Applied Time-Series Analysis
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Lecture – 62
Lecture 27B - Fourier Transforms for Deterministic Signals 4

Now, of course there are some rigorous questions that one have to ask. For example, the summation that I have here does it always converge, when does it converge, for what class of periodic signals can I construct such an expansion. That is one question. And the other question is in the analysis equation will this integral always yield a finite value, are there signals which will trouble me, it is always the one has to ask these questions and we have learnt to ask these questions.

The answers are not so binding, in other words for a large class of periodic signals all of this is possible. There are going to be some ill behaved signals, we call them ill behaved because they are not friendly enough to the Fourier analysis, but otherwise they are you know poor innocent signals. They are not amenable to Fourier analysis, but do not worry we do not run into such signals. I will nevertheless take the conditions soon. What is more important to us is to understand how to make use of the c n and that is we are actually going to do. And remember I am just repeating what I have said earlier whatever we are learning in the continuous time case whether it is periodic or a periodic it is useful in theoretical analysis only.

So, just to summarize here, I am going to skip this line here we have talked about it let us actually try to understand what c n has what information c n encodes. There are several ways several advantages of this c n in several different ways in which you can use c n. First of all remember that a periodic signal is a power signal, we have talked about this yesterday. And the beauty now comes about with this relation due to Parseval who showed that the power of a signal which we calculate using the expression that I gave yesterday can also be calculated using the Fourier coefficients. So, c n are called Fourier coefficients.

Please do note: call the expression that I have given for c n as Fourier transform we have just called Fourier coefficients whatever integral you want to call this as you can want to call as integral or whatever, but it is not yet Fourier transform. What is the beauty of this relation, why is it so nice? Can anybody explain I mean you can say- mathematically it is a beautiful result and you can keep looking at it and admire the beauty, but practically do you find any utility of this result? Sorry.

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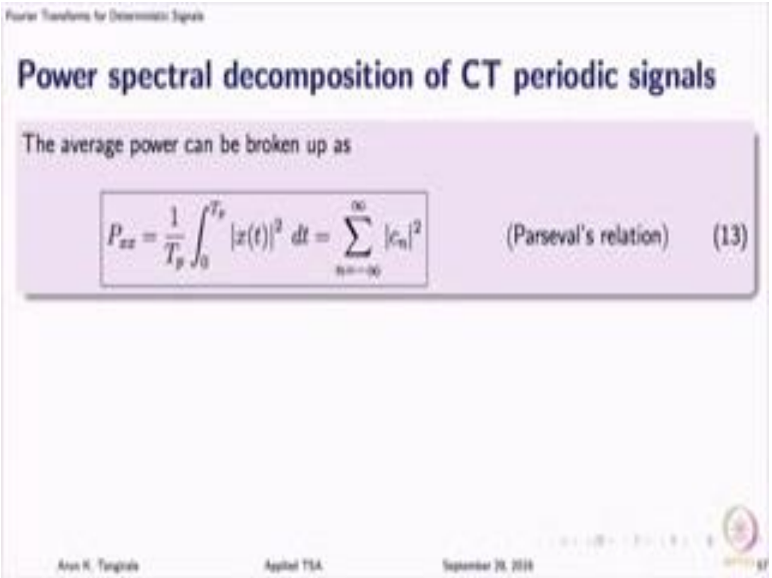

Power of each periodic signal, signal is x of t, it is ok. How do I make use of this result, what is it that I want to know of the signal. That is as far as calculation is concerned, but in signal analysis I am not so much worried about calculation.

Student: (Refer Time: 03:29).

Anybody else, sorry, it is not energy power; power is conserved, yes that is a beautiful result that Parseval showed. But, fine I mean do we stop there or do we go ahead and go further and make use of this. This is fairly innocuous integrals. Anybody else from that other room; what would be the utility of this Parseval's relation? If you understand the utility of this you will see similar relations coming up in all the remaining three cases continuous term aperiodic and the discrete and periodic and aperiodic; you will see similar relations. Here we are showing power is conserved or in both time and frequency domain. And how do you calculate power in both domains, but that is not that is a very superficial I would say utility of this result as far as signal analysis is concerned. Anybody from the other room; what is it that we want, yes.

Student: We can see with which frequency the power is concentrative.

Ok and how do we do that?

Student: we get to the (Refer Time: 04:54).

Ok

Student: plotting it on a graph where cm square.

