Applied Time-Series Analysis
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Lecture – 61
Lecture 27 A - Fourier Transforms for Deterministic Signals 3

Good morning, let us get started. In today's class we will learn the concepts of continuous time Fourier series and continuous time Fourier transform. There is a distinction between Fourier series and Fourier transform and will learn what the distinction is.

As I said yesterday any transform when you look at its application in a literature you will find predominantly two different kinds of applications: one is the analysis which is used to understand what features a signal contains. And that typically involves breaking up the signal into the so called building blocks or as we will call atoms into the respective atoms that we imagine that it is made up of. And then asking how much of those atoms are and so on. There are different names to these atoms, they are sometimes called building blocks, and they are sometimes called basis functions and so on.

Basically, you have to understand them as some elementary you can say constituents of the signal. And the other application of the transform is in filtering, which goes beyond analysis. So, you have analyzed the signal and you have decided what you want to retain and what you want to discard. Once you have broken up the signal you say well I do not want these components I know apriori these are not of interest to me and I am only going to retain this and then going to recover the signal; that is essentially filtering.

We will predominantly look at analysis, but also understand the filtering perspective, and it is very useful. Before we proceed to the Fourier series and transform, it is worthwhile talking about this concept of fixed versus adaptive basis.

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Just now I said the basic building blocks that are used that are used in a transform are called atoms or basis and so on. At this moment do not take the term basis in a linear algebra sense, we know what basis means in linear algebra sense. If I say a set of functions constitutes a basis for some space then they have to be linearly independent; that is a very important requirement. We are not really strictly speaking in a very strict sense here; the term basis should be interpreted as some building blocks.

Now, what is this fixed versus adaptive basis business? Whenever I am breaking up a signal I have two choices: I can break it up into known basis, in a sense known mathematical function. For example, in Fourier analysis I break up the signal into sines and cosines because that is what I imagine. Remember the synthesis precedes analysis, so I imagine that the signal has been synthesized from cosines and sines, whoever generated the signal whatever processes generated that signal that I am analyzing I am imagining it to be; I do not care really how it has been created, how it has been synthesized, but as far as the Fourier analysis is concerned I imagine the signal to be made up of sines and cosines.

Now, is it correct or wrong we will quickly talk about it, and there is nothing like correct or wrong it depends on the application. On the other hand I can imagine, I do not have to pre imagined the signal to be made up of something. I would rather figure that out by looking at the signal; that is I look at a signal say maybe this signal is made up of this,

this, this atoms. Like for example, I take the wall I do not know what it is made up of. I have two choices I can imagine always does not matter whichever wall you show me in any palace in any building; I always imagine it to be made up of a certain types of bricks which we call as a basis and that there is a mixing operation. It does not matter to me even if you show me the most beautiful palace in Rajasthan anywhere; I always imagine it to be that way

Now obviously, its common sense it is not correct to imagine because walls all need not be always made up of bricks they could be made up of other stones and so. On the other hand I look at the wall and figure out what it is made up of. Now when it comes to walls or tables and physical objects it is easier to figure out what it is made up of if I am allowed to really break it up physically. But when it comes to signals it is very hard to figure out what it is made up of. However, there is what elements must have gone in making up the signal. However, there exists some methods to be able to adaptively determine what the signal is made up of.

So, the difference between fixed and adaptive basis is; in the fixed basis approach I imagine the signal to be made up of always whatever I have in my bank in my basis bank. And there are certain advantages to it whereas, in a adaptive basis approach I do not have any pre determined basis bank with me, I will adapt according to the basis. And it is more or less like a chameleon approach, but it is useful.

