once again tells us what is the contribution of the n th harmonic towards the overall average power of the signal and once again I can give this interpretation that mode c n square versus n will is nothing, but the line spectrum it is a contribution of the n th harmonic towards the average power of the signal and as we said for the continuous time case, there is no notion of spectral density here; there is only a spectrum very often people again forget this fact. So, we have what is known as the line spectrum.

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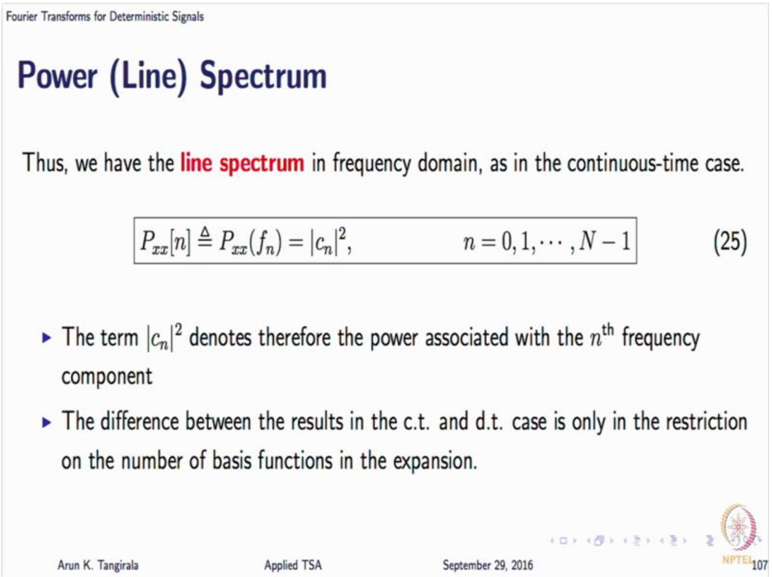

And the only difference here is again when you plot the line spectrum, you plot it up to a certain n. In fact, you can show that the power spectrum is symmetric although you are evaluating from n equal 0 up to n minus 1, after half way through it starts repeating; like we said in the continuous time case as well, it is symmetric with negative and positive. Here there is no negative and positive notion fortunately that negative and positive is taken care of in your n equal 0 to np minus 1 that is another advantage here when in the discrete time case.

So, what I just now said is when you plot the power spectrum or the line spectrum, you need to only plot up to half; well if it is odd then you go 1 more point, but if it is even you go up to 1 more point then if it is odd you just have an exact half there and it is going to be symmetric with respect to the n alright.

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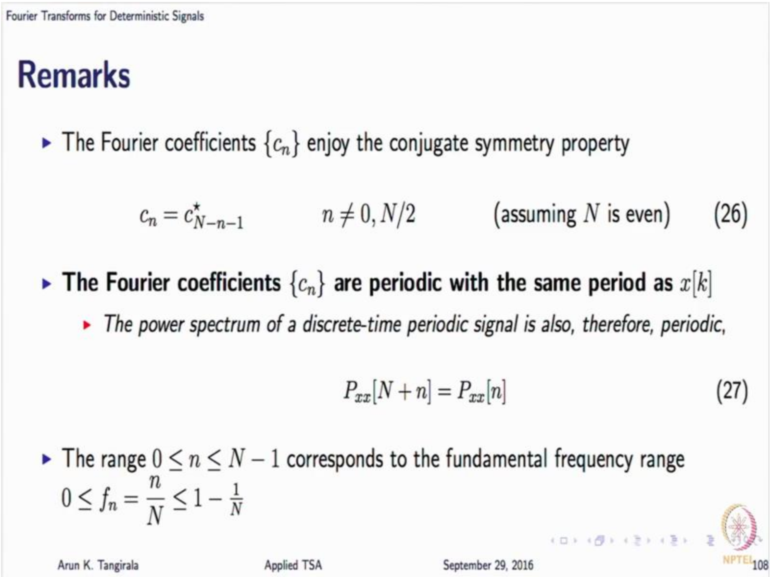

And that is what we mean in equation 26 here; that there is a certain conjugate symmetry property of the coefficients. In the continuous time case we said c n is c c minus n star, but here we are having a slightly different result because we are not looking at positive and negative frequencies, all you have to remember is there is a symmetricity in the coefficients as well, but it is not exact symmetry it is a conjugate symmetry that is all and because of the periodicity of your coefficients as well.