Right, versus n.

Student: (Refer Time: 05:06) what n we have maximum power points.

Good, very good so that is essence of this utility. What am I going to do with; I will be able to figure out now for example.

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In the simplest case maybe there is a peak only at this n let say n equals 2 and at all other ns you have the contribution to be 0 or you can say I have at n equals 1, n equals 1 is a fundamental and n equals is the harmonic, so there is only one fundamental and one harmonic.

Now, I know the cyclicity of the signal first of all. Of course here I am assuming a node t p, but I do not know how many are present. I only know the fundamental period I do not know exactly what are all the different frequency components that are present. The fact is if mod c n square is 0 then c n has to be 0. So, when I plot why am I plotting mod c n square because of Parseval's relation, why am I not plotting mod c n because I do not know exactly how to interpret that. The general tendency is to plot mod magnitude of c n, because c n is the weight it is telling me how much each sign is present in a signal.

So, the natural tendency is to plot magnitude of c n. This relation says look at magnitude squared magnitude of c n because it has a meaning and the squared magnitude of c n is the contribution of the nth harmonic to the overall power of the signal that is the main result that comes out. And that statement is based on a very beautiful property of the Fourier atoms which is a orthogonality property.

I cannot straight away make that statement just like that I cannot say my squared magnitude of c n is a contribution of the nth harmonic to the overall power. Why cannot I say that and when can I actually say that I can only say that if each atom is unique in the family. In other words what one they building block or atom or sign explains of the signal no other atom or member in the family should explain, then only I can claim mod c n square is the unique contribution of the nth harmonic to the power. If they are similar in some sense you can have, you can have in your building blocks that things that look similar. They may have their own unique things, but they may also be correlated in other words they are not orthogonal. Then it is hard to make this assertion, you can only say mod c n square is a unique contribution of the nth harmonic to the overall power if each harmonic is explains in uniquely something uniquely about the signal.

And that is why the orthogonality property is extremely useful in interpreting; I made the statement earlier I said, the orthogonality property of the Fourier atoms in this case in the Fourier series is useful in two different ways: one is in the contribution calculation of c n other is in the interpretation of the coefficients. So, the bottom line is mod c n square or square magnitude of c n is the contribution of the nth harmonic towards the overall power of the signal.

And when we plot such a signal that is magnitude c n square versus n we call this as a spectral plot, and because n is discrete we call this as a line spectral plot. You can never have a continuum continuous function mod c n square is not a continuous function. Very often you have to remember this is called spectrum this is no spectral density why cannot we say this is spectral density in frequency; I mean spectral density itself would mean density in frequency. Why cannot we use the term spectral density? So, frequency axis is discrete. So, you have to understand periodic signals, whenever comes to periodic signals you can never use the term density in a strict way, only for aperiodic signals you can think of densities. So, that is the first point to remember.

And the other thing that you should remember is this power spectral plot or the line spectral plot is symmetric with respect to n, whether I pot for negative n or positive n it looks the same. Therefore, it is customary to plot only for non negative n. Only the magnitude c n square is symmetric you can show that. Remember c n is a complex value number, therefore you can also look at the phase of c n that is argument of c n. And what is the phase of c n tell me, the phase of c n tells me when a particular sign that is that harmonic began in the signal the phase has got to do with how much things are offset.

Suppose, the phase is 0 for all the harmonics; that means, all your fundamentals and harmonics began at the same time as your x of t. Imagine always in this kinds of transform worlds that you are some kind of a mason who is building some wall or some house or so on and you have bricks with you and you are supposed to construct this house or construct this wall. Here the bricks are sines and cosines and you have to mange to explain construct anything only these bricks, you are not allowed to use any other bricks.

What Fourier series says is? If you are going to construct a periodic wall of a particular period you should only choose bricks of particular size. You cannot have bricks of any other size big and so on and the c n tells you how much these bricks are actually scaled, you can say how many such bricks are present in your signal and so on. So, some imagination always helps. So, I am going to skip past this and when I said power spectrum is symmetric it is true only for a real valued signal, but we are only going to deal with real valued signals.

