Applied Time-Series Analysis
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Lecture – 63
Lecture 28 A - Fourier Transforms for Deterministic Signals 5

Not much as I have explained basic idea is to break up the signal into sins and cosines. But one as to exercise some caution when it comes to the mathematics, depending on the nature of the signal, that is all you have to remember. The basic ideas remains the same whether you look at continuous time or discrete time signals or periodic and aperiodic signals. In fact, today very soon we will move on to the discrete time world, where there is going to be definitely an additional source of confusion and complication, but the underline principle is a same, it is just that the mathematics is slightly different and may be some practice will help you get over this confusion.

So, let us get started from where we left of yesterday. So, we started to look at Fourier transform and this Fourier transform was introduced in the context of continuous time aperiodic signals with finite energy.

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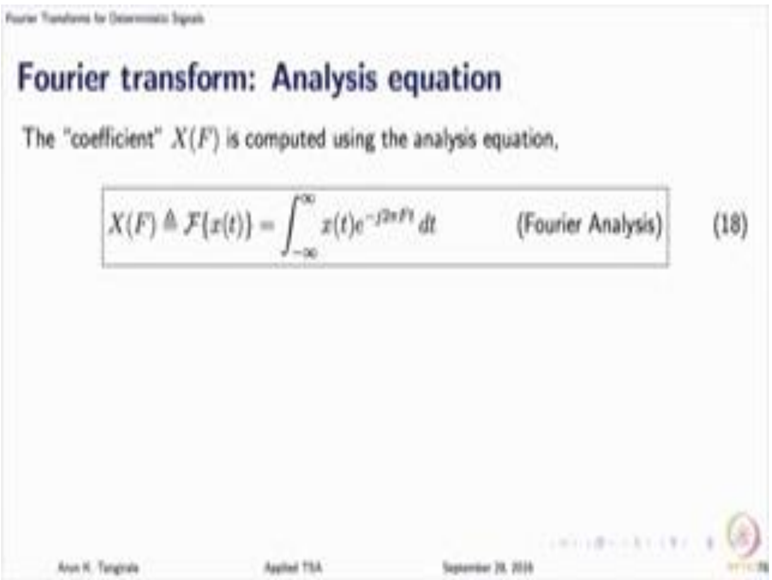

Now, why are we looking at finite energy signals? Basically, if you look at this equation here for the Fourier transform you can see and we have discussed this earlier as well, the Fourier transform itself exists first of all when this signal is absolutely convergent.

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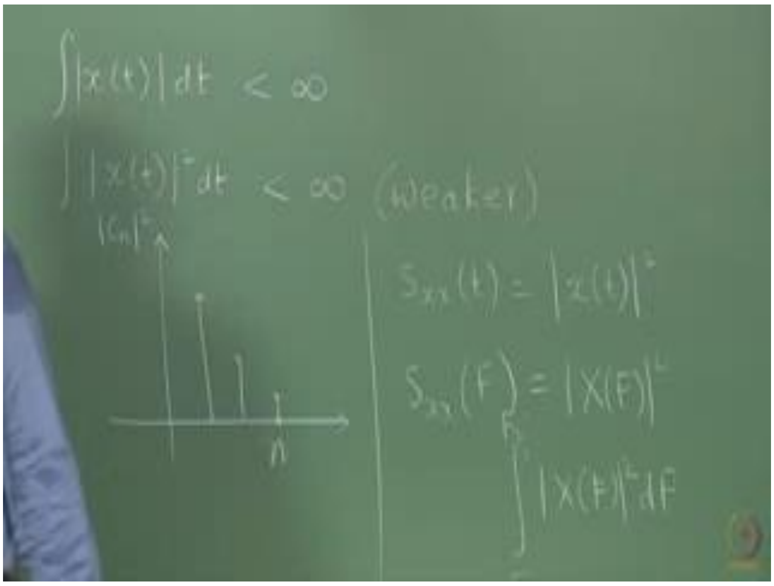

We have talked about this before, but in the context of discrete time signals. So, here we are extending the same to the continuous time signals. Of course, the integration runs from minus infinity to infinity; this is a necessary and sufficient condition, a slightly weaker condition for the existence of Fourier transform is that the signal has finite energy. So, this is a weaker requirement whereas the one on a top is a stricter requirement all right. And in any case if this is satisfied definitely the signal as finite energy, but not vice versa and as long as this is satisfied you can say that the Fourier series that we had here sorry the Fourier integral that we have here converges in a mean square error sense, what we mean by mean square is, as we include more and more frequencies, the error between the approximation of x of t remember we talked about this approximation yesterday.