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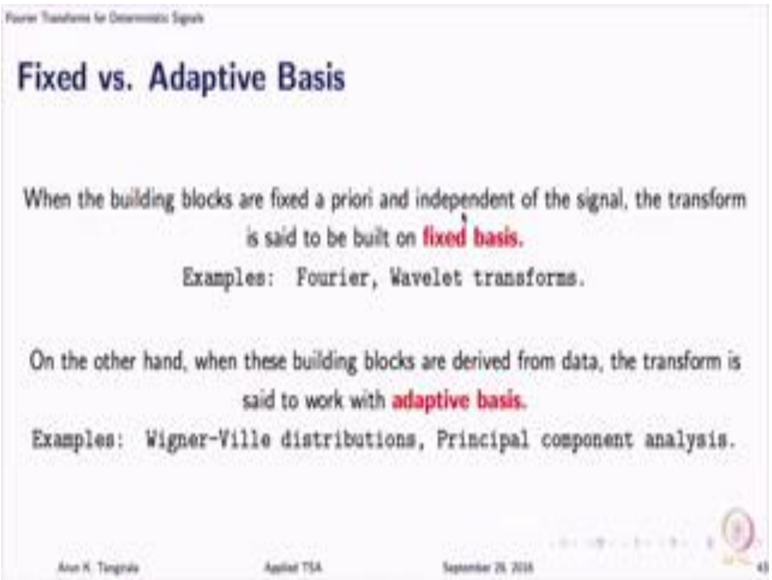

So, in the fixed basis approach you have the Fourier family, you have the wavelet family and so on. Whereas, in the adaptive basis approach you have a principal component analysis, for example; it figures out what is an appropriate basis for a given signal. And there are many others, I am just signal really distribution is not really a transform it basically looks at energy decomposition, but PCA is a very common tool you must have heard of it. Now you should not think that adaptive basis is superior always to fixed basis, and that the adaptive basis will always figure out correctly what the signal is made up of.

There are merits and demerits to both these approaches. In the fixed basis approach for example, Fourier analysis which is what we are going to focus on as I said assumes that the signal any signal you give to me I will assume it is made up of sines and cosines. Now, when is such an approach useful? Even if you take the Fourier family itself or the Fourier basis itself its useful when for example; I am interested in detecting periodicities I would like to know what frequencies are present or I would like to know what frequency content is present. On the other hand if I want to know what frequency is present at what time, over what time interval, because there are many signals over which frequencies do not the same frequency does not persist throughout. If you take change in colours of a flower in the morning or you take the sunset, sunrise colours or any many, even if you take speech signals,m music signals and so on; if you were to examine the frequency content they better change with time if somebody is speaking at a single frequency there are many speakers who are capable of that you are immediately put to sleep.

Now somehow that is how if a single frequency keeps hitting us at all the time. And if you speak like this for just 15-20 minutes that is it you are put to sleep there should be some modulation, there should be some variation in the frequencies and so on. And if that variation is pleasing we call it as music otherwise we call it as noise so or cacophonic. Now, when in analyzing such signals for Fourier basis functions are not suited. Then you have to turn to another class of functions which are wavelets for example; where the basis functions themselves have that characteristic. If you look at the Fourier family we have the sines and cosines as the atoms which persists forever; so they are suited for signals in which frequencies persist for ever.

Whereas if you are familiar with the wavelet family then the wavelet atoms are short lived they unlike your sines and cosines. By very their very nature they can capture short duration events or finite duration events. Again the wavelets family or the wavelet transform belongs to the fixed basis approach. In an adaptive basis approach there is one approach called empirical mode decomposition it is also known as the Hilbert Huang transform. There in what one does is breaks up the signal into its intrinsic functions mode functions using some algorithm, we will not go into that. There you are adaptively figuring out what the signal is made up of, but whether as well you have some assumption.

Now the advantage with an adaptive basis is you do not pre impose any your own bias or your own prejudice and rather look at the signal and figure out what is present which makes a lot more sense. However, the difficulty is in analyzing the properties of the broken up components. If I follow the fixed basis approach the advantage is I already know what the property of a sine or a cosine is. When I say property I know what the frequency is exactly, I know if I plot the power spectrum it looks like this and so on. So, I know what the properties of the atoms are. The moment I break up the signal into its constituent atoms if I see that certain atoms are present and others are not then I immediately say- you know these are the frequencies present in the signal it becomes very easy.