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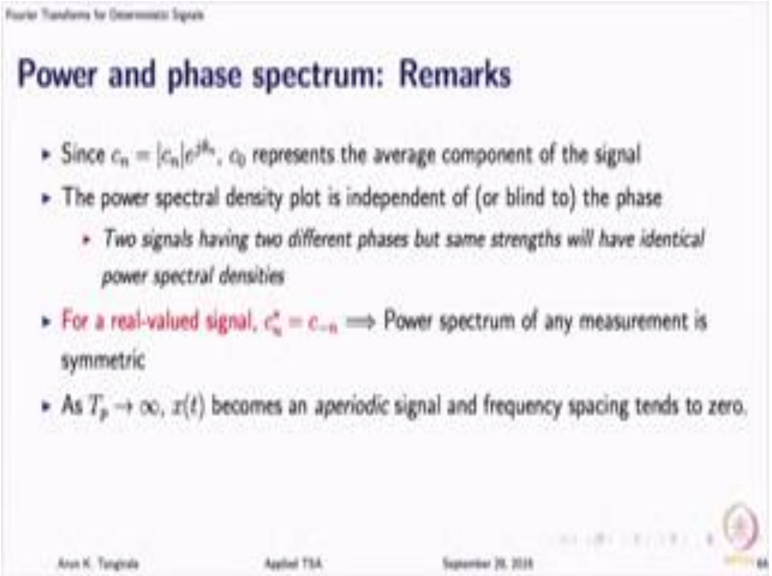

Now, very soon we will talk of aperiodic signals in a minute or so after we go through an example, where we take this Fourier series expansion and to the case of aperiodic signals and in doing so we say an aperiodic signal is a periodic signal with infinite period I can say its repetition. That is one way of imagining an aperiodic signal. That is my great great ancestors my great great grand children will never see its reputation. That is what is the definition of an aperiodic signal I cannot figure out I cannot find the time after which a repetition occurs. So, it is as good as saying t p is infinity.

So let us look at an example first here I have a periodic signal which has this values as a given on the screen over one period.

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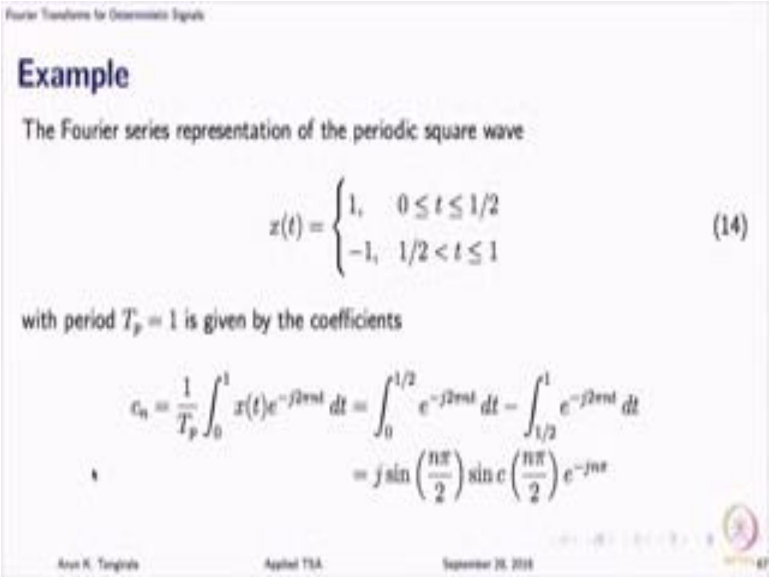

So, I have described the periodic signal over one period and one can then theoretically calculate the coefficients as I have shown on the screen I do not want to go over the sets you can go over the sets, it is fairly easy to go over the integrals it is not as difficult as one of you have said.

So, you can see if you were to look at the signal itself it has over one period looks like a pulse you know with differing signs half of the time it has one sign other half it has a different sign.

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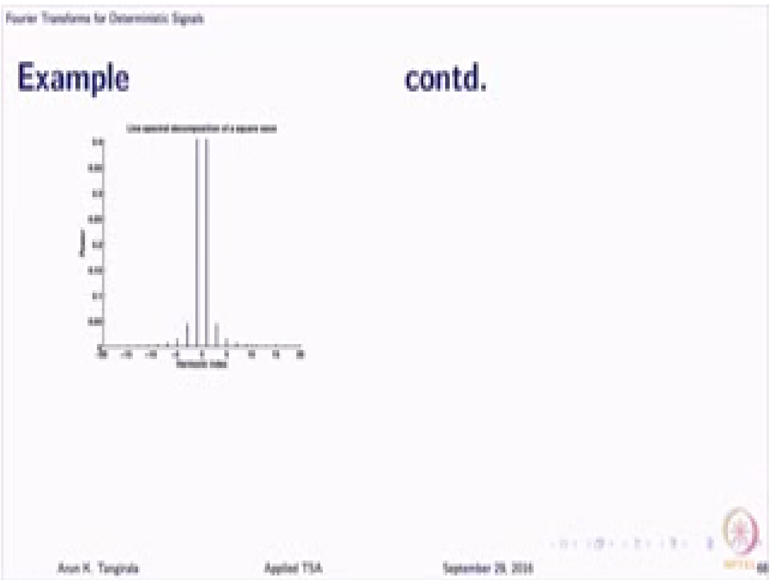