One can construct an approximation of a signal using Fourier transform by only concentrating on a select set of frequencies and that is the basic idea that is used whenever you want to approximate a signal using Fourier transform.

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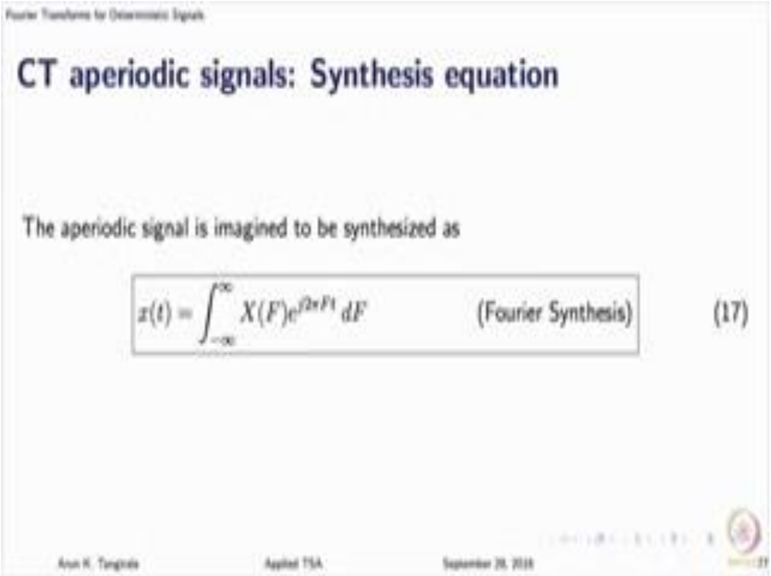

So here as well instead of including all the frequencies in this integral that you see in the synthesis equation, I can focus on a band of frequency and as I include more and more frequencies the error in the approximation starts to decrease and the weaker condition says that as long as the signal has finite energy the error will go squared integral squared error will go to 0 as of goes to infinity that is as you include more and more frequencies.

Whereas the first one is a strict one; it says basically as long as the signal is absolutely convergent, this integral will get you the signal of interest any way. So, those are existence conditions, what you have to remember practically is Fourier transform of any signal and here I also want to broaden your horizon, although we are talking of signals you should think of these as sequences as well because very soon we shall talk of Fourier transform of auto covariance functions and auto covariance function is not a signal per se, but it is a sequence.

So, whatever you are learning for signals equally applies to sequences, if just a different terminology. So, here we looking at continuous times sequences and what we are doing is we are saying I have continuous time a periodic finite energy signal and I am going to look at the frequency content of the signal, here we are not looking at fundamentals to harmonics we just want to know what frequencies are present when you think if the deterministic signal, but a better question to ask is what frequencies are contributing to the overall energy of the signal? Now why I say that this is a better question to ask is when we move to the world of random signals, we do not think of necessarily although it is possible to do that necessarily we do not think of signal reconstruction we do not think of necessarily signal decomposition, we are mostly focused on energy or power decomposition because we are going to talk of random signals and they are power signals.

We are going to talk about power decomposition rather than signal decomposition and that is why I been emphasizing this transition from signal decomposition to power or energy decomposition sorry. So, here let me now go further and show you the energy decomposition, all of this we have discussed I am going to move on; this again is due to a partial relation which shows that energy is conserved in both domains, but apart from that statement what this tells us is how frequent how the frequencies is a certain band are contributing towards the overall energy right, if you compare this with what you saw for periodic signals, we had first of all power decomposition earlier and secondly, while the signal still had in the time domain we had the integral, for the in the frequency domain we had a summation because we where only talking of fundamentals and harmonics where as now we have a continuum of frequencies and that is why we have now an integral in the for the frequency domain as well.

Now, earlier we talked about line spectrum; what I mean by earlier for periodic signals we talked about line spectrum, simply because when we plot the square of the Fourier coefficients verses the index, it is only defined at some at discrete values of n and so on right this a typical sketch of mod c n square verses and we call this as line spectrum and we said we cannot think of a density here, where as now with the a periodic signals since we have a band of frequencies or a continuum of frequencies, we can think of a density function what we mean by density is? Energy density here because we are looking focused on energy signals and this relation due to Perceval, tells us or gives us a hint of what could be an energy density in frequency.