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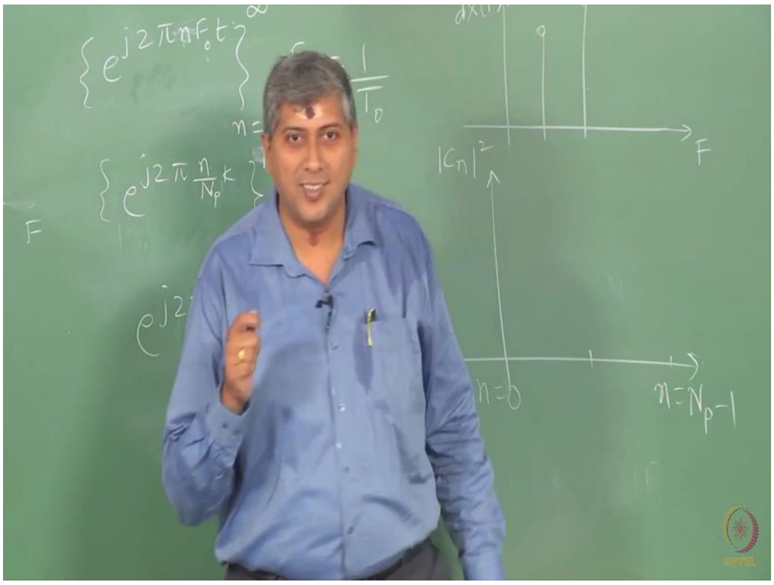

In fact, the coefficients if you have to ask remember we are only going to plot here, let us say I plot mod c n square up to n equals np minus 1. In fact, we do not have to do that we can stop half way through, but if you were to plot the complete mod c n square from n equal 0 to n minus 1, what we mean by periodicity here is I can continue to plot this beyond n as well, but it will repeat; it does not mean that the line spectrum is not defined beyond this point, you have to understand the c coefficients are defined beyond np minus 1, but they will repeat. So, what is a point in computing that is all and that is what equation 27 says that the line spectrum repeats itself after the period of the signal and once again this is a consequence of sampling in time causing periodicity in frequency.

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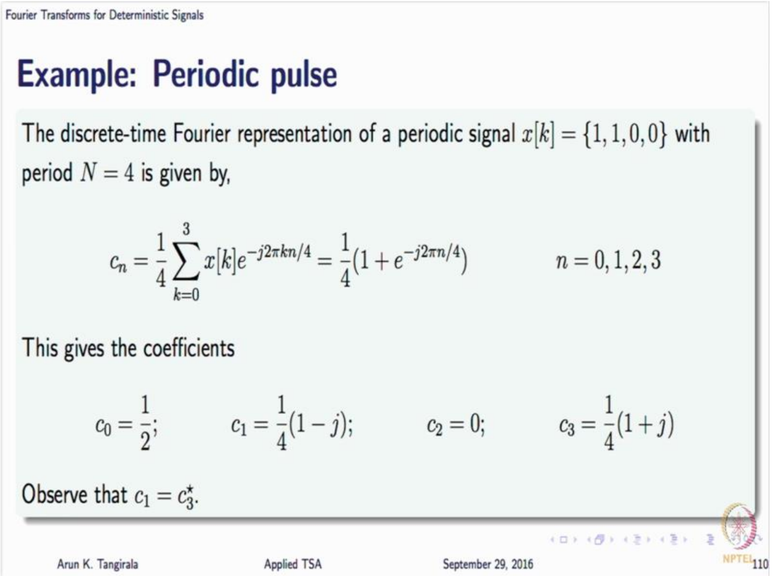

So, very simple example here I have a periodic pulse and I am just showing you how to calculate the Fourier coefficients and you can see the conjugate symmetry property here for the coefficients that is a very simple example to show you how Fourier coefficients are computed for periodic signal. All you have to do is in this case here again it assumes that you know the period, question is what do I do in practice in the practical aspects we will discuss next week, at this moment we are assuming I know the period of the signal and then I only want to know the contribution of different harmonics to the signal that that is the question that we are dealing with.