Now, how does; then by looking at the expression for c n there is not much appreciation, the moment you plot mod c n square versus n it has something to tell me. What is the period of the signal 1? So, what you see on the screen is a plot of mod c n square versus n of course, I am plotting it for both negative and positive because we are just beginning to understand and also to highlight the symmetricity of the power spectral plot.

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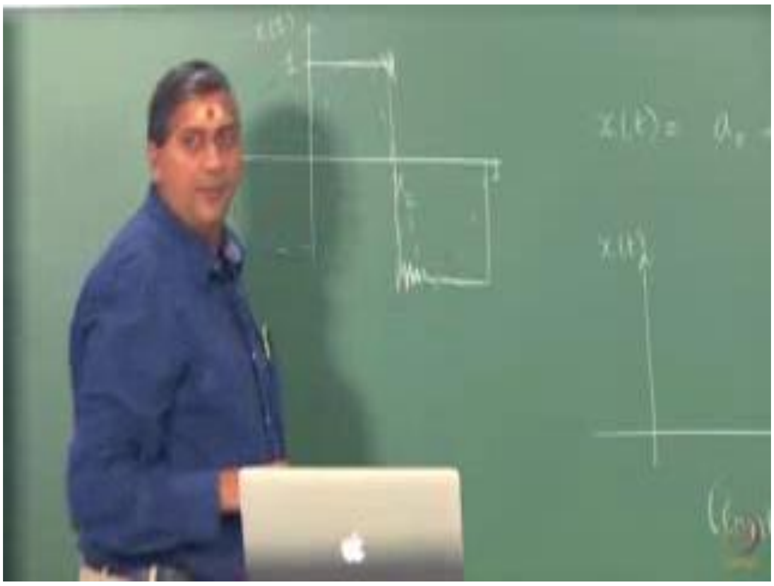

What does it tell me? It tells me that this signal which looks like this over one period, I am just only going to draw over one period. The signal has this expression from 0 to half it is 1 and from half to 1 it is minus 1. So, let say this is half this is your x of t this one it looks like this and then you have here minus 1. This is the behavior of the signal over one period. Just it is a pulse, and it says that to explain this pulse I need sinusoids of the fundamental frequency which is of period one and then I need some harmonics.

But, if you look at the expression itself and even if you look at the plot carefully you see that ideally you would require all harmonics. That is the c n does not go to 0 for any value of n identically 0 it may diminish, but it does not go to 0 an for any finite value of n right. So, what this means is to explain this kind of a pulse I need, fundamental frequency in fact let me show you here at least at the fundamental plus a few harmonics.

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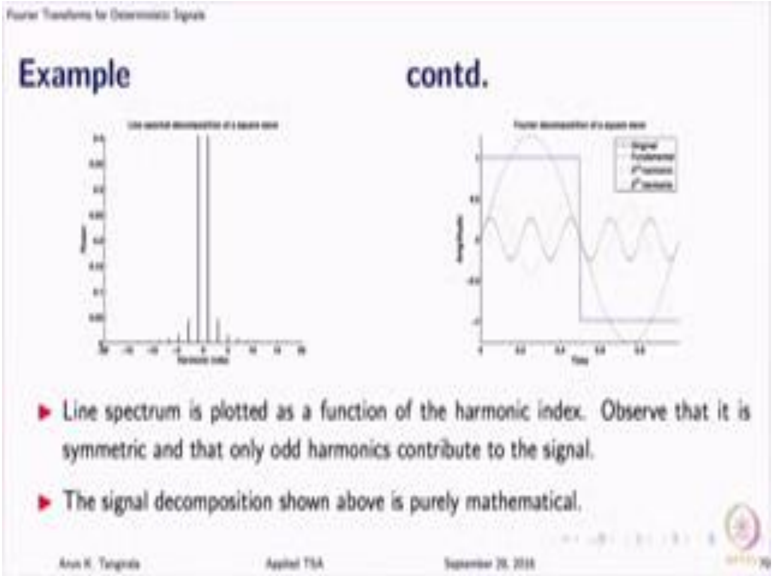