Earlier we had defined an energy density in time and the energy density in time was defined as simply mod x of t square. Why did we define this as a energy density, what was the reason for defining this as a energy density?

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Yeah the area under this will give me the energy right now we have a hint here, what would be the energy density in frequency. If I yet to ask, what is the energy density in frequency? That is energy per frequency that would be mod X of F square that is it. So, magnitude square of the Fourier transform is energy density. So, there is striking similarity in the expressions, but you have to remember after all that the area under this energy density should give me the energy area of course, the frequency is running from minus infinity to infinity. Now if I want to know, what is the energy contribution; or you can say the energy contain in a band of frequencies, then you can do that you can actually get the energy over a frequency band F 1, F 2 by simply integrating the energy density over that band naturally.

And whenever you have difficulty understand understanding these densities, go back to probability density functions; there the role of the probability density is to allow me to calculate probabilities, here they are allowing me to calculate energies that is all or power whatever soon will talk of power spectral density also. So, where ever you look at densities, the concepts remain the same the area under the density will give me some quantity for which it is a density function and then you can think of moments and so on. In this course we do not per se deal with moments of this densities, we are only talking about density functions that is all; in a more advanced course we go into the moments of this density and see how they are useful in signal analysis as I said in a joint time frequency analysis, we look at the moments namely duration bandwidth and so on or mean time mean frequency, center frequency and so on which will give us some valuable insides into the features of the signals ok.

So, any questions on this as I said now you can see a pattern whether it is continuous time periodic or a periodic, we come up with the synthesis equation, then we have an analysis equation and we briefly talk about it is existence and then move on to either power or energy decomposition, if a density function can be defined we define so otherwise not. But the utility of this analysis if what you have to understand; while utilities are manifold, what we are focused on is the decomposition of the energy or the power right.

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Once again a plot of mode of X of F square verses frequency; here once again we only plot over the non negative frequencies because it is symmetry, you can show it is symmetric with respect to frequencies tells us what are the predominant frequencies present in the signal right for example, mode X of F square can look like this flat what does it tell us? All frequencies are contributing uniformly to the overall energy, what kind of signals would have that?

Student: impulse.

Impulse right and impulse signal direct. In fact, it is continuous time function that we are looking at therefore, it is drag that you have to think of and let me tell you cannot straight away use this integral that you have used earlier to evaluate the for example, you cannot use this synthesis equation to come up with the Fourier transform of drag.

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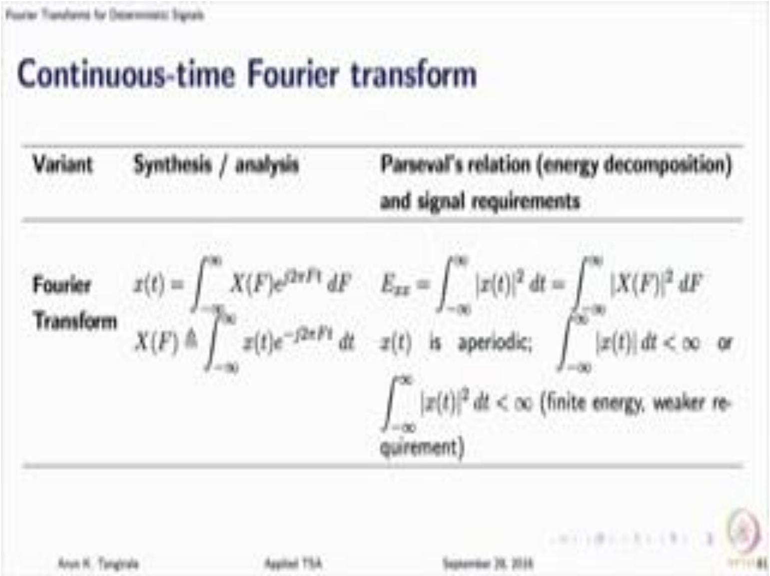

You need what is known as theory of generalized functions do not get scared we will not talk about it, but for special class of function, for which you cannot just ordinarily evaluate this integral there exist a theory of generalized functions, that will allow you to compute the Fourier transform in any case. So, this a very important thing to remember; the Fourier transform of an impulse whether it is continuous time or discrete time has a flat energy spectral density all right and there are many other possibilities it could have like you could have a density function that looks like this or this or may be this ones. So, there are numerous types of density functions that you can have depending on the frequency content of the signal, what makes up the signal and remember ultimately there is no signal without a system.