So, before I quickly move on to the discrete time aperiodic case; we now state a very important and fundamental result which we will see in the random signal domain as well; which is that the power spectrum not the power spectral density.

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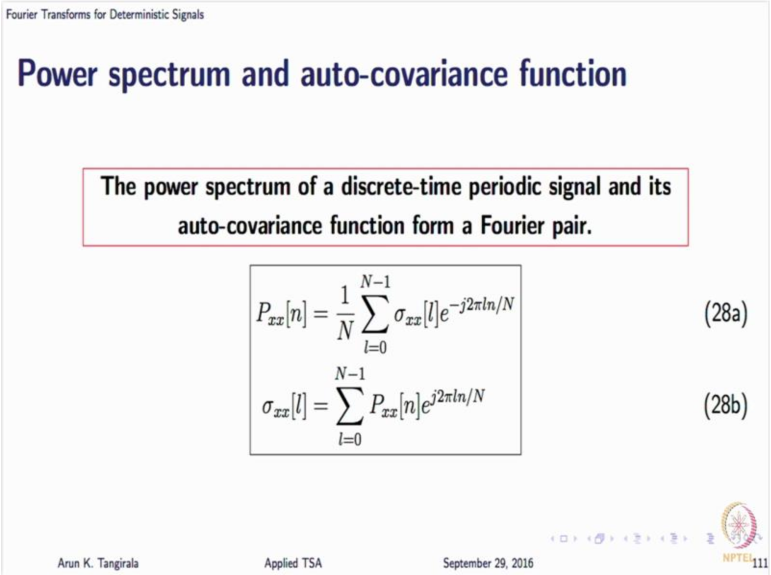

The power spectrum and the auto-covariance function, you must recall auto-covariance function that we defined two lectures ago for periodic signals. They form a Fourier pair, what do you mean by Fourier pair; the Fourier transform of the auto-covariance is the power spectrum and the inverse of the power spectrum is the auto-covariance function and that is what this equations 28 a and 28 b are telling you right; here you should now think of the auto-covariance has some sequence as that is why I have said earlier it is important to think of signals as sequences.

So, you can see that the first equation on the top is a decomposition of the auto-covariance function and the second equation is a synthesis equation; that is if you were to synthesis auto-covariance function sequence using the power spectrum that is what you would see and they look identical to the expressions that you have just seen for the signal; for the discrete time periodic signal, instead of thinking of the discrete time periodic signal as a signal; think of it has a sequence.

Then you will observe that the bottom equation for the auto-covariance function looks exactly similar to the one that we have been already for the synthesis equation for the periodic signal. All you have to verify is the auto-covariance function periodic, is it periodic for periodic signal that is it, the moment I have a sequence that is periodic; I can use all the concepts that I have learnt for discrete time periodic signal. The only difference is earlier we talked about signals, now we are talking of auto-covariance sequences; that is all. So, this will appear now later on even for the discrete time aperiodic case and even for the random signal case.