So, the first one is a fundamental, I do not know how well you can see there is a red line which is of the fundamental frequency that explains a lot. So, there is this sinusoid which does a good job of explaining this pulse that you have. And then there is a harmonic, and then there is another harmonic and so on which are trying to explain what has not being explained by the fundamental and the other harmonics. But, it turns out because of the nature of the signal I will require many many many harmonics to fully reconstruct a signal. In fact, this is a very special signal because it has sharp edges there it is not a smooth signal that we normally see for the periodicity periodic signals I mean as we imagine to be.

Now, there is something called Gibbs phenomenon in this Fourier series expansion I will talk about it in a minute, but what do you see here is I am if I were to for example, approximate this pulse with one or two or three signals, I just want to construct an approximation. I want to get rid of the sharp edges. I can actually only make use in my recovery equation. I can throw away all the negligible harmonics and only use the fundamentals and maybe the first two or three harmonics and come up with an approximate version of the signal, if I want to smoothen the signal, if I want to throw away the sharp edges and so on.

This is the basic idea in filtering, you break up and you say well I do not want certain features of the signal, but I want to retain the predominant features of the signal then you can construct approximations. The term filtering, approximation, prediction all of this actually fall into the broad banner of estimation. That anyway will talk about later. Any questions on this example; so what you see is now hopefully what you have a hopefully is a better feel of what Fourier series expansion is actually doing for you. There depends on what you want to do from a filtering view point I will look at it in particular way from a power spectral decomposition I will look at in different way and so on.

So, what you have now witnessed is a journey from signal decomposition to spectral decomposition. We started off by saying I will break up the signal, but gradually we moved on to power spectral decomposition because that is what is generally of interest to me. And that is going to be the case always.

Now, let us quickly talk about the existence of Fourier series as I said we need to at least know some answers if even though we may not look at the theoretical proofs and so on. When does the coefficient c n exists that is when does this integral that we have here for the analysis, when does it converge, for what class of signals does it converge.

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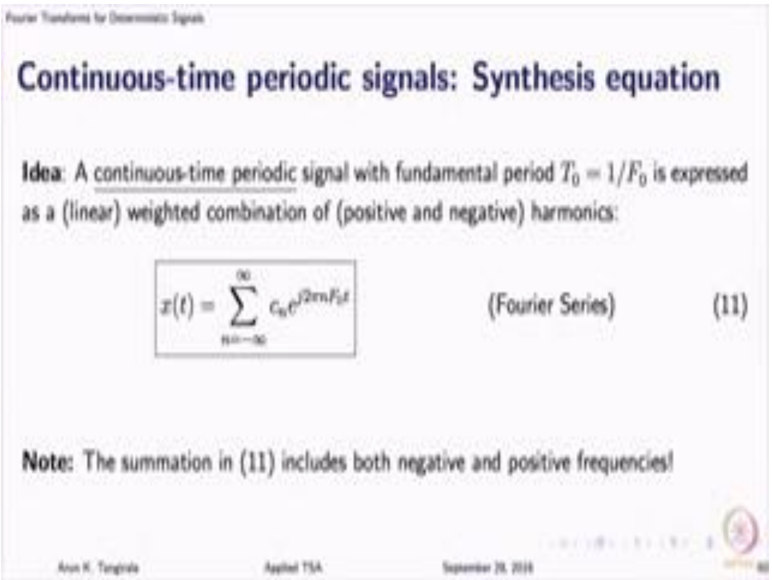

And likewise when does this summation converge, both questions have to answer. And it turns out that the c n exists so long as your signal is integrable over that one period. It is a fairly very very mild condition that x of t has to satisfy, it should be periodic for sure x of t has to be periodic, but otherwise you know as long as it is bounded and the integral the remain integral exists you should be ok; that is the first condition.