So, very soon we will talk about if this is some density of some function some signal x 2 here, we will may talk about this in the discrete domain, but does not matter the results are the same, you can think of if this is the density function of some signal x 2 and imagine that to be generated by G which is being driven by an input x 1; then you can use these densities to draw inference about what characteristics a system has, provided you know the frequency content of x 1 and that is what leads us to filtering ideas ok.

So, let us move on and now look at an example talked about all of this.

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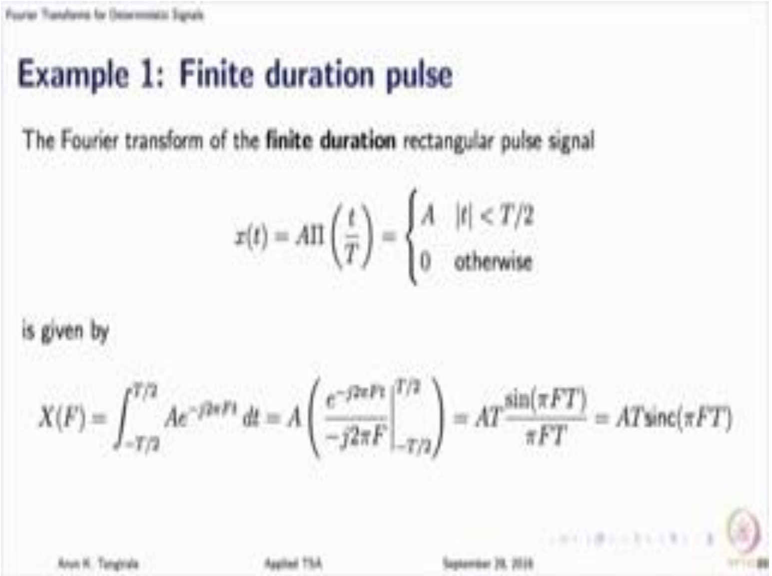

So, here is a finite duration rectangular pulse signal unlike the periodic case where as I gave you expression for one period, this is exactly exist only for the period that I have shown for the interval that I have shown, because we are looking at finite duration signals and periodic signals. So, the Fourier transform in fact, given by the sinc function; sinc function is your sin c function that you must have encountered in the different situations and this integral is very easy to evaluate is nothing to be scared about. How does sin c function look like? It looks like this crazy creature that you see in Hollywood some alien movies and so on which as it is hands spread all over as I say [FL] something like that.

So, that is what you see here if you look at the shape; on the left hand side we have the energy density in time.

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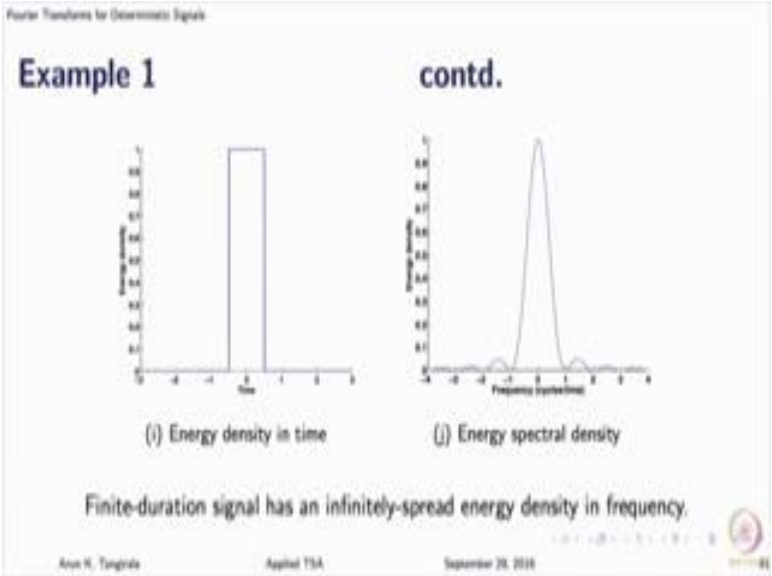

It is not the signal remember the signal also looks similar what we are showing is mode x of t square and on the right hand side you have mode X of F square, this is not the Fourier transform or the magnitude of the Fourier transform all right. Now what do you notice here? The finite duration signal that we had has an energy spread over a finite interval in time, that is the energy goes to zero after some finite time, let me say energy density goes to 0, let me not say energy exactly whereas, when you look at energy spectral density that is mod X of F square, it is it does not go to zero identically even after any finite frequency, it only goes to zero asymptomatically.