In an adaptive basis although I have an advantage of adopting to the signal, the disadvantage is I have to work on again figuring out what the properties of the broken up components are. This is only for your own perceptive and information maybe some of you are carrying out research and signal analysis and so on. So, you should see this broad classification of approaches in the transform world the Fourier, Laplace, wavelets and all they all belong to the fixed basis approach and we will stick to that in this course.

So, let us move on. And now ask what is a general idea in Fourier analysis, be it continuous time signal or discrete time signal the basic ideas remain the same, but the specifics of the signal synthesis an analysis change with the nature of the signal whether you are looking at continuous time or discrete time and whether you are looking at the periodic signal or an aperiodic signal. And gradually as we talk of signals we will also move to systems when the time comes. No signal exists without a system that is generating it.

So, any inference I draw of a signal is more or less an inference that I am making of the system that is generating here. So, when I talk of energy of a signal yesterday or a power of a signal one has to go beyond the signal and look at the system; what do I mean by energy of a signal for instance? The energy of a signal has to be understood as the energy expended by the system in order to generate that signal. You say this guy is has enormous energy keeps speaking all the day right, we do not say this guy is enormous power. So, how are you able to say that you are looking at the speech signal and you are saying this has a lot of energy and based on the speech signal you are drawing inferences about the speaker. Likewise, when we talk of energy or power of a signal we are explicitly or implicitly alluding to the process that is generating it. So, that should be kept in mind if you want suitable interpretation.

So, the general idea in Fourier analysis is to breakdown a signal into sines and cosines. Although, Fourier started off this with the idea of solving the heat conduction equation in an easy way; gradually as we all know Fourier analysis is more used today in analysis of signals measurements that you have and less in solving differential equations. Yes there are class of people who do that. But predominantly you see the Fourier analysis being used in data analysis or signal analysis.

Now, this is similar to expressing a signal as a combination of impulses, it is not that this idea was radically new, but there were some things which were radically new in terms of Fourier's claim that any periodic signal could be broken up into sines and cosines and so on. And it largely holds the claim holds.

Now, when you break up any signal into its atoms, typically you assume some mixing models you are assuming that these atoms are mixed up in a particular fashion to makeup this signal or to synthesize the signal. The purpose of signal analysis at least in a fixed basis approach like this is to figure out how much of each atom is present in a signal. So, these are called weights or coefficients. And they convey a lot of information about the signal as we shall learn. They can tell you when a particular feature began for example, or how much a particular frequency is contributing to the overall energy or power and so on. And that is essentially your spectral analysis.

And there is also this correlation perspective in signal analysis in a signal transform which is when we compute this weights as I just said or the mixing coefficients. You can think of these weights or coefficients as correlation between the signal and the atom that you have. If the correlation is very high, that is if the signal under analysis matches very well with the atoms that you have then like a bulb it glows very brightly otherwise it is going to be dim or even switched off. So that tells you essentially how much of each atom is present. This is a qualitative, but a very useful perspective. Sometimes you can quantitatively say that yes indeed the weight is nothing but some kind of correlation.

Let us move on some of these remarks I will actually visit once we talk about the respective methods. So, the first in class is the continuous time Fourier series. Now although I say this is continuous time Fourier series I would like you to remember this in a different way. What we are looking at is analysis of or Fourier analysis of continuous time periodic signals this is where the journey began and that are why historically I am just picking up this. Sometimes it may not be necessary to know the continuous time world, but it is good to begin with this so that certain terminologies and jargon becomes clear. So, the idea here in the continuous time Fourier series expansion is that I take a continuous time periodic signal and imagine it to be made up of sines and cosines. Now as I say sines and cosines I do not have the sines and cosines explicitly in the expression that I have for this synthesis equation.