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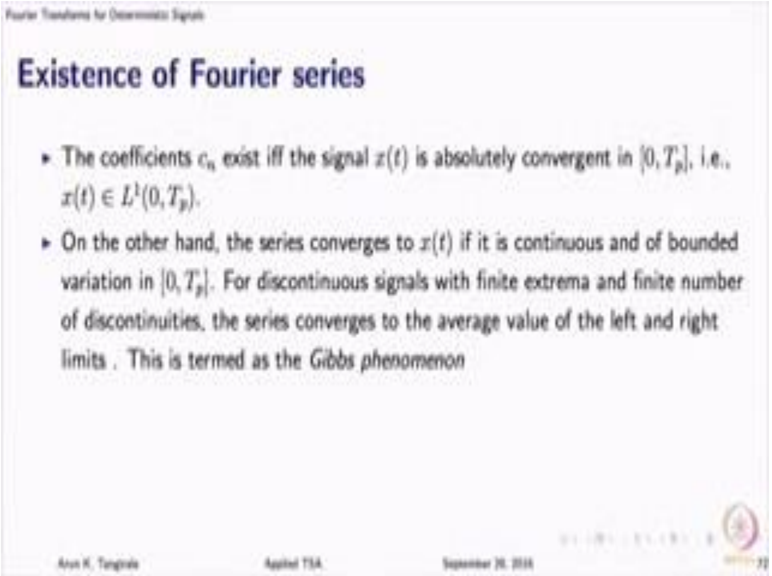

And then the other condition is on x of t; that it if you want the series in the synthesis equation to come there then the signal itself has to be continuous which is and should be of bounded variation in that one period. What happens is there are discontinuities? In the previous example that we saw was there a discontinuity, there was a discontinuity. What happens in such cases does the summation yield me x of t, it turns out that at the discontinuities it converges to the average of the values that it has at those points. And that was observed to that was named as a Gibbs phenomenon.

As you include more and more harmonics and so on still it does not converge. And what happens is at these points where you have discontinuities, yes you will be able to get fairly good approximations that at this point you will see this high frequencies coming in and that is what is Gibbs phenomenon. You can see that in my book I have shown you the example.

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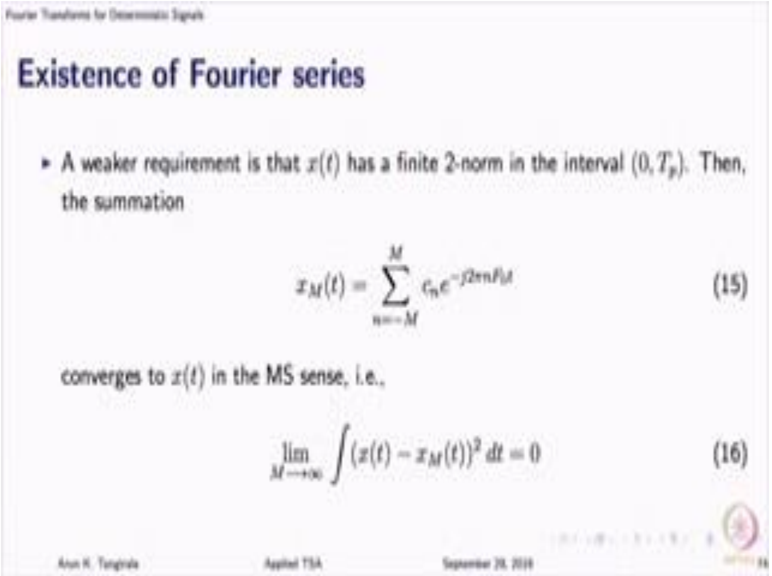

And then generally speaking sufficient condition is that for the series to converge in a mean square error sense is that the signal should have finite energy in one period, but this only for information.