Now, this has profound implications in time frequency analysis that is when you want to know for example, what frequencies where present at what time; we will talk about this now duration bandwidth product being bounded below by some quantity, what this result says is in fact I will show you couple of example as well, is that whenever I have a signal whose energy is are localized over a finite interval in time, you cannot have the energy also localized over a finite frequency band; any signal whose energy is localized over an interval in time, will have an infinite spread of energy density in frequency as well in frequency.

So, the vice versa applies if I have an energy dense spectral density that is only concentrated over a band and not like the one that you see here, then you are guaranteed that the energy density in time will be spread all over we just now saw right. So, this is what we have drawn here is a flat line is an extreme case of the example that we are looking at. The Dirac is extremely localized; you cannot have a more localized than a Dirac. So, let us at an example to understand this relation between so called duration; which has got which is a measure of the spread of energy density in time and bandwidth which is a measure of the energy spread in frequencies.

Bandwidth is a term that you would encounter in almost every field of signal analysis; in control, in signal processing and so on and typically we even in moderns when we talk of moderns we talk of bandwidth and so on. So, bandwidth as got to do here with this spread of energy in frequency or energy density in frequency.

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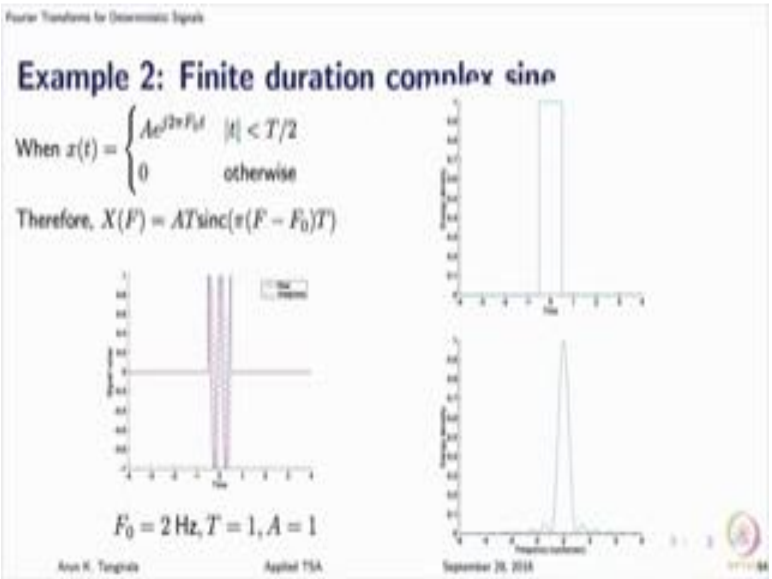

So, let us looks at a an example here another example where I have a finite duration sinusoid; not like your regular sine wave of course, I have taken a complex exponential and it turns out that the Fourier transform is once again a sin c function shifted in frequency. So, here is how the signal looks like in time, it is a finite duration sine wave you see both the real and imaginary parts that is what I am showing you here because that is a complex sine wave and what you see on the right now on the top you see the energy density of the signal in time and what you see at the bottom is the energy density of the signal in frequency ok.

So, you can see that the energy density is exist only over a finite interval in time whereas, the energy density is spread all over in frequency. Now suppose I increase a width of the signal, what should happen to the energy density in time? It will spread then one should aspect shrinkage in the energy spread or the bandwidth of the signal, let us see if that happens.