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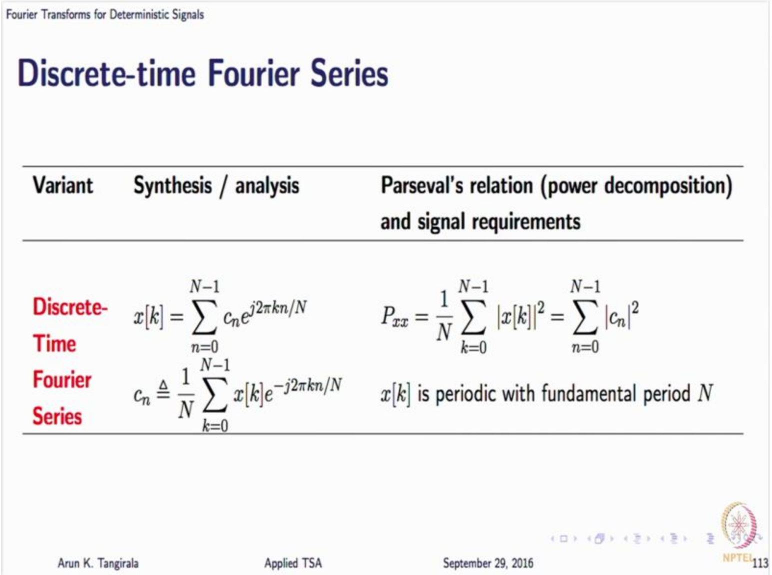

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Now, this is just a summary of what we have leant and I am going to move on quickly to the discrete time Fourier transform, we will discuss half of it today and then continue with it next week. So, what is the difference now, how do I go from discrete time Fourier series to Fourier transform same story, what did I do in the continuous time case; I thought of the continuous time aperiodic signal as a periodic signal with infinite period same story here, but keeping in mind this fact. This integers still remains now as np goes to infinity then this within this set the it becomes a continuum.

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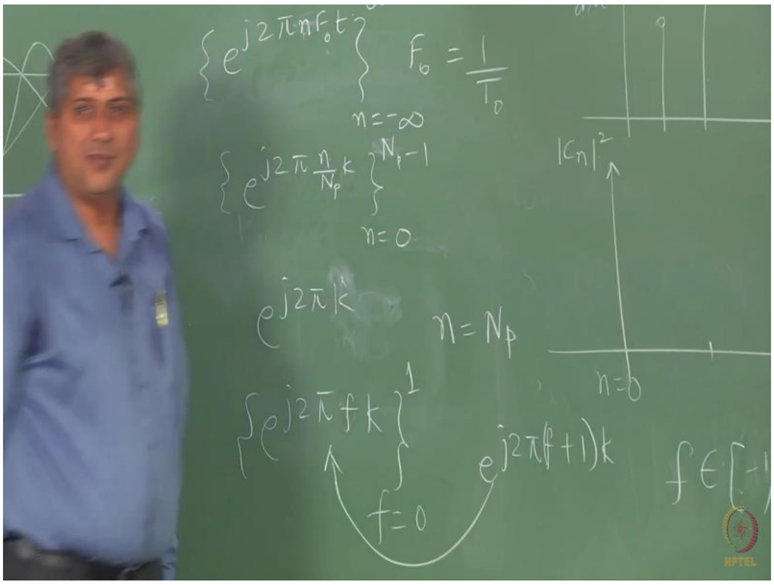

The still is repetitive; in the sense now the set of atoms that I have are going to be e power j 2 pi f k and this is again periodic what is the period of e power j 2 pi f k.

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1; just 1 in cyclic frequency is 1 right; e power j 2 pi f k is the same as e power j 2 pi f plus 1. So, e to the j 2 pi f plus 1 k is the same as this am I right, so the periodicity of e to the j 2 pi f k is 1 in cyclic frequency. Now I am using a small f look at that I am using the lower case and not using the upper case, just to distinguish the continuous time from the discrete time case alright.

So, here as well I have periodicity, but now I do not talk of this situation here and in terms of harmonics; I just talk in terms of frequencies. In other words, we call this whatever this interval 1 that you have after which the atoms repeat themselves is called the fundamental frequency range, that is it is sufficient to focus on frequencies of any interval minus half to half for example, or 0 to 1; whatever it is you take any frequency of interval 1 and consider all the complex sine waves over that interval; that is sufficient to explain a discrete time aperiodic signal with finite energy. The same here as well, but the only difference here is; here you have a discrete set whereas, here you have a continuum.

So, in other words now f runs for example, either from minus half to half or 0 to 1 that is all or in terms of omega the angular frequency runs from minus pi to pi or 0 to 2 pi and that is why you see this synthesis and analysis equations.