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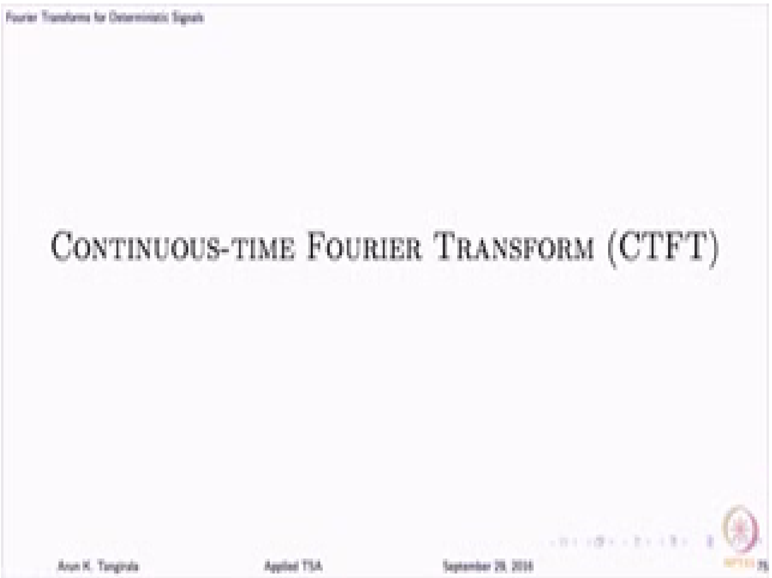

Lets quickly move on to the Fourier transform; continuous time Fourier transform I will spend maybe one or two minutes and we will adjourn we will continue our discussion tomorrow. So, we have now discussed how to analyze periodic signals using the Fourier series expansion. We have learnt there is a synthesis equation there is an analysis equation. We have also learnt that although we begin with signal decomposition by virtues of Parseval's relation we can move on to power spectral decomposition.

Now, we move on the class of aperiodic signals, and you may wonder why I should look at aperiodic signals at all. Because after all we set out by saying I want to detect periodicities. So, it is a valid question to have in our minds. Now the answer to that question is; we use Fourier analysis not only for the analysis of signals, but also for the analysis of systems. And what we mean by systems is linear time invariant system. So it turns out that when I construct a Fourier analysis, when I perform a Fourier analysis of aperiodic signals it becomes extremely useful in understanding the frequency characteristics of any LTI system what we mean by frequency characteristics is what frequency does it allow, does it filter, what frequencies does it attenuate, does it amplify; all such questions can be answered with a use of Fourier transform.

Now, we now we actually enter the Fourier transform business and two things we should observe as we move from periodic to aperiodic signals as I have said earlier it amounts to saying that now signals are infinite period. That means, we let t p go to infinity. As a result what happens to this frequency spacing, remember here in the periodic case there is a certain spacing in frequency what is that spacing equal to; that is a fundamental frequency. Now we are saying t p goes to infinity that means there is no periodicity. So, your fundamental frequency is heading to 0. I mean you can say that this delta f is actually going to 0 spacing itself is going to 0, in other words we should expect now a continuum.

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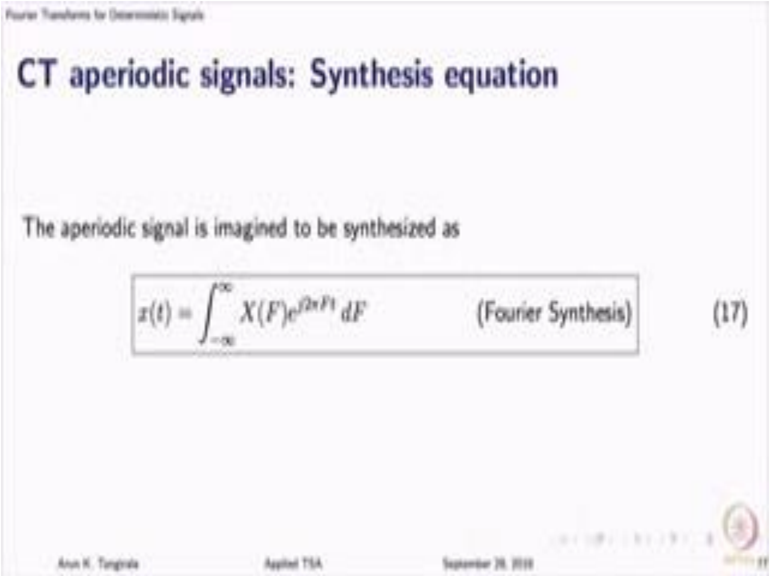

So, to express an aperioidic signal in terms of sines and cosines I will now have to admit all frequencies not just fundamentals and harmonics there is no notion of fundamental and harmonics and that is what leads us to this synthesis equation where we have now replaced c n by x of capital F. I use capital F to distinguish between a continuous time world and the discrete time world. When we move to the discrete time world we will use small f.

So, the big f is the frequency now it is a continuous valued quantity, and we have now replaced summation by an integral, and c n we have replaced it with x of f and rest of the expression looks very similar. There we had summation now we have integral, the summation there ran from minus infinity to infinity integral also runs from minus infinity to infinity; which means I have to include all frequencies but now it is a continuum.