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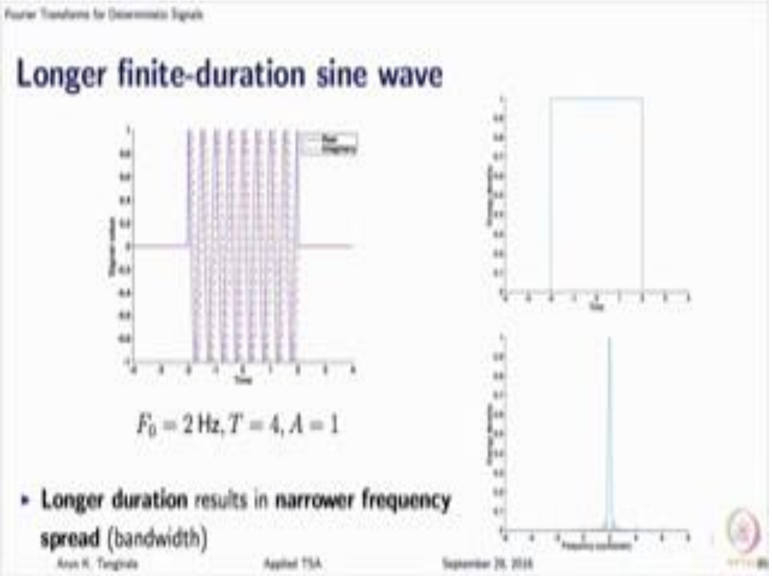

So, here I have now extended the duration of this sinusoids complex sine wave and as you as expected the energy density in time has dilated has increased, but what you see for the energy density spectral density? It as shrink right you can compare. So, here you have a larger bandwidth meaning more and more frequencies are coming in to explain the signal and as I expand the signal, fewer and fewer frequencies are contributing and that is got to do with the very nature of the Fourier atoms itself, we will not go deeper into why this is happening, but the intuitive answer to that is it is just the nature of the bases functions or sorry building blocks that we are using that the in a continuous time Fourier transform case, strictly speaking we cannot use a term bases function we can only use atoms; where as in the periodic case we can any way.

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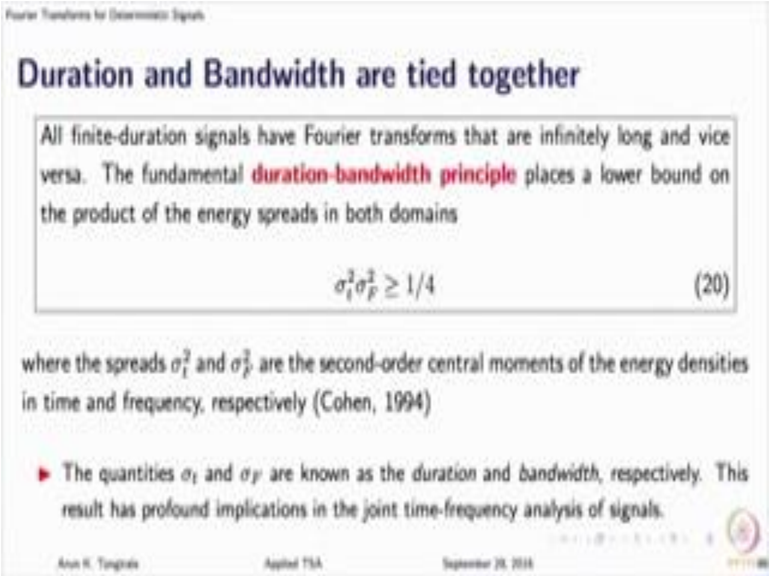

So, what you see here is that longer duration results in narrower frequency spread and this is just a general information for your results, this is known this is a fundamental result in time and frequency analysis which is also known as the uncertainty principle for signal analysis, although there is nothing uncertainty here there is no randomness or probability here, the only reason why this is known as uncertainty principle relation in signal analysis is because of it is striking similarity with some other result that you see in physics what is that?

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The Heisenberg's uncertainty depends; which says that you cannot really locate the position and the velocity you cannot really locate the position and I have measure the position and velocity of a particle with arbitrary accuracy at the same time. If you have located the position then you have lost you have huge uncertainty about it is velocity and vice versa. So, the sigma square t that you see here although it is not corresponding to any random event, it is still corresponds to second moment the sigma square t is a second central moment of the energy density in time and the physical meaning of that is a duration; it gives you an idea of how long the signal exit it exist in time, it is not exactly equal to the duration of the signal, but it is a measure of that pretty much like your variance. Variability gives you an idea of the spread of outcomes and sigma square omega is the bandwidth, it is once again a measure of the spread of energy in frequency.