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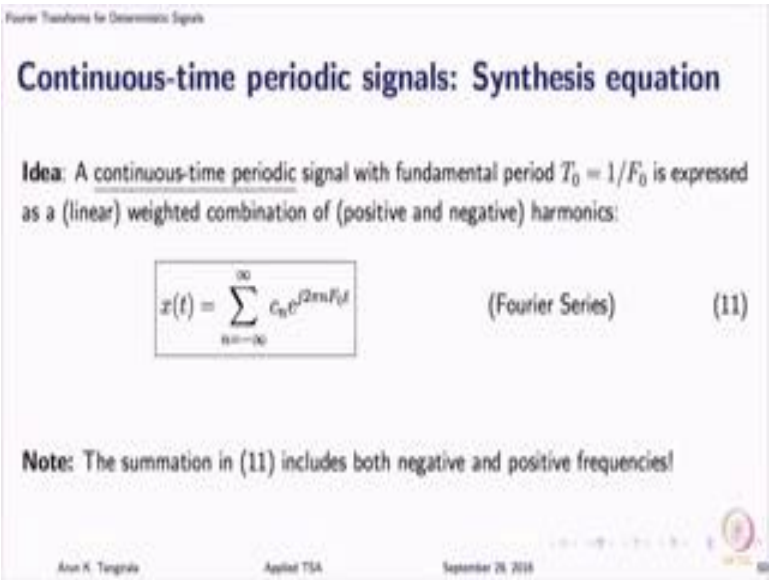

So, what you see on the screen is the synthesis equation, it is called the Fourier series or series expansion of x of t. And x of t is assumed to have a fundamental period t naught, this is the continuous time signal, therefore the period of the signal is a real value number and the fundamental frequency is 1 over t naught. You can think of f naught as 1 over t naught or t naught as 1 over f naught. And what we are doing in this series is expressing the period, sorry signal as a sum of all signals that have the same period.

So, if I take the fundamental frequency that is present here, in fact when n equals 1 in the series expansion that is where I have the fundamental frequency. When I move on to n equals 2 then I am including the first harmonic which also has a same period it may have the twice of frequency, it does have the twice the frequency of the fundamental harmonic but the period is the same, the only difference is for the first harmonic the fundamental period is not t naught. But it is still periodic with the period t naught.

So, it intuitively if you look at it the signal has a period of t naught so it makes sense only to include atoms or building blocks that have the same period. If I include any other function in, that is the function which has a different period in the expansion then I will run into problem because that would not contribute; that is the nature of the sines and cosines. So, that is the basic idea you are expressing a periodic signal in terms an addition of the fundamentals plus harmonics.

Now there are two points to keep in mind: there is also n equals 0 which corresponds to the d c component of the signal. And if the average of the signal is 0 then the corresponding coefficient c naught would turn out to be 0. And secondly, what keeps bothering everyone is this negative frequency business. So, you see that in the summation we have both negative frequency and positive frequency.

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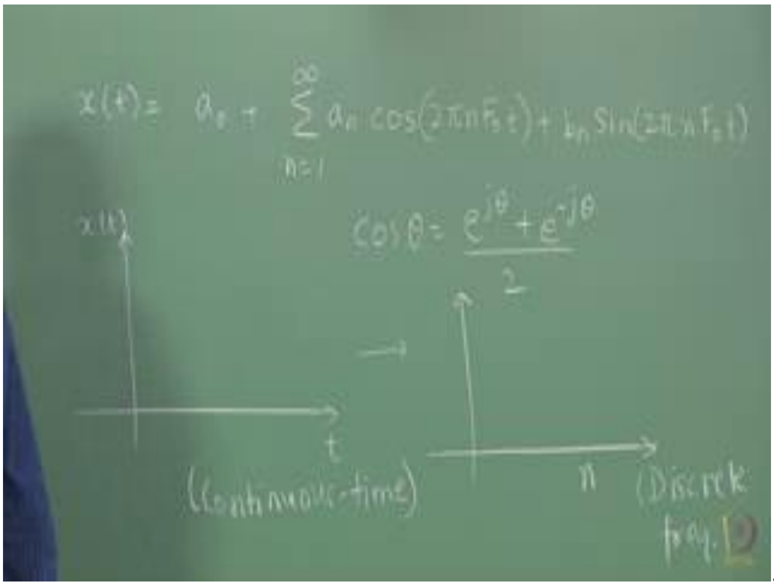

However, if you look at the original proposition that Fourier made, there was nothing like this Fourier kept it a bit simpler let us say you have 2 pi n f naught t plus b n sin 2 pi n f naught t; n here running from 1 to infinity. So in this expansion, this was the original expansion that was proposed there is nothing like a negative frequency. So, where did the negative frequency come in? When we wanted mathematical elegance and convenience and compactness and so on where we said we will replace these cosines and sines with the complex exponentials using Euler's formula.