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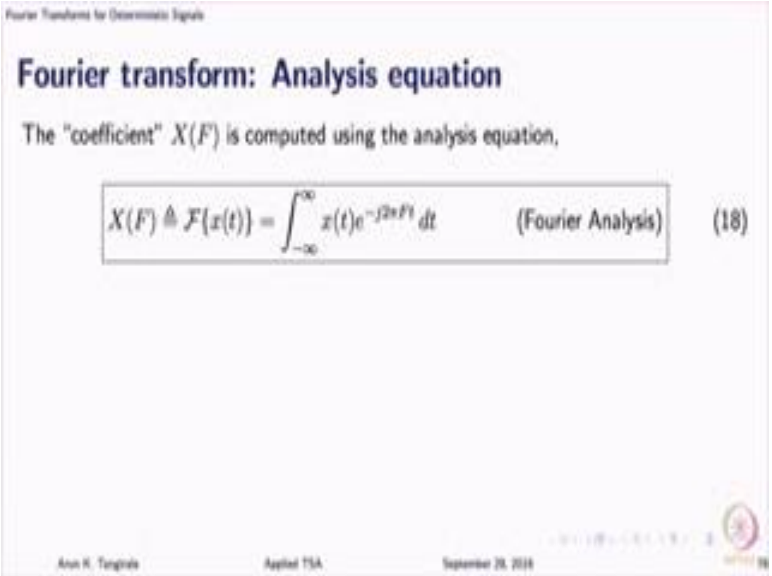

And likewise, the expression for the computation of so called Fourier coefficient; now we do not call it as coefficient per se we call this as a Fourier transform. This x of f now is a Fourier transform you can still call it as coefficient is now the integral x of t e to minus j 2 pi f t d t.

Now, we still have an integral, but what is the difference between this analysis equation and the previous one in the periodic case. Both are integrals, but what is the difference. We are now integrating over the entire x of t, there it was sufficient to integrate over one period, here we are integrating over the entire existence of x of t. Clearly that tells us this integral can diverge, there it was not you know the conditions were very mild, but now there is a possibility that this integral can diverge unless x of t has certain property. So, this is what is continuous time Fourier transforms. You should keep telling that this is continuous time Fourier transform because later on we will learn tomorrow we will learn discrete time Fourier transform. I know that as I promised to you will be confused towards the end, so this is second such contribution to your confusion.

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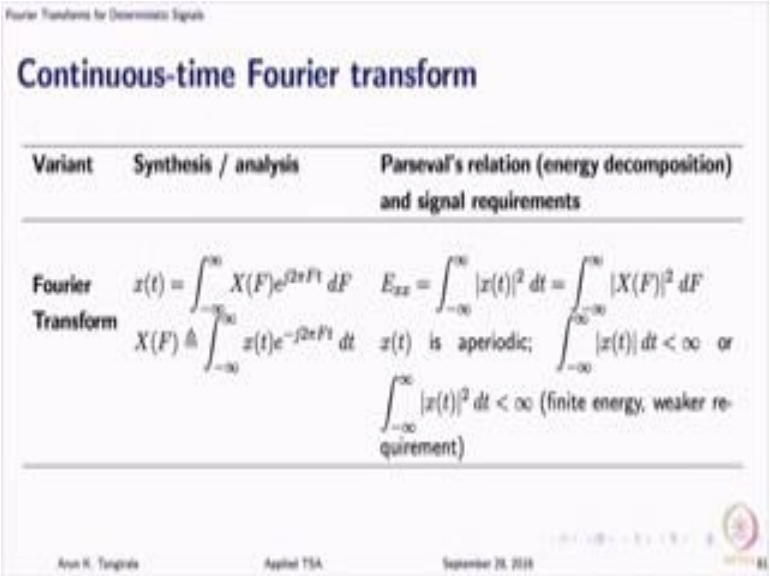

What we will do is tomorrow will go over an example and understand what this Fourier transform has to offer and that there is a Parseval’s relation as well which will go over similar to what we had for periodic signals. But, remember now we are dealing with aperiodic energy signals, we cannot have any class of signals coming in here; therefore our Parseval’s decomposition will give us a decomposition of the energy no longer power. Fine, we will meet tomorrow.