So, what it says is the energy spreads in times and frequency cannot be arbitrary small if one shrinks the other as to expand and vice versa what is implication again we are going to talk about the implications, we may not even be hampered by this result in this course, but in a broad scheme of things we are limited by this result in any joint time frequency analysis; when I say joint I would like to know for example, over what duration what frequencies existed contributed, when I am doing that see you think of time domain analysis as looking at this direction and frequency domain analysis as this direction they are kind of orthogonally you can think of and joint time frequency analysis is looking at some diagonal trying to figure out what is happening here and there and it says you are limited when the moment you do a joint time frequency analysis; any way so we will move on and before we move on to the discrete time world, I want to introduce what is known as Fourier stieltjes transform. It is not a new transform let me tell you do not get scared remember you have two different expressions in a continuous time case, where depending on whether it is periodic or a periodic.

In the case of periodic we do not use the term Fourier transform for we just say the Fourier coefficients and then we have an expression for the Fourier coefficients that we saw in yesterdays class and today we talked about Fourier transform, which is an integral. So, on one hand you have summation and the other hand you have integral, is it possible to use both this expressions and that is what Fourier stieltjes transform does. It says define an increment d x of f as x as x of f times d f. So, you have this equation that you saw for Fourier transforms right integral X of F sorry.

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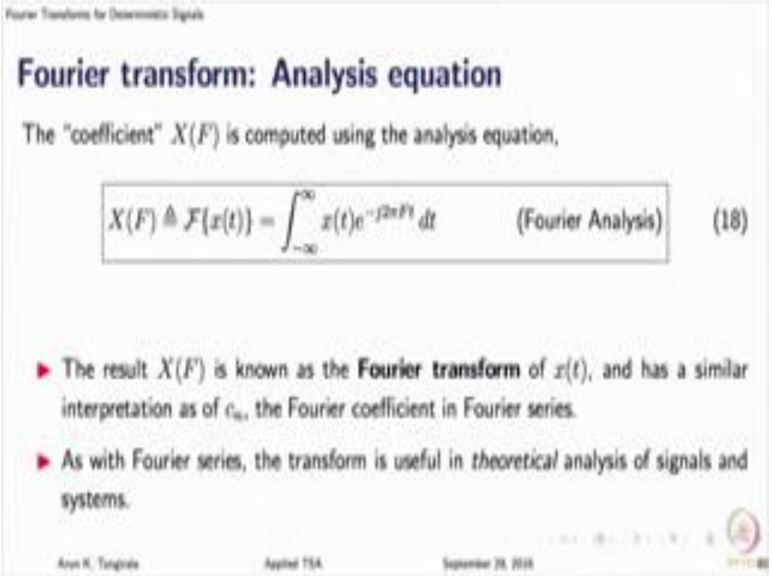

X of t e to the minus j two pi f t d t and go back to the synthesis equation.

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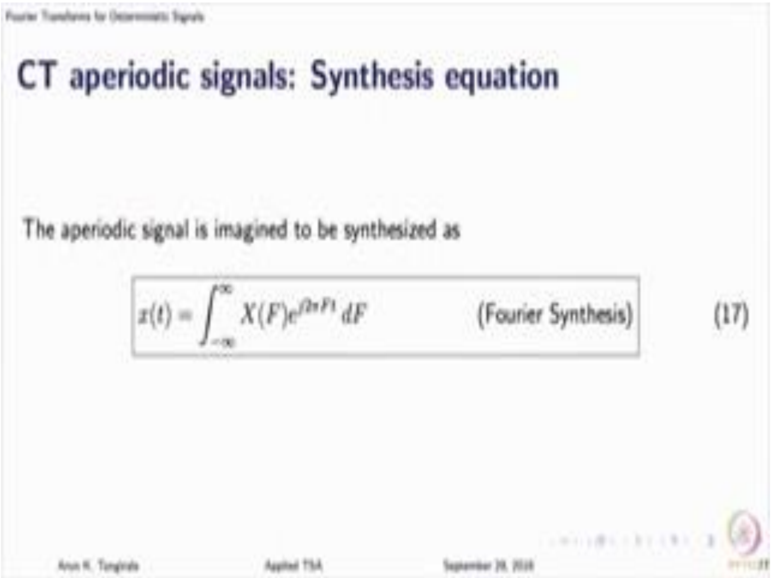

So, you have this X of F e to t he j 2 pi F t d F.