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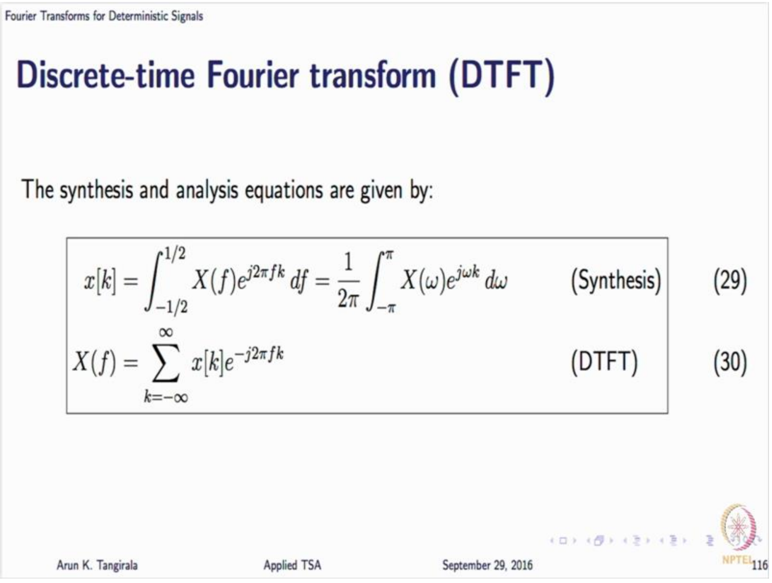

In the synthesis equation you see that the integral runs from minus half to half, what it means is it is sufficient to consider the sine waves in that interval, the discrete time sinusoids. The signals are discrete in nature in time, but not in frequency, you have to understand. Whereas, for the periodic case both the signal was discrete in time as well as in frequency, do you realize that in a continuous time periodic case the signal was continuous in time, but the frequency axis was discrete whereas, for the discrete time periodic case, the signal is discrete in time as well as in frequency.

Now, when we move on to the aperiodic case; obviously, for the discrete time signal you are still looking at discrete time, but the frequency axis is continuous, so that is the difference and. So, the simple rule to remember is whenever you have periodic signals continuous time or discrete time, the frequency axis is discrete and whenever you have aperiodic case the frequency axis is continue that is all; does not matter whether you have continuous or discrete time and notice the difference in the synthesis equation depending on whether you use cyclic or angular frequency and what you see at the bottom is the standard discrete time Fourier transform; when you say discrete time Fourier transform it is of the discrete time signal and you can see once again that x of f which is the discrete time Fourier transform is also periodic that is x of f plus 1 is a same as x of f x of f plus 2 is x of f and so on that is got to do with the nature of the atoms alright.

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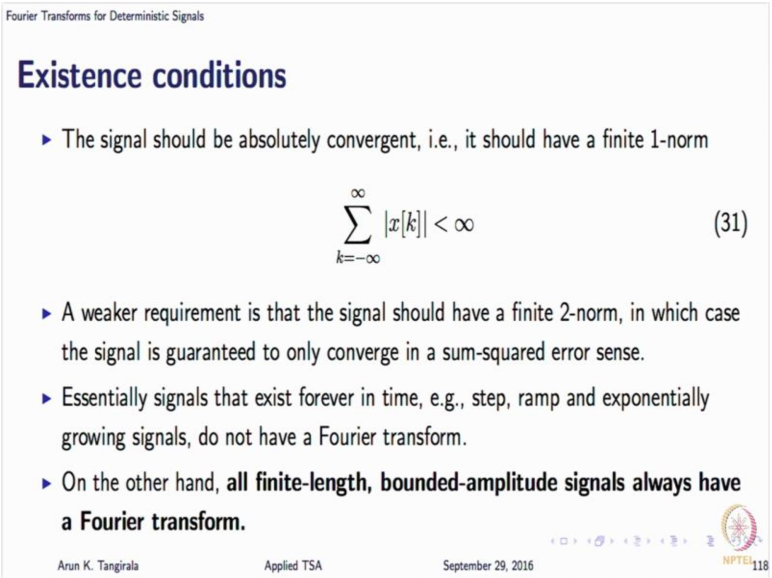