And that is when the notion of negative frequencies actually came in. Otherwise you can see here if x is real value all your coefficients a's and b's are going to be real value, there is no issue there. Whereas, in the Fourier series expansion that we see on the screen which is also used extensively the coefficient c are complex valued in general. Now, it is hard to imagine a mixing coefficient being complex valued. What do you mean by take a I take a complex valued atom? First of all imagining a complex valued atom itself is difficult and then to top it you have a complex valued coefficient, but if you make it a practice whenever you have this trouble in your mind go back to this and say- it is only a mathematical artifact that that makes this coefficients complex and the atoms complex valued you should be ok.

Therefore, negative frequencies come up only because you are replacing the cosine and sines with a complex valued, you are replacing real valued atoms with complex valued atoms. Naturally, you need for example if you replace cosine with the complex valued you know from Euler's formula that cosine theta is e to the j theta plus e to the minus j theta by 2. And that is how the story of negative frequencies begin.

In certain textbooks you see that positive frequencies correspond to clockwise motion and negative frequencies correspond to counter clockwise, I do not really subscribe to that. And there is no base, there is no really basis to believe all of that, they is just interpretations retrofitting and so on. But the fact is you have negative frequencies coming in because of this. There are other explanations, but anyway let us not go into that. So, what you have to remember is that the coefficients of this expansion are in general going to be complex valued. Therefore, there is a magnitude and phase of these coefficients and we will soon talk about that.

So, this is a synthesis equation, this is not going to be useful in practices for me. At least anyway this entire continuous time case is not useful in practice let me tell you that, it is only useful for theoretical analysis or if you are given a signal x of t you can actually perform a Fourier analysis; you cannot use it on data let me tell you that; so will not spend too much time on this. Nevertheless it is extremely useful in a lot of other applications. Sometime you must have seen this kind of application used in summing up series for example; if you are summing up infinite series the Fourier series expansion comes very handy.

Now let us ask, how these coefficients are calculated? It is not sufficient and to just say this is a synthesis equation without telling you how to compute the coefficients. Now the coefficients are computed using an integral. So, notice that the synthesis equation has summation why does it have a summation you have to be very clear in your mind, because we are not including all the frequencies, we are only including in our bank of atoms we have only fundamentals and harmonics. So, the frequency axis is not a continuum, whereas the time axis is a continuum. So, that part you have to understand.

If you were to plot here the time, this is your x of t here it is a continuous domain, but when I move to the frequency domain here where the axis is now n will I will tell you what to plot here, but let us say you are image you are plotting magnitude of c n does not matter. But the x axis is going to be now the index, the index of the atom that will help you keep track of which atom is present in or which set of atoms are present or basis functions are present in your signal.

This n is an integer. So, here you are in discrete frequency domain not discrete time, whereas the signal itself is in continuous time, but the signal is periodic. Very soon when we move to aperiodic signals we will realize although the signal is continuous time the frequency axis is also continuum. The main reason why the frequency axis is discrete is because the signal is periodic with some period and it makes sense only to include those signs or complex signs which have the same period. Therefore, not all frequencies are admitted only the specific ones and you have a discrete index or discrete frequency domain. And that is why this subscript is also n for the coefficient.

And now it explains also, if you look at the expression for the coefficients you have an integral. This integral has come about because you are integrating x of t, you are dealing with x of t or figuring out what that c n is. How does one arrive at this expression you might wonder, that is what is usual but it is very straight forward; all you do is you go back to the synthesis equation. And then multiply both sides with e to the minus j right 2 pi n f naught t. Notice that in the synthesis equation I assume it is e to the j, whereas in the analysis equation I have e to the minus j; there is a mistake there I will correct it should have e to the minus j, I will make that correction.