Replace this X of F d F with some increment of think of it as an increment in X of F that is what it says. So, when you do that you get this expression all we are doing this we are rewriting the synthesis equation using this d X of F that is all we are doing.

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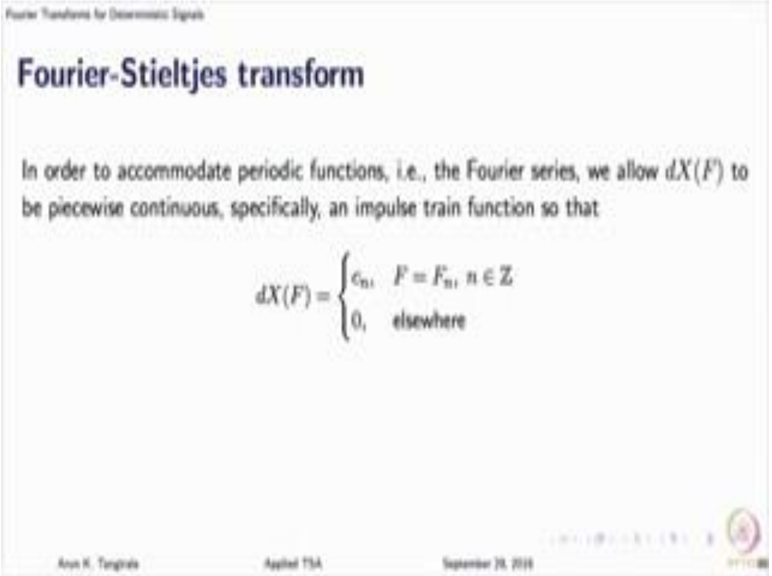

Now what is a advantage of this, how does it help me if we fuse both this results? It helps me in fusing this result by saying that whenever the signal is periodic for example, then I let d of d X of F to be this that is it is it looks like spikes.

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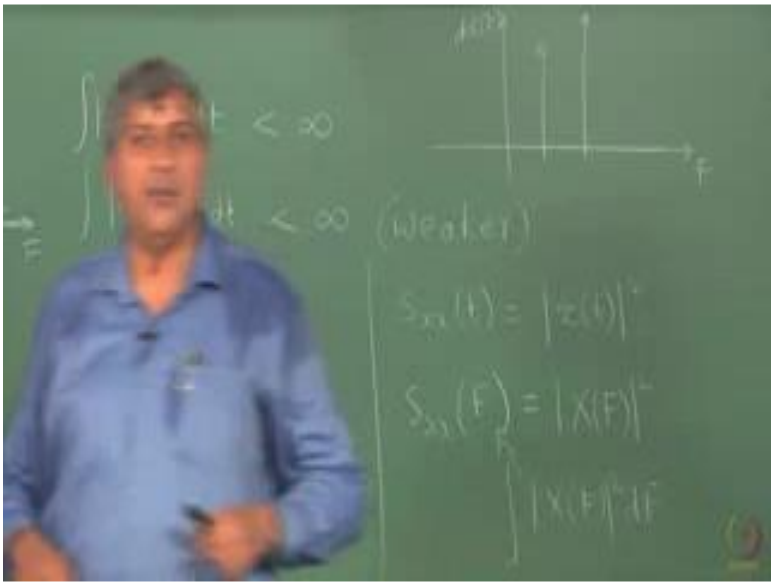

So, the d X of F would only have c ns right this is how d X of F would be it is not defined at intermediate frequencies at all, it is only defined at discrete set of frequencies so that the integral boils down to summation.

When it comes to aperiodic signals, I am going to let d X of F being defined as X of F times d F then I get the Fourier synthesis equation. So, that is the basic idea you see that and this is a standard trick that is used also in probability density functions and random variables and so on; although I did not discuss that you can write the expression for the probabilities for both discrete valued and continuous valued case by adopting this kind of a strategy. In the continuous valued random variable case we said the probability is simple the integral f of x f x right; whereas when it comes to discrete valued case we do not use that integral we simply use the mass function and say it is sigma f of x; you can fuse both those expression using the same trick or we have not discuss that, but you can now think of the same way.

So, this is a useful in fusing both words, but otherwise nothing much more we will make use of this expression a bit later on when it comes to random signals. So, let us move on now to the world of discrete time signals unless you have any questions.