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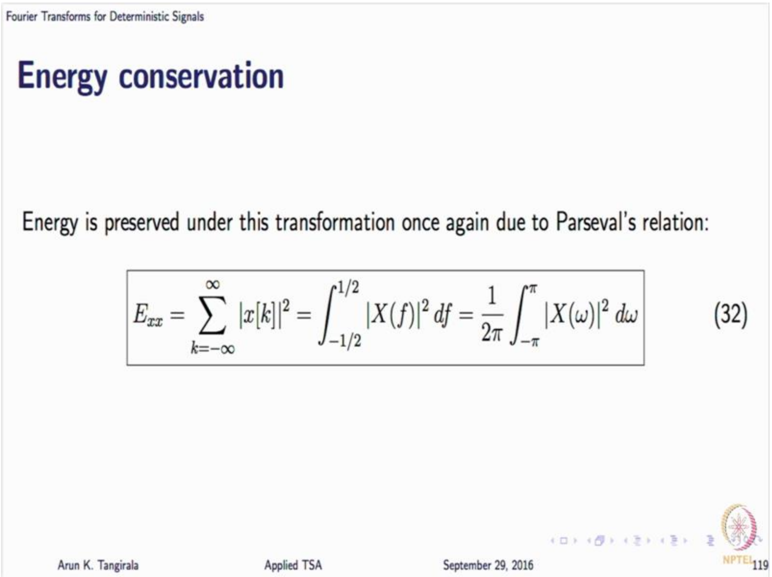

So, existence conditions are more or less the same; I am not going to talk about it. I just wanted to talk about the energy conservation; we will adjourn in a minute and half. So, the energy conservation looks pretty much similar to what you saw in the continuous time case. The only difference is now the energy for the signal in time is computed using a summation as against and integral. In the frequency domain, you still have an integral and what is the difference between the continuous and the discrete time case. In the continuous time case, in the frequency domain you had integral running from minus infinity to infinity whereas, in the discrete time case you have frequencies running from minus half to half because that is your fundamental frequency range.

And please remember again depending on whether you use spectral energy density whether you express it in terms of cyclic or angular frequency, the expression is slightly different there is a 1 over 2 pi factor there.

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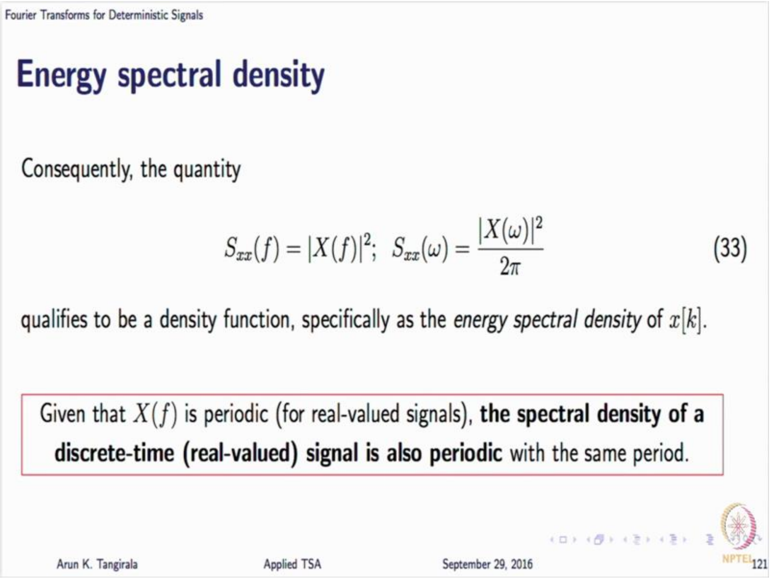

And let me just point that out, so if you have to use the spectral energy density; if you have to express it in cyclic frequency, it is simply mode x of f square, but if you have to change it per angular frequency; obviously, then you have to (Refer Time: 20:35) the conversion from cyclic triangular frequency, that is the only difference and I just wanted to show you 1 example and then will conclude alright.