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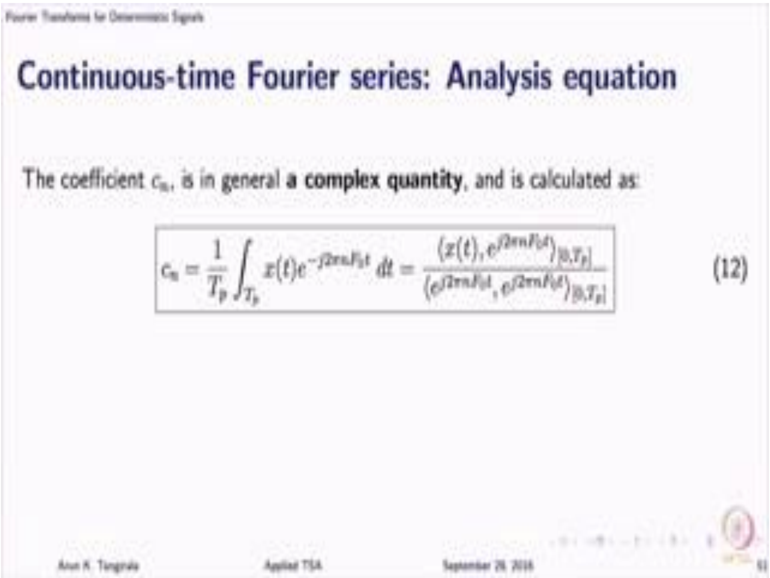

Sorry, it does have perfect. So, in the analysis equation you have e to the minus j, how do you arrive at this equation? Multiply both sides of your Fourier series with it with e to the minus j and use the so called orthogonality property of this complex exponentials. What we mean by orthogonality property is if you talk of in terms of inner products the inner product between two complex sin waves is 0 if they are of different frequencies, if they are of identical frequencies their value turns out to be 1. And the inner products for functions continuous time functions are essentially defined in terms of integrals. So, you can see this derivation anywhere, but intuitively what turns out is I just have to multiply both sides with the conjugate of the atom and integrate, the moment I integrate only one term remains on the right hand side which corresponds to the value of n, all the other terms turn out to be 0. That is because of the orthogonality property of these atoms, and that is a beautiful property of the Fourier atoms.

This orthogonality property makes life a lot easier in terms of calculations and interpretation and so on, I will tell you why the orghogonality property is useful in interpretation, but you can see in terms of calculation its very straight forward. What I mean by calculation is calculation of c m, all I have to do is multiply both sides by conjugates I do not know if you are familiar with inner products of functions that do have inner products. If I look at if I were to evaluate inner products of two functions f 1 and f 2 the appropriate limits these are periodic functions let us say, so I have here f 1 of t times f 2 star of t dt. This is called the inner product.

What is f 2 star conjugate, conjugate of f 2; that is exactly what we are doing here. And these complex signs have the property that if f 1 is of a particular frequency let us say n times f naught is one harmonic and f 2 is of a different is a different harmonic then integral works out to be 0. If they are same then it turns out to be 1. And that is why you see in this expression for c n an inner product notation, in a numerator normalized with the value of the inner product between e to the j 2 pi and f naught and itself.

So, you can therefore give it what is known as a projection perspective, I am not going to talk of projections; but projections in functional analysis in linear algebra basically involves inner products. You are projecting essentially x of t onto the family of cosines and sines. Projection in a loose sense can be thought of and its very useful perspective is nothing but a shadow, so when we are walking outside on the road in the daylight or in bright moonlight the entire 3D image actually is projected onto the two dimensional road, and that is essentially a projection. And what we call is daily language is a shadow, mathematically it is a projection. Here you have x of t being made up of several sines and cosines in general and you are projecting x of t onto one or each of these atoms, and asking how much of this atom is present. And that projection is nothing but your inner product; normalized inner product you can say so.

So, to summarize we imagine in Fourier series expansion the periodic signal to be made up of fundamentals and harmonics and the coefficients are calculated by evaluating this integral over one period, that integral T p would mean that I can choose any interval of period T p or t naught. Please understand that this T p is the same as t naught that is it. So, that is what your Fourier synthesis and analysis is.