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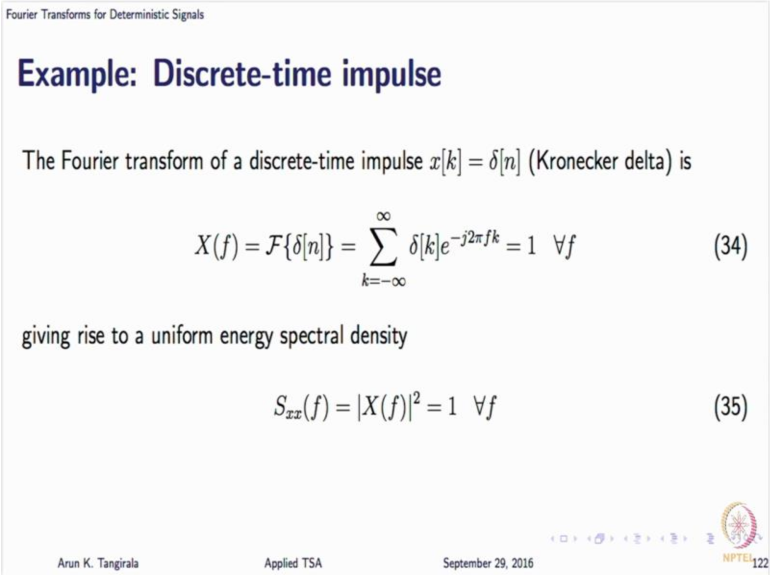

And it is very easy to evaluate unlike the continuous time case where you if you have to look at the dirac as I said you need some special theory to evaluate the integral. Here it is a Kronecker delta function that we are looking at; all you have to do is plug in this kronecker delta into the expression for dt f t and you get this expression and what you get here the energy spectral density is flat it is uniform and you will see something like this coming up when we look at the random signals white noise, where we will see the auto-covariance function is an impulse and we already have a hint that the power spectral density is a Fourier transform of the auto-covariance function and therefore, the power spectral density there you do not compute Fourier transform and then take magnitude square. You directly use auto-covariance function and apply Fourier transform, take a Fourier transform of it; you get the power spectral density and it turn out to be 1.

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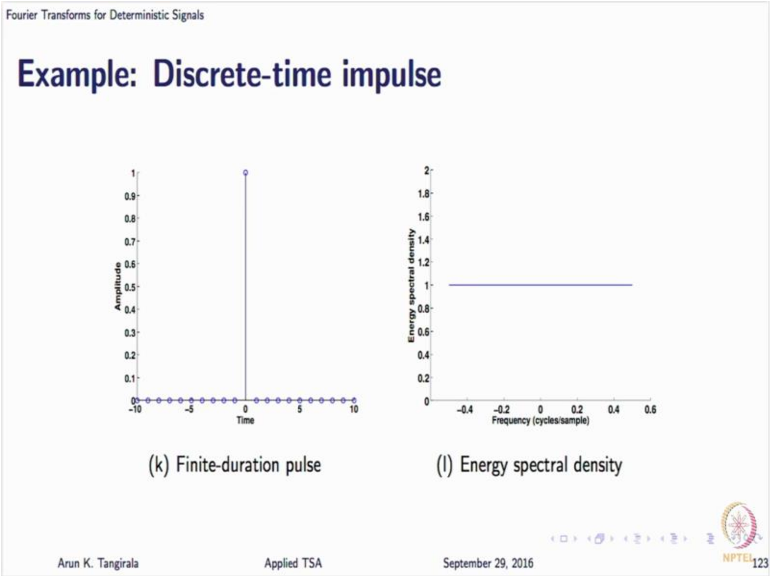

So, is the case here and this is how it looks like. Do you see the duration band width principle again showing its face here, it says you have a signal highly localized in time, its energy spread is also highly localized therefore, whereas its energy is spread in over all frequencies, so that is it for today you have to move to the next class. So, let you go when we come back; next week will discuss more examples and then discuss d f t.